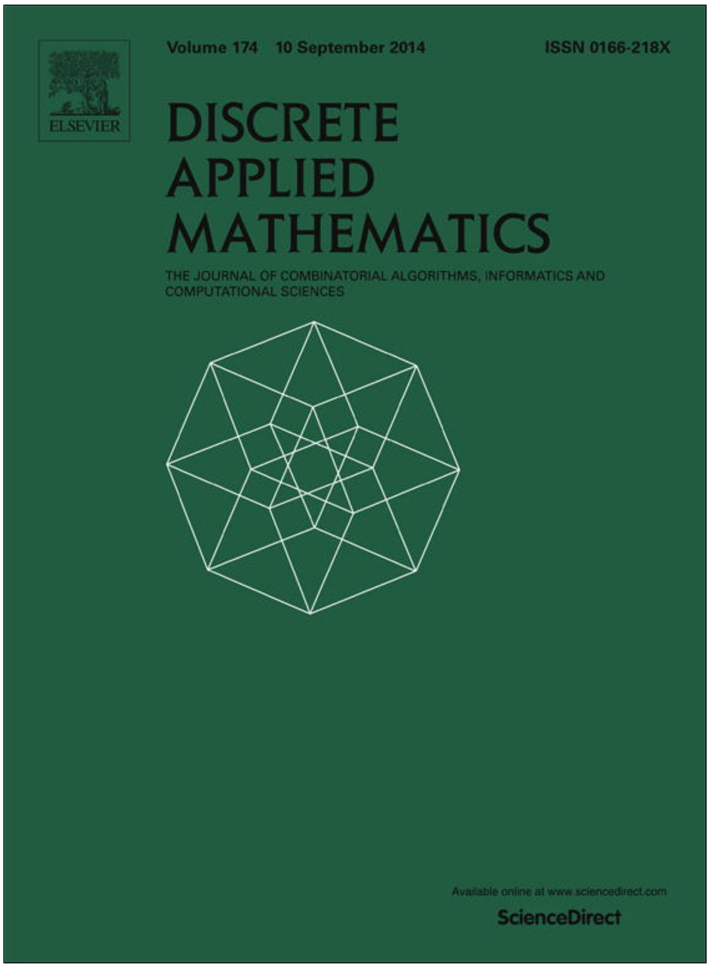
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On strongly $\mathbb{Z}_{2s+1}$-connected graphs

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ABSTRACT

An orientation of a graph G is a mod$(2s+1)$-orientation if under this orientation, the net out-degree at every vertex is congruent to zero mod$(2s+1)$. If for any function $b : V(G) \to \mathbb{Z}_{2s+1}$ satisfying $\sum_{v\in V(G)} b(v) \equiv 0 \pmod{2s+1}$, G always has an orientation D such that the net out-degree at every vertex v is congruent to $b(v) \mod (2s+1)$, then G is strongly $\mathbb{Z}_{2s+1}$-connected. In this paper, we prove that a connected graph has a mod$(2s+1)$-orientation if and only if it is a contraction of a $(2s+1)$-regular bipartite graph. We also proved that every $(4s-1)$-edge-connected series–parallel graph is strongly $\mathbb{Z}_{2s+1}$-connected, and every simple $4p$-connected chordal graph is strongly $\mathbb{Z}_{2s+1}$-connected.

1. Introduction

We consider finite graphs without loops, but multiple edges are allowed, and we follow [1] for undefined terms and notations. In particular, for a graph G, $\kappa(G)$ and $\kappa'(G)$ denote the connectivity and edge-connectivity of G, respectively. If H_1 and H_2 are subgraphs of a graph G, then $H_1 \cap H_2$ and $H_1 \cup H_2$ are the intersection and the union of H_1 and H_2, respectively, as defined in [1]. For subsets $S, S' \subseteq V(G)$, $[S, S']$ denotes the set of edges of G with one end in S and the other in S'. If $X \subseteq E(G)$ is an edge subset, then the *contraction* G/X is obtained by identifying the two ends of each edge in X and then deleting all the resulting loops. As shown on p. 55 of [1], the contraction does not delete resulting multiple edges. If H is a subgraph of G, we use G/H for $G/E(H)$. Throughout this paper, $\mathbb{Z}$ denotes the set of all integers. For an $m \in \mathbb{Z}$, $\mathbb{Z}_m$ denotes the set of integers modulo m, as well as the additive cyclic group on m elements. For a graph G, and for any integer $i \geq 0$, define

$$V_i(G) = \{v \in V(G) : d_G(v) = i\}.$$

Let D denote an orientation of G. Following [1], for an edge $e = uv \in E(G)$, if e is oriented from u to v under D, we use (u, v) to denote this arc (directed edge). For each $v \in V(G)$, $d^+_D(v)$ and $d^-_D(v)$ denote the out-degree and the in-degree of v

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under this orientation, respectively. When the orientation D is clear in the context, we use d^+ and d^- to denote d_D^+ and d_D^-, respectively. If a graph G has an orientation D such that at every vertex $v \in V(G)$, $d_D^+(v) - d_D^-(v) \equiv 0 \pmod{2s+1}$, then we say that G admits a mod$(2s+1)$-*orientation*. The set of all graphs which have mod$(2s+1)$-orientations is denoted by M_{2s+1}.

Let A be an (additive) abelian group and G be a graph with an orientation $D = D(G)$. For any vertex $v \in V(G)$, let $E_D^+(v)$ denote the set of all edges directed out from v, and $E_D^-(v)$ the set of all edges directed into v. For a function $f : E(G) \to A$, define $\partial f : V(G) \to A$, called the *boundary of* f, as follows:

$$\text{for any vertex } v \in V(G), \quad \partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e).$$

A function $b : V(G) \to A$ is a *zero-sum function* on A if $\sum_{v \in V(G)} b(v) \equiv 0$, where 0 denotes the additive identity. The set of all zero-sum functions on A of G is denoted by $Z(G, A)$. Let A' be a subset of A. We define $F(G, A') = \{f : E(G) \to A'\}$. For any zero-sum function b on A of G, a function $f \in F(G, A')$ satisfying $\partial f = b$ is referred to as an (A', b)-flow. When $b = 0$, an $(A - \{0\}, 0)$-flow is known as a nowhere zero A-flow in the literature (see [4,5], among others). Following [5], if for any zero-sum function b on A of G, G always has an $(A - \{0\}, b)$-flow, then G is *A-connected*.

Our research is motivated by the study of group connectivity initiated in [5]. Let G be a graph under a given orientation D. A *unitary $\mathbb{Z}_m$-flow* is a function $f \in F(G, \{\pm 1\})$ such that $\partial f = 0$. Given any unitary $\mathbb{Z}_m$-flow f under an orientation D, by keeping the orientation of each edge with $f(e) = 1$ and reversing the orientation of each edge with $f(e) = -1$, we then obtain a mod m-orientation D_f, on which the constant function that assigns every edge with the value 1 is a unitary $\mathbb{Z}_m$-flow of G. Thus a graph G has a unitary $\mathbb{Z}_m$-flow if and only if G has a mod m-orientation.

The concept of group connectivity can be extended also. A graph G is *strongly $\mathbb{Z}_m$-connected* if, under a given orientation D, for any zero-sum function b on $\mathbb{Z}_m$ of G, there exists a function $f \in F(G, \{\pm 1\})$ such that $\partial f = b$. Again, for a given $b \in Z(G, \mathbb{Z}_m)$ and an $f \in F(G, \{\pm 1\})$ with $\partial f = b$, one can keep the orientation of each edge with $f(e) = 1$ and reverse the orientation of each edge with $f(e) = -1$ to obtain a new orientation D' of G such that for any vertex $v \in V(G)$, $d_{D'}^+(v) - d_{D'}^-(v) = b(v) = \partial f(v)$. This orientation D' will be referred to as a *$(\mathbb{Z}_m, b)$-orientation* of G. Thus a graph G is strongly $\mathbb{Z}_m$-connected if and only if for any $b \in Z(G, \mathbb{Z}_m)$, G always has a $(\mathbb{Z}_m, b)$-orientation. We use M_m^o to denote the collection of graphs that are strongly $\mathbb{Z}_m$-connected.

Tutte and Jaeger proposed the following conjectures concerning mod$(2s+1)$-orientations. A conjecture on strongly $\mathbb{Z}_{2s+1}$-connected graphs has also been proposed recently.

Conjecture 1.1. *Let $s \geq 1$ denote an integer.*

(i) *(Tutte [13]) Every 4-edge-connected graph has a mod 3-orientation.*
(ii) *(Jaeger [3,4]) Every 4s-edge-connected graph has a* mod$(2s+1)$-*orientation.*
(iii) *(Jaeger [3,4]) Every 5-edge-connected graph is strongly $\mathbb{Z}_3$-connected.*
(iv) [9,10] *Every $(4s+1)$-edge-connected graph is strongly $\mathbb{Z}_{2s+1}$-connected.*

Conjecture 1.1(i) is well-known as Tutte's 3-flow conjecture. Conjecture 1.1(ii) is an extension of Tutte's 3-flow conjecture, which includes Conjecture 1.1(i) as the special case of $p = 1$. In [7], Kochol showed that to prove Conjecture 1.1(i), it suffices to prove that every 5-edge-connected graph has a mod 3-orientation. Consequently, Conjecture 1.1(iii) implies Conjecture 1.1(i). To the best of our knowledge, all these conjectures remain open. The best known results so far have been recently obtained by Thomassen [12], and by Lovász, Thomassen, Wu and Zhang [11].

Theorem 1.2 (*Thomassen, [12]*)**.** *Every 8-edge-connected graph is strongly $\mathbb{Z}_3$-connected.*

Theorem 1.3 (*Lovász, Thomassen, Wu and Zhang [11], Wu [14]*)**.** *Every 6s-edge-connected graph is strongly $\mathbb{Z}_{2s+1}$-connected.*

The main results of this paper are the following.

Theorem 1.4. *A connected graph admits a unitary $\mathbb{Z}_{2s+1}$-flow if and only if it is a contraction of a $(2s+1)$-regular bipartite graph.*

Theorem 1.5. *Every $(4s-1)$-edge-connected series–parallel graph (graph with no K_4-minor) is strongly $\mathbb{Z}_{2s+1}$-connected.*

Theorem 1.6. *Every simple 4s-connected chordal graph is strongly $\mathbb{Z}_{2s+1}$-connected.*

The bounds in Theorems 1.5 and 1.6 are best possible in some sense. We shall show that there exist infinitely many $(4s-2)$-edge-connected K_4-minor free graphs that are not strongly $\mathbb{Z}_{2s+1}$-connected; and there exist $(4s-1)$-connected chordal graphs that are not strongly $\mathbb{Z}_{2s+1}$-connected.

We shall present some of the useful facts and preliminary results in the next section. The proofs for Theorems 1.4–1.6 will be in the subsequent sections.

2. Some useful facts

In this section, we review some of the useful properties to be applied in our arguments, introduce the mod$(2s+1)$-closure of a graph, and investigate the distribution of the in-degrees and out-degrees of certain vertices in a graph with a mod$(2s+1)$-orientation.

The statements (i) and (ii) of Proposition 2.1 below are proved in Proposition 2.2 of [9].

Proposition 2.1 *([9]). For any integer $p \geq 1$, each of the following holds.*

(i) *If G is strongly $\mathbb{Z}_{2s+1}$-connected and e is an edge of G, then G/e is strongly $\mathbb{Z}_{2s+1}$-connected.*
(ii) *If H is a subgraph of G, and both H and G/H are strongly $\mathbb{Z}_{2s+1}$-connected, then so is G.*

Given a graph G and an integer $m > 0$, the graph $G^{(m)}$ is obtained by replacing each edge e of G by m parallel edges joining the same two end vertices of e. The next lemma presents some examples of graphs that are in M^o_{2s+1}, the set of all strongly connected graphs, and examples of graphs that are not in M^o_{2s+1}.

Lemma 2.2. *Let G be a graph, and $m, s \geq 1$ be integers. Each of the following holds.*

(i) *If G is strongly $\mathbb{Z}_{2s+1}$-connected, then G is $2s$-edge-connected.*
(ii) *$K_2^{(m)}$ is strongly $\mathbb{Z}_{2s+1}$-connected if and only if $m \geq 2s$.*

Proof. (i) By contradiction, assume that G is in M^o_{2s+1} with $\kappa'(G) < 2s$. Then G has an edge cut X with $|X| < 2s$. Let G_1, G_2 denote the two components of $G - X$. By Proposition 2.1(i), $G' = G/G_1 \in M^o_{2s+1}$. Let v denote the vertex of G' onto which G_1 is contracted. Then $d_{G'}(v) = |X| < 2s$.

Suppose first that $d_{G'}(v) = 2k < 2s$. Pick a zero-sum function $b \in Z(G', \mathbb{Z}_{2s+1})$ with $b(v) \equiv 1 \pmod{2s+1}$. As $G' \in M^o_{2s+1}$, G' has a $(\mathbb{Z}_{2s+1}, b)$-orientation $D = D(G')$. Under this orientation, $d^+(v) + d^-(v) = 2k$ and $d^+(v) - d^-(v) \equiv 1 \pmod{2s+1}$. It follows that $2d^+(v) \equiv 2k + 1 \pmod{2s+1}$. Since $0 < k < s$, and since $0 \leq d^+(v) \leq s - 1$, we have $2d^+(v) = 2k + 1$, which is impossible.

Next we assume that $d_{G'}(v) = 2k+1 < 2s$. Let $b \in Z(G', \mathbb{Z}_{2s+1})$ be a function with $b(v) \equiv 0 \pmod{2s+1}$. As $G' \in M^o_{2s+1}$, G' has a $(b, \mathbb{Z}_{2s+1})$-orientation $D = D(G')$. Under this orientation, $d^+(v)+d^-(v) = 2k+1$ and $d^+(v)-d^-(v) \equiv 0 \pmod{2s+1}$. It follows again that $2d^+ \equiv 2k + 1 \pmod{2s+1}$. Since $0 < k < s$, and since $0 \leq d^+ \leq s - 1$, we have $2d^+ = 2k + 1$, which is impossible.

(ii) First assume that $m = 2s$. By Part (i), it suffices to show that $K_2^{(m)} \in M^o_{2s+1}$. Let $V(K_2^{(m)}) = \{v_1, v_2\}$, and $b(v_1) \equiv b' \pmod{2s+1}$ with $0 \leq b' \leq m$. Then exactly one member of $m - b'$ and $b' - 1$ is an even number $2t$ with $0 \leq t \leq s$. Orient $K_2^{(m)}$ such that exactly t edges are directed from v_2 to v_1 if $m-b'$ is even; or such that exactly t edges are directed from v_1 to v_2 if $b'-1$ is even. This yields a $(\mathbb{Z}_{2s+1}, b)$-orientation of $K_2^{(m)}$, and so $K_2^{(m)} \in M^o_{2s+1}$. If $m \geq 2s+1$, then $K_2^{(m)}/K_2^{(2s)} = K_1 \in M^o_{2s+1}$, and so by Proposition 2.1(ii), $K_2^{(m)} \in M^o_{2s+1}$. This completes the proof of the lemma. □

Definition 2.3. Let H be a subgraph of G, and let $s > 0$ be an integer. The mod$(2s+1)$-*closure* of H in G, denoted by $cl_G^{2s+1}(H)$ or $cl(H)$ when G and s are understood from the context, is the (unique) maximal subgraph of G that contains H such that $V(cl(H)) - V(H)$ can be ordered as a sequence $\{v_1, v_2, \ldots, v_t\}$ such that $|[\{v_1\}, V(H)]| \geq 2s$ and for each i with $1 \leq i \leq t-1$,

$$|[\{v_{i+1}\}, V(H) \cup \{v_1, v_2, \ldots, v_i\}]| \geq 2s. \tag{1}$$

Any sequence $\{v_1, v_2, \ldots, v_t\}$ satisfying (1) will be referred to as a *closure sequence* of H in G.

Proposition 2.4. *Let H be a subgraph of G, and let $s > 0$ be an integer, and let $cl(H) = cl_G^{2s+1}(H)$. If H is strongly $\mathbb{Z}_{2s+1}$-connected, then each of the following holds.*

(i) *$cl(H)$ is strongly $\mathbb{Z}_{2s+1}$-connected.*
(ii) *The graph G is strongly $\mathbb{Z}_{2s+1}$-connected if and only if $G/cl(H)$ is strongly $\mathbb{Z}_{2s+1}$-connected.*
(iii) *The graph G admits a unitary $\mathbb{Z}_{2s+1}$-flow if and only if $G/cl(H)$ has a unitary $\mathbb{Z}_{2s+1}$-flow.*

Proof. Let $(v_1, v_2, \ldots, v_t)$ denote a closure sequence of H in G. Let $H_i = G[V(H) \cup \{v_1, v_2, \ldots, v_i\}]$ with $H_0 = H$. We argue by induction on $0 \leq i \leq t$ to show that $H_i \in M^o_{2s+1}$. As $H \in M^o_{2s+1}$, we assume that $H_{i-1} \in M^o_{2s+1}$ with $i \geq 1$. By (1), v_i is adjacent to $m \geq 2s$ vertices in H_{i-1}. Thus $H_i/H_{i-1} \cong K_2^{(m)}$ with $m \geq 2s$, and so by Lemma 2.2(ii), $H_i/H_{i-1} \in M^o_{2s+1}$. Then by Proposition 2.1(ii), $H_i \in M^o_{2s+1}$, and so $cl(H) = H_t \in M^o_{2s+1}$ follows by induction. This proves Part (i).
Parts (ii) and (iii) follow from Proposition 2.1(i) and (ii), and by Part (i) above. □

Lemma 2.5. *Let $s > 0$ be an integer.*

(i) *(Corollary 3.4 in [10]) K_{4s+1} is strongly $\mathbb{Z}_{2s+1}$-connected.*
(ii) *For any $n \geq 4s + 1$, K_n is strongly $\mathbb{Z}_{2s+1}$-connected.*

Proof. To prove Part (ii), we view $H = K_{4s+1}$ as a subgraph of $G = K_n$. By Lemma 2.5(i), $H \in M^o_{2s+1}$. Since $cl(K_{4s+1}) = K_n$, it follows from Proposition 2.4(i) that $K_n \in M^o_{2s+1}$. □

Let G be a connected graph with a mod$(2s+1)$-orientation D. For every vertex $v \in V_{4s-1}(G)$, if $d_D^+(v) = 3s$ (or if $d_D^+(v) = s - 1$, respectively), then v is called a *positive vertex* of D (or a *negative vertex* of D, respectively). Part (i) of the following lemma follows immediately from the definition.

Lemma 2.6. *Let G be a connected simple graph with a $\bmod(2s+1)$-orientation D, and let $X \subseteq V_{4s-1}(G)$ be a set of positive (or negative) vertices of G such that $G[X]$ is a complete subgraph of G. Each of the following holds.*

(i) *For every vertex $v \in V_{4s-1}(G)$, v is either a positive vertex or a negative vertex of D.*
(ii) $|V(G)| - |X| \geq 2s+1$.
(iii) $|X| \leq 2s-1$.

Proof. (ii) We assume that there exists a set X of positive vertices with $|V(G)| - |X| \leq 2s$ such that $H' = G[X]$ is a complete graph. Then $D' = D(H')$ is a subdigraph of $D = D(G)$. At each vertex $x \in X$, since x is a positive vertex, then by Lemma 2.6(i) and by the assumption of $|V(G)| - |X| \leq 2s$, D' has at least s edges directed out from x, and at most $s-1$ edges directed into x. This leads to a contradiction: $s|X| \leq |E(H')| \leq (s-1)|X|$.

(iii) By contradiction, we assume that $|X| \geq 2s$. Let $X' \subseteq X$ with $|X'| = 2s$. Then $H' = G[X']$ is a complete graph and $D' = D(H')$ is a subdigraph of $D = D(G)$. At each vertex $x \in X$, since x is a positive vertex, it follows from Lemma 2.6(i) that $d^-_{D'}(x) \leq s-1$, and so $s(2s-1) = |E(H')| = \sum_{x\in X} d^-_{D'}(x) \leq (s-1)2s$, a contradiction. □

Lemma 2.7. *Let $n, s \geq 1$ be integers. Each of the following holds.*

(i) *K_n is not strongly $\mathbb{Z}_{2s+1}$-connected for any n with $3 \leq n \leq 4s$.*
(ii) *A complete graph K_n is strongly $\mathbb{Z}_{2s+1}$-connected if and only if $n \geq 4s+1$.*

Proof. As Part (ii) of this lemma follows from Part (i) and Lemma 2.5, it suffices to show Part (i).

First, we show that for any positive integer s, K_{4s} has no $\bmod(2s+1)$-orientation, and so $K_{4s} \notin M^o_{2s+1}$. Let $G = K_{4s}$ and suppose that G has a $\bmod(2s+1)$-orientation $D = D(G)$. Let V_P denote the set of all positive vertices of $D(G)$. By the lemma above, since $V(G) - V_P$ is the set of all negative vertices, $|V_P| \geq 2s+1$. By the same reason, $|V(G) - V_P| \geq 2s+1$, which leads to a contradiction:

$$4s = |V(G) - V_P| + |V_P| \geq 2(2s+1) = 4s+2.$$

Now let n be an integer with $3 \leq n \leq 4s-1$. By Lemma 2.2(i), we may assume that $2s+1 \leq n \leq 4s-1$. View K_n as a subgraph of K_{4s}. Since $n \geq 2s+1$, $cl^{2s+1}_{K_{4s}}(K_n) = K_{4s}$. Thus if $K_n \in M^o_{2s+1}$, then by Proposition 2.4(ii), we would have $K_{4s} \in M^o_{2s+1}$, contrary to the fact that $K_{4s} \notin M^o_{2s+1}$. Hence $K_n \notin M^o_{2s+1}$. □

Lemma 2.8. *Let G be a connected graph. Each of the following holds.*

(i) *If D is a $\bmod(2s+1)$-orientation of G, then for any vertex $v \in V_{2s+1}(G)$, either $d^+_D(v) = 2s+1$ or $d^-_D(v) = 2s+1$. In particular, $G[V_{2s+1}(G)]$ must be a bipartite graph, with $\{v \in V_{2s+1}(G) : d^+_D(v) = 2s+1\}$ and $\{v \in V_{2s+1}(G) : d^-_D(v) = 2s+1\}$ being a bipartition of its vertices.*
(ii) *Suppose that G is a $(2s+1)$-regular graph. Then G has a $\bmod(2s+1)$-orientation if and only if G is bipartite.*
(iii) *If G is a bipartite graph with a vertex bipartition (X, Y) such that for every vertex $x \in V(G)$, $d_G(x) \equiv 0 \pmod{2s+1}$, then G has a $\bmod(2s+1)$-orientation.*
(iv) *If G has a $\bmod(2s+1)$-orientation, then for any $v \in V(G)$, either $d^+_D(v) = d^-_D(v)$ or $d_G(v) \geq 2s+1$.*

Proof. The verifications for (i)–(iii) are straightforward, so they will be omitted. We will only show (iv).
(iv) Let $v \in V(G)$, let $d^+ = d^+_D(v)$ and $d^- = d^-_D(v)$. If $d^+ \neq d^-$, then since $d^+ - d^- \equiv 0 \pmod{2s+1}$, either $d^+ - d^- \geq 2s+1$ or $d^- - d^+ \geq 2s+1$, and so $d_G(v) \geq 2s+1$. □

3. A characterization of graphs with mod$(2s+1)$-orientations

The main result in this section is Theorem 1.4, restated as follows.

Theorem 3.1. *A connected graph admits a $\bmod(2s+1)$-orientation if and only if it is a contraction of a $(2s+1)$-regular bipartite graph.*

Proof. Suppose first that G is the contraction of a $(2s+1)$-regular bipartite graph G'. By Lemma 2.8(ii), G' has a $\bmod(2s+1)$-orientation, and so G has a $\bmod(2s+1)$-orientation.

Conversely, we assume that G has a $\bmod(2s+1)$-orientation. We shall fix this $\bmod(2s+1)$-orientation D (say) in the discussion below. If G is $(2s+1)$-regular, then by Lemma 2.8(ii), G is bipartite and we are done. Therefore, assume that G is not regular. Define

$$h_1(G) = |\{v \in V(G) : d_G(v) \equiv 0 \pmod 2\}|, \qquad h_2(G) = \sum_{v\in V(G) \text{ and } d_G(v)\geq 2s+2} d_G(v).$$

By Lemma 2.8(iv), if $v \in V(G)$ has degree at most $2s$, then $d_G(v) \equiv 0 \pmod 2$. Therefore G is $(2s+1)$-regular if and only if $h_1(G) + h_2(G) = 0$. We shall argue by induction on $h_1(G) + h_2(G)$, and assume that $h_1(G) + h_2(G) > 0$ and that Theorem 3.1 holds for graphs G with smaller values of $h_1(G) + h_2(G)$.

Since $h_1(G) + h_2(G) > 0$, G has a vertex u with

$$d_G(u) \neq 2s+1. \tag{2}$$

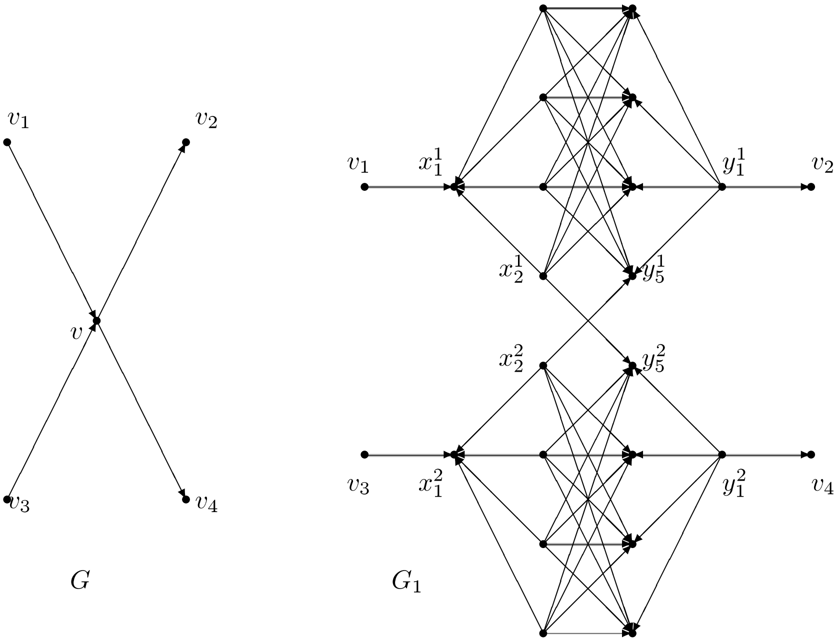

Fig. 1. Part of the graphs G and G_1 when $d_G(v) = 4$ and $s = 2$ (and so $2s + 1 = 5$).

Claim 1. $h_1(G) = 0$.

By the definition of $h_1(G)$*, it suffices to show that* G *has no vertex* v *with* $d_D^+(v) = d_D^-(v)$ *under the orientation* D*. By contradiction, we assume that* G *has a vertex* v *with* $d_D^+(v) = d_D^-(v) = m > 0$*. We shall show that* G *is a contraction of a* $(2s+1)$*-regular bipartite graph. Let* $v_1, v_2, \ldots, v_{2m}$ *denote the vertices adjacent to* v *in* G *such that* (v_{2l-1}, v) *and* (v, v_{2l}) *are in* D*, for* $1 \le l \le m$*. (Note that we allow* $v_i = v_j$ *when* $i \ne j$*. This could happen when* G *has multiple edges.) For each* l*, let* $x_1^l, x_2^l, \ldots, x_{2s+1}^l$*,* $y_1^l, y_2^l, \ldots, y_{2s+1}^l$ *be* $2(2s+1)$ *new vertices. Let* $K_{2s,2s}(l) - x_2^l y_{2s+1}^l$ *denote the complete bipartite graph with bipartition*

$$\{x_2^l, x_3^l, \ldots, x_{2s+1}^l\} \quad \text{and} \quad \{y_2^l, y_3^l, \ldots, y_{2s+1}^l\}$$

minus an edge $x_2^l y_{2s+1}^l$*. Let* $H(x_1^l, y_1^l)$ *denote the graph obtained from* $K_{2s,2s}(l) - x_2^l y_{2s+1}^l$ *by adding the vertex* x_1^l *that is adjacent to all* $x_2^l, x_3^l, \ldots, x_{2s+1}^l$ *and by adding the new vertex* y_1^l *that is adjacent to all* $y_2^l, y_3^l, \ldots, y_{2s+1}^l$*. Obtain a new graph* G_1 *from* $G - v$ *and* $H(x_1^l, y_1^l)$*,* $(1 \le l \le m)$*, by joining* v_{2l-1} *to* x_1^l*, and* v_{2l} *to* y_1^l*, and* x_2^{l+1} *to* y_{2s+1}^l*, where the superscripts are taken modulo* m*. Orient the edges in* $E(G_1) - E(G)$ *such that for each* $l = 1, 2, \ldots, m \pmod m$*,* (x_2^{l+1}, y_{2s+1}^l)*,* (v_{2l-1}, x_1^l)*,* (x_j^l, x_1^l)*,* (y_1^l, v_{2l})*,* (y_1^l, y_j^l)*,* $(2 \le j \le 2s+1)$ *are arcs in this orientation of* G_1*, and such that all the vertices* $x_2^l, \ldots, x_{2s+1}^l$ *are directed to all the vertices* $y_2^l, \ldots, y_{2s+1}^l$ *in* $K_{2s,2s}(l) - x_2^l y_{2s+1}^l$*. See Fig. 1 for an example.*

Thus the $\bmod(2s+1)$*-orientation of* $E(G)$ *together with the orientation on the edges* $E(G_1) - E(G)$ *is a* $\bmod(2s+1)$*-orientation of* G_1*. Since the newly introduced vertices are all of degree* $2s+1$ *in* G_1*, and since* v *satisfies* $d_D^+(v) = d_D^-(v) = m > 0$*, we have* $h_1(G_1) = h_1(G) - 1$ *and* $h_2(G_1) = h_2(G)$*. It follows by induction that* G_1 *is the contraction of a* $(2s+1)$*-regular bipartite graph. Since* G *can be obtained from* G_1 *by contracting* $\bigcup_{l=1}^m H(x_1^l, y_1^l)$*,* G *is also a contraction of a* $(2s+1)$*-regular bipartite graph. This completes the proof of the claim.*

By Claim 1, $\delta(G) \ge 2s+1$. By (2), $d_G(u) \ge 2s+2$. Without loss of generality, we may assume that $d_D^+(u) > d_D^-(u)$. Since $d_D^+(u) - d_D^-(u) \equiv 0 \pmod{2s+1}$, we must have $d_D^+(u) > 2s+1$. Let $h = d(u)$ and let $w_1, w_2, \ldots, w_h$ be the vertices adjacent to u in G, and assume that each directed edge (u, w_i) is oriented from u to w_i, for any i with $1 \le i \le 2s+1$. (Note that for each i with $h \ge i \ge 2s+2$, either (u, w_i) or (w_i, u) is an arc of D.) Obtain a new graph G_2 from G by first splitting u into two vertices u', u'' such that u' is adjacent exactly to $w_1, w_2, \ldots, w_{2s}$, and u'' is adjacent to $w_{2s+1}, w_{2s+2}, \ldots, w_h$, and by adding a new edge $e' = (u', u'')$. Thus we can view $E(G_2) - \{e'\} = E(G)$.

Assign an orientation of G_2 such that the orientation of edges in $E(G_2) - \{e'\}$ is identical with that in D, and such that (u', u'') is an arc in this orientation of G_2. See Fig. 2 for an example.

Then the $\bmod(2s+1)$-orientation D of G plus the orientation of e' is a $\bmod(2s+1)$-orientation of G_2. By the construction of G_2, $h_1(G_2) = h_1(G) = 0$. As $h_2(G_2) = h_2(G) - 2s + 1$, it follows by induction that G_2 is a contraction of a $(2s+1)$-regular bipartite graph. Since $G = G_2/e_u$, G is also a contraction of a $(2s+1)$-regular bipartite graph. This completes the proof of the theorem. □

Recall that by the definition of contraction in [1], contractions of graphs do not delete resulting multiple edges. By Theorem 3.1, Jaeger's conjecture (Conjecture 1.1(ii)) can now be restated as follows.

Conjecture 3.2. *Every* $4s$*-edge-connected graph is a contraction of a* $(2s+1)$*-regular bipartite graph.*

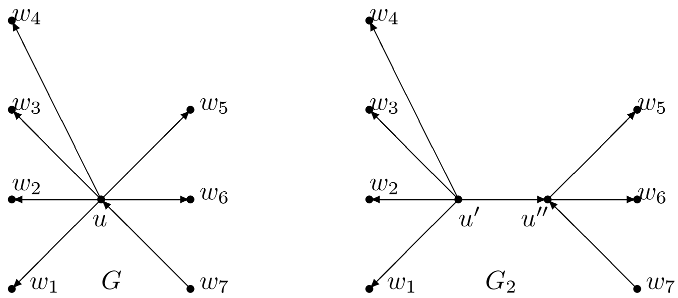

Fig. 2. Part of the graphs G and G_2 when $2s+1=5$, $d_D^+(v)=6$ and $d_D^-(v)=1$.

4. Proof of Theorem 1.5

A graph G is K_4-*minor free* if K_4 cannot be obtained from G by contraction and by deleting edges or vertices. As shown on p. 275 of [1], 2-connected graphs without a K_4-minor are also called serial–parallel graphs. In this section, we shall show a sharp lower bound of edge-connectivity for a K_4-minor free graph to be in M^o_{2s+1}, the collection of all strongly $\mathbb{Z}_{2s+1}$-connected graphs. We need a former theorem of Dirac.

Theorem 4.1 (*Dirac [2]*)**.** *If G is a simple K_4-minor free graph, then G has a vertex of degree at most* 2.

Corollary 4.2. *Every $(4s-1)$-edge-connected K_4-minor free graph is strongly $\mathbb{Z}_{2s+1}$-connected.*

Proof. Let G be a $(4s-1)$-edge-connected K_4-minor free graph, and let G_0 denote the underlying simple graph of G (see p. 47 of [1]). By the definition of strongly $\mathbb{Z}_{2s+1}$-connectedness, $K_1 \in M^o_{2s+1}$. Hence we assume that $|V(G)| > 1$ and the conclusion of the corollary holds for graphs with smaller order.

Since G has no K_4-minor, G_0 does not have a K_4-minor either. By Dirac's Theorem, G_0 must have a vertex w of degree 1 or 2. If w has degree 1 and is incident with the only edge e in G_0, then since $\kappa'(G) \geq 4s-1$, G must have a subgraph H isomorphic to $K_2^{(4s-1)}$. If w has degree 2 and is incident with the edges e_1 and e_2 in G_0, then since $\kappa'(G) \geq 4s-1$, one of e_1 and e_2 must be in a set of at least $2s$ parallel edges, and so G must have a subgraph H isomorphic to $K_2^{(2s)}$. In either case, by Lemma 2.2(ii), $H \in M^o_{2s+1}$. Since G has no K_4-minors, G/H also has no K_4-minors. By the definition of contractions, we have $\kappa'(G/H) \geq \kappa'(G)$. It follows by induction that $G/H \in M^o_{2s+1}$. Since $H \in M^o_{2s+1}$ and by Proposition 2.1(ii), $G \in M^o_{2s+1}$, and so the corollary is proved by induction. □

The next example indicates that the edge-connectivity condition cannot be relaxed.

Example 4.3. Let k, s be positive integers, $m = 2s-1$ and let $G = C^{(m)}_{2k+1}$. Choose the constant function $b \in Z(G, \mathbb{Z}_{2s+1})$ such that for any vertex $v \in V(G)$, $b(v) \equiv 1 (\mathrm{mod}\ 2s+1)$. Assume that G has a $(\mathbb{Z}_{2s+1}, b)$-orientation D. Then for any vertex $v \in V(G)$, we have

$$\begin{cases} d^+(v) + d^-(v) = 4s-2 \\ d^+(v) - d^-(v) \equiv 1 (\mathrm{mod}\ 2s+1). \end{cases}$$

It follows that either $d^+(v) = 3s$ and $d^-(v) = s-2$ (referred to as a positive vertex) or $d^-(v) = 3s-1$ and $d^+(v) = s-1$ (referred to as a negative vertex). It follows that no two positive vertices are adjacent, and no two negative vertices are adjacent. This implies that G must be bipartite, contrary to the fact that G has an odd cycle of length $2k+1$. Hence G does not have a $(\mathbb{Z}_{2s+1}, b)$-orientation, and so $G \notin M^o_{2s+1}$.

5. Proof of Theorem 1.6

Throughout this section, s denotes a positive integer, and a graph $H \in M^o_{2s+1}$ will be referred to as an M^o_{2s+1}-*graph*. A simple graph G is *chordal* if every cycle of length greater than 3 possesses a chord. Equivalently speaking, a simple graph G is chordal if every induced cycle of G has length at most 3. In Theorem 4.2 of [8], it has been proved that every 4-connected chordal graph is in M^o_3. The purpose of this section is to extend this Theorem 4.2 of [8] to the main result of this section below.

Theorem 5.1. *Every simple 4s-connected chordal graph is strongly $\mathbb{Z}_{2s+1}$-connected.*

To prove this theorem, we need some lemmas.

Lemma 5.2 (*Lemma 2.1.2 of [6]*)**.** *A graph G is chordal if and only if every minimal vertex cut induces a complete subgraph of G.*

Lemma 5.3. *Let T be a connected spanning subgraph of G. If for each edge $e \in E(T)$, G has a subgraph $H_e \in M^o_{2s+1}$ with $e \in E(H_e)$, then $G \in M^o_{2s+1}$.*

Proof. We argue by induction on $|V(G)|$. Since K_1 is strongly $\mathbb{Z}_{2s+1}$-connected, the lemma holds trivially if $|V(G)| = 1$. Assume that $|V(G)| > 1$ and pick an edge $e' \in E(T)$. Then G has a subgraph $H' \in M^o_{2s+1}$ such that $e' \in E(H')$. Let $G' = G/H'$ and let $T' = T/(E(H') \cap E(T))$. Since T is a connected spanning subgraph of G, T' is a connected spanning subgraph of G'. For each e in $E(T')$, e is also in $E(T)$, and so by assumption, G has a subgraph $H_e \in M^o_{2s+1}$ with $e \in E(H_e)$. By Proposition 2.1(i), $H'_e = H_e/(E(H_e) \cap E(H')) \in M^o_{2s+1}$ and $e \in H'_e$. Therefore by induction $G' \in M^o_{2s+1}$. Then by Proposition 2.1(ii), and by the assumption that $H' \in M^o_{2s+1}$, $G \in M^o_{2s+1}$. □

Proof of Theorem 5.1. Let G be a $4s$-connected chordal graph. If G itself is a clique, then as $\kappa(G) \geq 4s$, $G \cong K_m$ for some integer $m \geq 4s + 1$, and so by Lemma 2.5, $G \in M^o_{2s+1}$. Thus throughout the rest of the proof, we assume that G is not a complete graph.

By Lemma 5.3, it suffices to show that every edge $e \in E(G)$ lies in a subgraph H_e of G with $H_e \in M^o_{2s+1}$. Let $e = xy$ be an edge in G. For any vertex $v \in V(G)$, let $N(v)$ denote the vertices adjacent to v in G. We shall show that in each of the following two cases concerning the possibilities of the end vertices of e, a subgraph $H_e \in M^o_{2s+1}$ can always be found such that $e \in E(H_e)$.

Case 1: $N(x) \neq V(G) - \{x\}$ or $N(y) \neq V(G) - \{y\}$.

Without loss of generality, we assume that $N(x) \neq V(G) - \{x\}$. Then G has a vertex z such that $xz \notin E(G)$. Since G is 2-connected and not a complete graph, $N(x)$ contains a minimal vertex cut X of G which separates x and z. By Lemma 5.2, $G[X]$ is a complete graph. Since x is adjacent to every vertex in $N(x)$, $G[X \cup \{x\}] \cong K_{m_x}$ is a complete subgraph of G with order $m_x = |X| + 1 \geq \kappa(G) + 1 = 4s + 1$. It follows that $m_x \geq 4s + 1$ and so by Lemma 2.5, $G[X \cup \{x\}] \in M^o_{2s+1}$. If $y \in X$, then we define $H_e = G[X \cup \{x\}] \in M^o_{2s+1}$.

Hence we assume that $y \notin X$ for any minimal vertex cut $X \subseteq N(x)$. If there exists $t \in V(G) - (N(x) \cup \{x\})$ such that $yt \in E(G)$, then there is a minimal vertex cut of $N(x)$ containing y which separates x and t, contrary to the assumption that $y \notin X$ for any minimal vertex cut contained in $N(x)$. Hence $N(y) \subseteq N(x) \cup \{x\}$. Since $z \notin N(x) \cup \{x\}$, $yz \notin E(G)$, and so $N(y)$ contains a minimal vertex cut Y separating y and z. By Lemma 5.2 and by the assumption of $\kappa(G) \geq 4s$, $G[Y \cup \{y\}]$ is a complete graph of order at least $4s + 1$, and so by Lemma 2.5, $G[Y \cup \{y\}] \in M^o_{2s+1}$.

If $x \in Y$, then we define $H_e = G[Y \cup \{y\}] \in M^o_{2s+1}$. Hence we assume further that $x \notin Y$ for any minimal vertex cut $Y \subseteq N(y)$, and so x and y must be in the same component of $G - Y$. For any such vertex cut Y of G contained in $N(y)$, by Lemma 5.2 and by $\kappa(G) \geq 4s$, $G[Y]$ is a complete subgraph of G with order at least $4s$. Note that $Y \subseteq N(y) \subseteq N(x) \cup \{x\}$ and $x \notin Y$. It follows that $G[Y \cup \{x, y\}]$ is a complete subgraph of G with order at least $4s + 2$, and so by Lemma 2.5, $G[Y \cup \{x, y\}] \in M^o_{2s+1}$. Therefore in this final subcase of Case 1, we define $H_e = G[Y \cup \{x, y\}]$.

Case 2: Both $N(x) = V(G) - \{x\}$ and $N(y) = V(G) - \{y\}$.

Since G is not a complete graph itself, G has vertices $v, v' \in V(G) - \{x, y\}$ such that $vv' \notin E(G)$. Therefore, $N(v)$ contains a minimal vertex cut X' separating v and v' in G. By Lemma 5.2 and by the assumption of $\kappa(G) \geq 4s$, $W = G[X' \cup \{v\}]$ is a complete graph of order at least $4s + 1$, and so by Lemma 2.5, $W \in M^o_{2s+1}$. Since both $N(x) = V(G) - \{x\}$ and $N(y) = V(G) - \{y\}$, both x and y must be in X', and so $e = xy \in W$. It is now natural to define $H_e = W$.

Since in either case, we can always find a subgraph $H_e \in M^o_{2s+1}$ such that $e \in E(H_e)$, it follows by Lemma 5.3 that $G \in M^o_{2s+1}$. □

Definition 5.4 (*Definition 2.1.8 in [6]*). Let $k > 0$ be an integer. A clique with order $k + 1$ is a k-tree; given a k-tree T_n on n vertices, a k-tree with $n + 1$ vertices is constructed by taking T_n and creating a new vertex x_{n+1} which is made adjacent to a k-clique of T_n, and non-adjacent to any of the other $n - k$ vertices of T_n.

Corollary 5.5. *Every k-tree with $k \geq 4s$ is in M^o_{4s+1}.*

Proof. We may assume that G is a k-tree but not a clique. By Lemma 5.2, every k-tree is also a chordal graph. By the definition of a k-tree, it is routine to verify that $\kappa(G) \geq k$. It now follows by Theorem 5.1 that, if $k \geq 4s$, every k-tree must be in M^o_{2s+1}. □

By Lemma 2.7, the complete graph K_{4s} is a $(4s-1)$-tree which is not in M^o_{2s+1}. This shows that Corollary 5.5 is best possible.

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