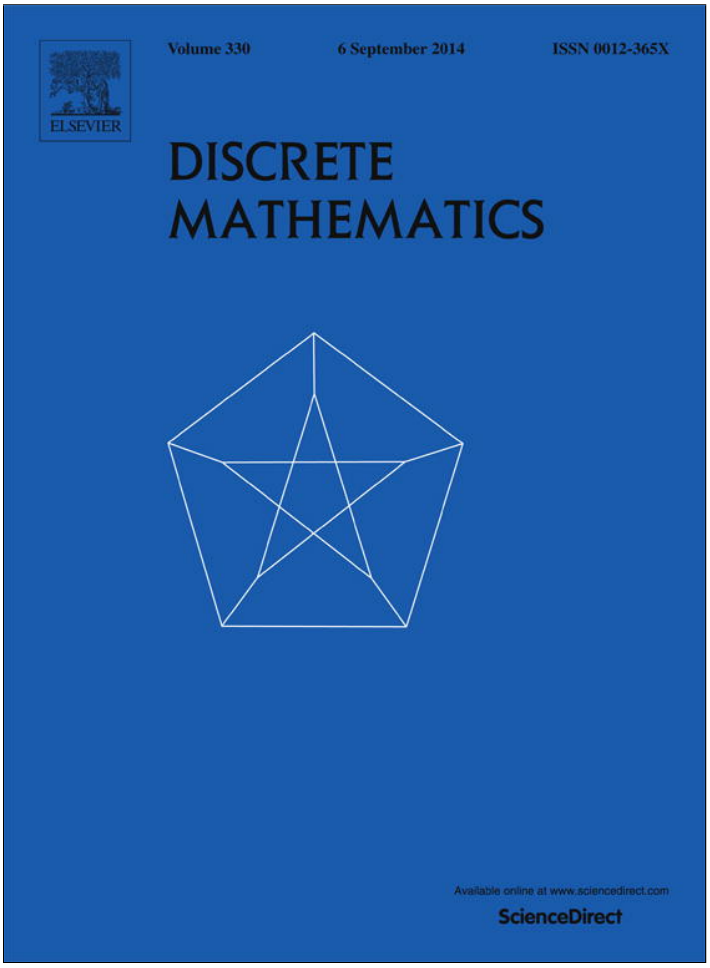
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Supereulerian digraphs[☆]

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ABSTRACT

A digraph D is supereulerian if D has a spanning directed eulerian subdigraph. We give a necessary condition for a digraph to be supereulerian first and then characterize the digraph D which are not supereulerian under the condition that $\delta^+(D)+\delta^-(D) \geq |V(D)|-4$.

1. Introduction

We consider finite digraphs that do not have loops or parallel arcs (bi-direction edges are allowed). For undefined terms and notations, refer to [3] for graphs and [1] for digraphs. To avoid possible confusion, we use ditrails, dipaths and dicycles to mean directed trails, paths, and cycles, while trails, paths and cycles refer to undirected graph terminology. Let D be a digraph. We use both notations uv and (u, v) to denote an arc oriented from a vertex u to a vertex v. We use $G = G(D)$ to denote the underlying graph of D. If X and Y are disjoint subsets of $V(D)$, then $\lambda_D(X, Y)$ denotes the maximum number of arc-disjoint dipaths from X to Y in D. As in [1], $A(D)$ denotes the set of arcs in D, $c(D)$ denotes the number of components of the underlying graph of D, and $\delta^+(D)$, $\delta^-(D)$ denote the minimum out-degree and the minimum in-degree of D, respectively. For a pair of disjoint sets $X, Y \subset V(D)$, define

$$(X, Y)_D = \{(x, y) \in A(D) : x \in X \text{ and } y \in Y\}.$$

When $Y = V - X$, we use

$$\partial^+_D(X) = (X, V - X)_D, \quad \text{and} \quad \partial^-_D(X) = (V - X, X)_D.$$

When $X = \{v\}$, we also use $\partial^+_D(v) = \partial^+_D(\{v\})$ and $\partial^-_D(v) = \partial^-_D(\{v\})$. As in [1], we denote

$$N^+_D(v) = \{u \in V(D) : (v, u) \in A(D)\} \quad \text{and} \quad N^-_D(v) = \{u \in V(D) : (u, v) \in A(D)\}.$$

A graph G is eulerian if G is connected without vertices of odd degree, and G is supereulerian if G has a spanning eulerian subgraph. In [2], Boesch et al. raised the problem to determine when a graph is supereulerian, and they remarked that such a problem would be a difficult one. In [6], Pulleyblank confirmed the remark by showing that the problem to determine if a graph is supereulerian, even within planar graphs, is NP-complete. For more literature on supereulerian graphs, see Catlin's excellent survey [4] and its supplement [5].

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The purpose of this paper is to investigate the digraph version of the supereulerian problem. A digraph D is strongly connected if there is a (u, v)-dipath for any two vertices u, v. Furthermore, D is said to be *eulerian* if D is strongly connected and for every vertex $v \in V(D)$, $d_D^+(v) = d_D^-(v)$. Thus D is eulerian if and only if D itself is a closed ditrail. A digraph D is *supereulerian* if D has a spanning eulerian subdigraph H. The main result of this paper determines a best possible lower bound of the minimum degree to assure a simple digraph to be supereulerian, and to characterize all the extremal digraphs.

In Section 2, we derive a necessary condition for a digraph to be supereulerian, and apply it to find candidates of the extremal graphs for the main result. The proof of the main result is stated and proved in Section 3.

2. A necessary condition

Let D be a strong digraph and $U \subsetneq V(D)$. Then $D[U]$, the digraph induced by U, has ditrails $P_1, \ldots, P_t$ such that $\bigcup_{i=1}^t V(P_i) = U$ and $A(P_i) \cap A(P_j) = \emptyset$ for any $i \neq j$. Let $\tau(U)$ be the minimum value of such t. Then $c(G(D[U])) \leq \tau(U) \leq |U|$. For any $A \subseteq V(D) - U$, denote $B := V(D) - U - A$ and let

$$h(U, A) := \min\{|\partial_D^+(A)|, |\partial_D^-(A)|\} + \min\{|(U, B)_D|, |(B, U)_D|\} - \tau(U), \quad \text{and}$$
$$h(U) := \min\{h(U, A) : A \cap U = \emptyset\}.$$

Then we have the following proposition.

Proposition 2.1. *If D has a spanning eulerian subdigraph, then for any $U \subset V(D)$, $h(U) \geq 0$.*

Proof. Suppose that D has a spanning eulerian subdigraph H but for some U, $h(U) < 0$. Without loss of generality, we may assume that for some vertex set A disjoint with U, $h(U, A) < 0$. Let $B := V(D) - U - A$. Then

$$\min\{|\partial_D^+(A)|, |\partial_D^-(A)|\} + \min\{|(U, B)_D|, |(B, U)_D|\} < \tau(U).$$

Since H is spanning and eulerian, H has a closed ditrail visiting every vertex in U, and so by the definition of $\tau(U)$,

$$|\partial_H^+(U)| = |\partial_H^-(U)| \geq \tau(U).$$

It follows that

$$\begin{aligned} |\partial_H^-(A)| = |\partial_H^+(A)| &\geq \max\{|(U, A)_H|, |(A, U)_H|\} \\ &= \max\{|\partial_H^+(U)| - |(U, B)_H|, |\partial_H^-(U)| - |(B, U)_H|\} \\ &= |\partial_H^-(U)| - \min\{|(U, B)_H|, |(B, U)_H|\} \\ &\geq \tau(U) - \min\{|(U, B)_D|, |(B, U)_D|\} \\ &> \min\{|\partial_D^+(A)|, |\partial_D^-(A)|\}, \end{aligned}$$

a contradiction. ■

The proposition above can be used to show that there exists a family of strong digraphs each of which has a large minimum degree but contains no spanning eulerian subdigraphs.

Example 2.2. Let $k_1, k_2, l \geq 2$ be integers, and D_1 and D_2 be two disjoint complete digraphs of order $k_1 + 1$ and $k_2 + 1$, respectively, and let U be an independent set disjoint from $V(D_1) \cup V(D_2)$ with $|U| = l$. Let $\mathcal{F}(k_1, k_2, l)$ denote the family of digraphs such that $D \in \mathcal{F}(k_1, k_2, l)$ if and only if D is the digraph obtained from $D_1 \cup D_2 \cup U$ by adding all arcs directed from every vertex in U and D_2 to every vertex in D_1, and all arcs directed from every vertex in D_2 to every vertex in U, and then by adding an set of $l - 1$ arcs directed from some vertices in D_1 to some vertices in D_2.

Assume $k_1, k_2 \geq l - 1$. For any $D \in \mathcal{F}(k_1, k_2, l)$, D has $n = k_1 + k_2 + l + 2$ vertices, and is a strong digraph with minimum degree $\delta^+(D) = k_1$ and $\delta^-(D) = k_2$. Let $A = V(D_1)$. Then $h(U, A) = |\partial_D^+(A)| + |(U, V(D) - U - A)_D| - \tau(U) = (l - 1) - l < 0$. By Proposition 2.1, D does not have a spanning eulerian subdigraph. By direct computation, for each $D \in \mathcal{F}(k_1, k_2, 2)$, $\delta^+(D) + \delta^-(D) = |V(D)| - 4$. Let $\mathcal{F}_0(k_1, k_2, 2)$ be the set of spanning subdigraphs D' of the digraphs in $\mathcal{F}(k_1, k_2, 2)$ which satisfy $\delta^+(D') + \delta^-(D') = |V(D')| - 4$.

Then no digraph in $\mathcal{F}_0(k_1, k_2, 2)$ has a spanning eulerian subdigraph. In the next section, we will show $\mathcal{F}_0(k_1, k_2, 2)$ is the only counterexample under the condition $\delta^+(D) + \delta^-(D) \geq |V(D)| - 4$ and $\delta^+, \delta^- \geq 4$.

Also, if we do not assume that the digraph is strong, we can find non-supereulerian digraphs with a higher minimum degree sum.

Example 2.3. Let $k_1, k_2 > 1$ and let D_1 and D_2 be two disjoint complete digraphs of order k_1 and k_2, respectively. Obtain $D(k_1, k_2)$ from $D_1 \cup D_2$ by adding all arcs directed from every vertex in D_2 to every vertex in D_1. Then $\delta^+(D) = k_1 - 1$, $\delta^-(D) = k_2 - 1$, and so $\delta^+(D) + \delta^-(D) = |V(D)| - 2 > |V(D)| - 4$. However, D is not strong and so cannot be supereulerian.

3. A degree condition for the existence of a spanning Eulerian subdigraph

In this section, we shall show that for a strong digraph D, if $\min\{\delta^+(D), \delta^-(D)\} \geq 4$ and $\delta^+(D) + \delta^-(D) > |V(D)| - 4$, then D is supereulerian. Throughout this section, D denotes a digraph on n vertices, $\delta^+ = \delta^+(D)$ and $\delta^- = \delta^-(D)$.

Definition 3.1. Let H be an eulerian subdigraph of a digraph D. Suppose for some disjoint vertex subsets $X, Y \subseteq V(H)$, Q is an (X, Y)-ditrail of H. Let H' be the connected component of the underlying graph of $H - A(Q)$ containing both ends of Q. Define $I_Q = V(H) - V(H')$, called the *increment of* Q *with respect to* H. If the eulerian subdigraph H is clear from context, we also say I_Q is the increment of Q.

Suppose Q is from $x \in X$ to $y \in Y$. Since H is eulerian, H has a minimum (x, y)-ditrail that contains all arcs in $H[I_Q]$ and Q. This ditrail is denoted by $\bar{Q}$. Note that it is possible that $\bar{Q} = Q$. Also, the underlying graph of $H[I_Q]$ might not be connected.

Using these definitions and notations, we have the following observations, stated as a lemma below.

Lemma 3.2. *Let D be a digraph, H be an eulerian subdigraph of D, and $X, Y \subseteq V(H)$ be two disjoint vertex subsets. Then each of the following holds.*

(i) *If Q is an (X, Y)-ditrail of H then $(V(H - I_Q), I_Q)_H \cup (I_Q, V(H - I_Q))_H \subseteq A(Q)$.*

(ii) *If Q is an (X, Y)-ditrail such that $|A(\bar{Q})|$ is minimized, then $I_Q \cap (X \cup Y) = \emptyset$.*

(iii) *If Q_1 and Q_2 are two arc-disjoint (X, Y)-ditrails of H, then either $I_{Q_1} \cap I_{Q_2} = \emptyset$.*

Proof. By the definition of I_Q, all the arcs of H between $V(H - I_Q)$ and I_Q lie in Q. So (i) follows. For (ii), suppose $\bar{Q}$ is from $x \in X$ to $y \in Y$. As H is eulerian, by the definition of $\bar{Q}$, $\bar{Q}$ visits every vertex of I_Q. If $I_Q \cap (X \cup Y) \neq \emptyset$, then there exists a vertex $x' \in X \cap I_Q$ or a vertex $y' \in Y \cap I_Q$. Thus $\bar{Q}[x', y]$ or $\bar{Q}[x, y']$ is an (X, Y)-ditrail with fewer arcs, a contradiction.

For (iii), suppose, to the contrary, that $I_{Q_1} \cap I_{Q_2} \neq \emptyset$. Let $X_1 = I_{Q_1} - I_{Q_2}$, $X_2 = I_{Q_2} - I_{Q_1}$, $X_3 = I_{Q_1} \cap I_{Q_2}$ and $X_4 = V(D) - I_{Q_1} \cup I_{Q_2}$. Then by (i), $\partial^+_D(X_i \cup X_3) \subseteq A(Q_i)$ for $i = 1, 2$. Also, as Q_1 and Q_2 are arc-disjoint, $(X_3, X_4)_D = \emptyset$ and $V(Q_i) \cap X_i \neq \emptyset$ for $i = 1, 2$. By the definitions of I_{Q_1} and I_{Q_2}, $(X_i, X_3)_D \cup (X_3, X_i)_D \neq \emptyset$ for $i = 1, 2$. So, $(X_1, X_3)_D \subseteq A(Q_2)$ (and thus $(X_1, X_3)_D \cap A(Q_1) = \emptyset$) and $(X_3, X_2)_D \subseteq A(Q_1)$, contradicts the fact Q_1 is a dipath. ■

Lemma 3.3. *Let D be a strong digraph of order n and minimum out-degree $\delta^+ \geq 2$ and minimum in-degree $\delta^- \geq 2$, H be an eulerian subdigraph of D with maximum order. If $n \geq \max\{\delta^+, \delta^-\} + 3$, then $|V(H)| \geq \max\{\delta^+, \delta^-\} + 3$.*

Proof. Without loss of generality, we may assume $\delta^+ \geq \delta^-$. Suppose, to the contrary, that $|V(H)| \leq \delta^+ + 2 < n$. Let P be a ditrail of D such that $|V(P)|$ is maximized and subject to this, $|A(P)|$ is minimized. Assume P is an (x, y)-ditrail for some vertices x, y. Then, by the choice of P, $N^+_D(y) \subseteq V(P)$ and

$$\partial^+_D(y) \cap A(P) = \emptyset, \tag{3.1}$$

since otherwise, there exists an out-arc of y in P. Then y lies in $P[x, y']$, where y' be the immediate predecessor of y in P. Thus, $P[x, y']$ is an (x, y')-ditrail with $|V(P)|$ vertices and $|A(P)| - 1$ arcs, a contradiction to the choice of P.

Let $y_0 \in V(P)$ such that there exists a (y, y_0)-ditrail Q with $V(Q) \subseteq V(P[y_0, y])$ and $A(Q) \cap A(P) = \emptyset$, and furthermore, we choose y_0 so that $|V(P[y_0, y])|$ is maximized. Note that such y_0 does exist, as any out-neighbor of y is such a candidate by (3.1).

Denote $K = V(P[y_0, y])$. By the choice of y_0, yQy_0Py is an eulerian subdigraph of D of order $|K|$. By assumption, $n > |H| \geq |K|$. Then, by the strongness of D, there exist $y_1 \in K$ and $z_1 \notin K$ such that $y_1z_1 \in A(D)$. In this proof, for any vertex $w \in V(P)$, denote by w^+ the next vertex of w in P and by w^- the previous vertex of w in P.

Proposition 1. *$yy_1 \notin A(D)$ and $yw \in A(D)$ for any $w \in K - \{y, y_1\}$.*

Suppose, to the contrary, that $yy_1 \in A(D)$. If $z_1 \notin V(P)$, then $xPyy_1z_1$ is a ditrail of D of order $|V(P)| + 1$, a contradiction to the choice of P. So, $z_1 \in V(P)$. This, together with the fact $z_1 \notin K$, forces $z_1 \in V(P[x, y_0]) - \{y_0\}$. However, as $z_1 \notin K$ and $y_1 \notin V(P[x, y_0]) - \{y_0\}$ (by the choice of y_0), $y_1z_1 \notin A(P)$. Thus, $z_1 \in V(P)$ is a vertex satisfying that yy_1z_1 is a ditrail arc-disjoint with P and $P[z_1, y]$ is longer than $P[y_0, y]$, contrary to the choice of y_0. Hence, $yy_1 \notin A(D)$. Then, by the choice of P, $|K| \geq |N^+_D(y) \cup \{y, y_1\}| \geq \delta^+ + 2$. This, together with the assumption that $|K| \leq |H| \leq \delta^+ + 2$, forces the proposition.

From Proposition 1 and the assumption, $|K| = |H| = \delta^+ + 2$.

Proposition 2. *$N^-_D(y) \subseteq K$.*

Suppose, to the contrary, that there exists a vertex $y' \in N^-_D(y) - K$. Then by the strongness of D, there is a dipath, denoted by P', from K to y'. By the arbitrariness of y_1 and by Proposition 1, P' is orientated from y_1 and $yy_1^+ \in A(D)$ if $y \neq y_1^+$. Thus $y_0Py_1P'y'yy_1^+Pyy_0$ is an eulerian subdigraph of order at least $|K|+1 = \delta^+ + 3$ vertices, contrary to the assumption. The proposition is proved.

Since D is strong, there is a dipath P' from z_1 to K for $z_1 \notin V(H)$. Denote by y_2 the endpoint of P'. If there is no such a dipath P' such that $y_2 \in V(P[y_0, y_1]) - \{y_1\}$, then $y_2 \in V(P[y_1, y])$ and $y_1P'y_2$ is arc-disjoint with P, and so $xPy_1P'y_2Pyy_1^+Py_2^-$ is a ditrail with at least $|V(P)| + 1$ vertices, contrary to the choice of P. Hence, D has such a dipath P' such that $y_2 \in V(P[y_0, y_1] - \{y_1\})$. By the arbitrariness of y_1 and by Proposition 1, $N^+_D(y_2) \subseteq K$.

If $y_2y \in A(D)$, then $y_2Py_1z_1P'y_2yy_1^+Pyy_0Py_2$ is an eulerian subdigraph of order at least $|K| + 1$, contrary to the assumption. So, $y_2y \notin A(D)$. Moreover, if $y_2y_1^+ \in A(D)$, then $y_0Py_1z_1P'y_2y_1^+Pyy_0$ is an eulerian subdigraph of order at least $|K|+1$, contrary to the assumption again. Hence, $y_2y_1^+ \notin A(D)$. Thus, $N^+_D(y_2) \subseteq K - \{y_2, y_1^+, y\}$. This, together with the fact $|K| = \delta^+ + 2$, forces $y_1^+ = y$.

By $\delta^- \geq 2$ and by Proposition 2, there exists $y_3 \in K - \{y_1, y\}$ such that $y_3y \in A(D)$. Then $y_3 \in V(P[y_0, y_1] - \{y_1\})$. Also, by Proposition 1, $yy_3 \in A(D)$. So, if $z_1 \notin V(P)$ then $xPy_3yy_3Py_1z_1$ is a ditrail with $|V(P)| + 1$ vertices, contrary to the choice of P, and if $z_1 \in V(P)$ then $z_1 \in V(P[x, y_0]) - \{y_0\}$ and thus $z_1Py_3yy_3Py_1z_1$ is an eulerian subdigraph with at least $|K| + 1 = \delta^+ + 3$ vertices, contrary to the assumption. ■

Theorem 3.4. *Let D be a strong digraph of order n and minimum out-degree $\delta^+ \geq 4$ and minimum in-degree $\delta^- \geq 4$. If $\delta^+ + \delta^- \geq n - 4$, then the following are equivalent.*
(i) *G has a spanning eulerian subdigraph.*
(ii) *Either $\delta^+ + \delta^- > n - 4$, or for some integer k_1, k_2, $\delta^+ = k_1$, $\delta^- = k_2$ but $D \notin \mathcal{F}_0(k_1, k_2, 2)$.*

Proof. By Example 2.2, it suffices to prove that (ii) implies (i). By the definition of $\mathcal{F}_0(k_1, k_2, 2)$, for every digraph $D \in \mathcal{F}_0(k_1, k_2, 2)$, $\delta^+(D) + \delta^-(D) = n - 4$. So, we may assume that

$$D \text{ is not supereulerian,} \tag{3.2}$$

to prove that $D \in \mathcal{F}_0(k_1, k_2, 2)$ for some integer k_1, k_2. Choose an eulerian subdigraph H of D such that

$$|V(H)| \text{ is maximized,} \tag{3.3}$$

and subject to (3.3),

$$|A(H)| \text{ is maximized.} \tag{3.4}$$

Let P be a ditrail of $D - V(H)$ with $p = |V(P)|$ vertices such that $|V(P)|$ is maximum. Assume P is from u to v. By (3.2), $p \geq 1$. By the choice of P, $N_D^-(u) \cup N_D^+(v) \subseteq V(H) \cup V(P)$. Define

$$A = N_D^+(v) \cap V(H) \quad \text{and} \quad B = N_D^-(u) \cap V(H).$$

Claim 1. *We may choose a ditrail P such that each of the following holds.*
(i) *$A \cap B = \emptyset$.*
(ii) *$(A, B)_D \subseteq A(H)$ and $(B, A)_D \cap A(H) = \emptyset$.*
(iii) *$|A| \geq \delta^+ - p + 1$, $|B| \geq \delta^- - p + 1$.*
(iv) *$A \neq \emptyset$ and $B \neq \emptyset$.*

Claim 1(i)–(ii) follow from (3.3), and definition of A and B. For (iii), by definition of A, $|A| \geq d_D^+(v) - |N_D^+(v) \cap V(P)| \geq \delta^+ - (p - 1) = \delta^+ - p + 1$. Similarly, $|B| \geq \delta^- - p + 1$. Thus, (iii) holds.

For (iv), let P_0 be a ditrail of $D - V(H)$ such that $|V(P_0)|$ is maximized and subject to this, $|A(P_0)|$ is minimized. Assume P_0 is from u_0 to v_0. Similar to (3.1) in the proof of Lemma 3.3,

$$\partial_D^-(u_0) \cap A(P_0) = \emptyset \quad \text{and} \quad \partial_D^+(v_0) \cap A(P_0) = \emptyset.$$

If there exist vertices $v_1 \in \left(N_D^+(v_0) \cap V(P_0)\right) \cup \{v_0\}$ and $u_1 \in \left(N_D^-(u_0) \cap V(P_0)\right) \cup \{u_0\}$ such that $N_D^+(v_1) \cap V(H) \neq \emptyset$ and $N_D^-(u_1) \cap V(H) = \emptyset$, then $u_1u_0Pv_0v_1$ is a candidate of ditrail P satisfying (iv). So, it suffices to show the existence of such vertices u_1 and v_1.

By contradiction and without loss of generality, we may assume no such v_1 exists. Then $N_D^+(v_0) \cap V(H) = \emptyset$. So, $|\left(N_D^+(v_0) \cap V(P_0)\right) \cup \{v_0\}| \geq \delta^+ + 1$. Also, by Lemma 3.3, $|V(H)| \geq \max\{\delta^+, \delta^-\} + 3$. So, $|V(P_0)| \leq n - |V(H)| \leq n - \max\{\delta^+, \delta^-\} - 3 \leq \min\{\delta^+, \delta^-\} + 1$. It follows that $\left(N_D^+(v_0) \cap V(P_0)\right) \cup \{v_0\} = V(P_0)$. Thus, the existence of v_1 is ensured by the strongness of D. This finishes the proof of Claim 1.

Since H is an eulerian subdigraph of D, H contains a (B, A)-dipath Q. Choose such a (B, A)-dipath Q of H such that

$$|I_Q| \text{ is minimized and subject to this, } |A(\bar{Q})| \text{ is minimized.} \tag{3.5}$$

Then, by Lemma 3.2(ii), I_Q is disjoint with $A \cup B$. Assume Q is from $z_1 \in B$ to $z_2 \in A$. Then $V(Q) \cap (A \cup B) = \{z_1, z_2\}$. Let z_1' be the first vertex of $\bar{Q}$ in I_Q and z_2' the last vertex of $\bar{Q}$ in I_Q. Note that it is possible that $z_1' = z_2'$.

For simplicity, we denote $H \ominus Q$ to be the subgraph of H by removing all the arcs of Q and then removing the increment I_Q.

Let $q = |I_Q|$. As z_1uPvz_2 is a (z_1, z_2)-ditrail of D arc-disjoint with H, if $q < p$, then $H \ominus Q + P'$ is an eulerian subdigraph with order $|V(H)| - q + p > |V(H)|$, contrary to (3.3). Hence $q \geq p$. Define

$$R = V(D) - V(H) - V(P), \qquad T = V(H) - A - B - I_Q, \qquad r = |R| \quad \text{and} \quad t = |T|.$$

Then by Claim 1,

$$\begin{aligned} n &= |V(D)| = |A| + |B| + p + q + r + t \\ &\geq \delta^+ + \delta^- + 2 - p + q + r + t \geq n + q + r + t - p - 2. \end{aligned} \tag{3.6}$$

Thus we have obtained

$$p \leq q, \quad q + r + t \leq p + 2 \quad \text{and} \quad r + t \leq 2. \tag{3.7}$$

Claim 2. *Both $(A, (I_Q - \{z_2'\}) \cup \{z_1'\})_D = \emptyset$ and $((I_Q - \{z_1'\}) \cup \{z_2'\}, B)_D = \emptyset$.*

By symmetry, it suffices to show that $(A, (I_Q - \{z_2'\}) \cup \{z_1'\})_D = \emptyset$. By contradiction, assume that D has an arc $az_3 \in (A, (I_Q - \{z_2\}) \cup \{z_1\})_D$. By Lemma 3.2, $az_3 \notin A(H)$. We first establish each of the following.

(2A) $N_D^-(z_1') \cap A = \emptyset$ and $N_D^+(z_2') \cap B = \emptyset$.

By symmetry, it suffices to show $N_D^-(z_1') \cap A = \emptyset$. If $N_D^-(z_1') \cap A \neq \emptyset$, then we may assume that $z_3 = z_1'$. Hence $P' = z_1uPvaz_1'\bar{Q}z_2$ is a (z_1, z_2)-dipath, edge-disjoint from $H \ominus Q$. It follows that $(H \ominus Q) + P'$ is an eulerian subdigraph of order $|V(H)| - q + (p + q) > |V(H)|$, contrary to (3.3). Therefore (2A) must hold.

(2B) $N_D^+(z_3) \cap B = \emptyset$ and $N_D^+(z_3) \cap V(P) = \emptyset$.

If there exists a vertex $b \in N_D^+(z_3) \cap B$, then by Lemma 3.2, $z_3b \notin A(H)$, and so $H + az_3buPva$ is an eulerian subdigraph of order at least $|V(H)| + 1$, violating (3.3). If there exists a vertex $w \in N_D^+(z_3) \cap V(P)$, then $H + az_3wPva$ is an eulerian subdigraph with at least $|V(H)| + 1$ vertices, contrary to (3.3). Hence (2B) holds.

(2C) $z_3a \notin A(D)$ and $N_D^+(z_3) \cap I_Q = \{z_2'\}$.

Suppose $z_3a \in A(D)$. If $z_3a \notin A(H)$, then $H + az_3a$ is an eulerian subdigraph with exactly $|V(H)|$ vertices and $|A(H)| + 2$ arcs, contrary to (3.4). Hence $z_3a \in A(H)$. By Lemma 3.2, $z_3a \in A(Q)$. It follows that $a = z_2$ and $z_3 = z_2'$. This, together with the fact $z_3 \in (I_Q - \{z_2'\}) \cup \{z_1'\}$, forces $z_3 = z_1'$. Thus $a \in N_D^-(z_1')$, contrary to (2A). Hence, $z_3a \notin A(D)$.

To show $N_D^+(z_3) \cap I_Q = \{z_2'\}$, we first show that $z_2' \in N_D^+(z_3) \cap I_Q$. If $z_3z_2' \notin A(D)$, then $H \ominus Q + z_1uPvaz_3\bar{Q}z_2$ is an eulerian subdigraph of order $|V(H)| - q + p + |V(\bar{Q}[z_3, z_2'])| \geq |V(H)| - q + p + 3 > |V(H)|$, contrary to (3.3).

Then, we show that $N_D^+(z_3) \cap I_Q \subseteq \{z_2'\}$. If there exists another vertex $z_4 \in (N_D^+(z_3) - \{z_2'\}) \cap I_Q$, then $D[I_Q] - z_3z_4$ has a (z_4, z_2')-dipath Q', and so $H \ominus Q + z_1uPvaz_3z_4Q'z_2'\bar{Q}z_2$ is an eulerian subdigraph of order $|V(H)| - q + p + |V(Q')| + 1 \geq |V(H)| - q + p + 3 > |V(H)|$, contrary to (3.3). Thus (2C) must hold.

(2D) $z_3z_2' \in A(D)$, $q = 3$, $p = 1$, $t = 0$, $|A| = \delta^+$ and $|B| = \delta^-$. Moreover, if there is an arc $zb \in ((I_Q - \{z_1'\}) \cup \{z_2'\}, B)_D$, then $z_1'z \in A(D)$.

By (2C), we have $z_3z_2' \in A(D)$ directly. Moreover, by (2A)–(2C), we have shown that $N_D^+(z_3) \subseteq A \cup R \cup T \cup \{z_2'\}$, and so $d_D^+(z_3) \leq |A - \{a\}| + r + t + 1 = |A| + r + t$. It follows that $|A| \geq d_D^+(z_3) - r - t \geq \delta^+ - r - t$, and so

$$\begin{aligned}
n &= |A| + |B| + p + q + r + t \\
&\geq \delta^+ - r - t + \delta^- - p + 1 + p + q + r + t \\
&= \delta^+ + \delta^- + q + 1 \\
&\geq n + q - 3.
\end{aligned}$$

It follows that $q \leq 3$. Moreover, $H \ominus Q + z_1uPvaz_3z_2'\bar{Q}z_2$ is an eulerian subdigraph of D of order $|V(H)| - q + p + 2$. By (3.3), $|V(H)| - q + p + 2 \leq |V(H)|$ and so $q \geq p + 2$. This, together with (3.7), forces $q = p + 2 \geq 3$ and $r = t = 0$, and so we must have $q = 3$, $p = 1$, $z_3z_2' \in A(D)$ and $|A| = \delta^+$, $|B| = \delta^-$. Arguing similarly, we conclude that if there is an arc $zb \in (I_Q, B)_D$, then $z_1'z \in A(D)$, and so (2D) follows.

Since $t = 0$, H has exactly one (B, A)-dipath, and so by H being eulerian and by Menger's Theorem (Page 170, Theorem 7.16 of [3]), $|(A, B)_H| \leq \lambda_H(A, B) = \lambda_H(B, A) = 1$. By Claim 1(ii), $|(A, B)_D| \leq 1$. Hence, there is a vertex $b \in B$ such that $N_D^-(b) \cap A = \emptyset$. Also, as $p = 1$, by Claim 1(i), $N_D^-(b) \cap V(P) = \emptyset$. So, $N_D^-(b) \subseteq B \cup I_Q$. By $|B| = \delta^-$, there must be a $z_4 \in N_D^-(b) \cap I_Q$. By Lemma 3.2, $z_4b \notin A(H)$.

By (2A) and as $b \in B$, $z_4 \neq z_2'$. Also, by (2B) $z_4 \neq z_3$. Moreover, if $z_4 = z_1'$ then $H \ominus Q + z_1Qz_1'buPvaz_3z_2'Qz_2$ is an eulerian subdigraph of D violating (3.3). Hence $z_4 \in I_Q - \{z_1', z_2', z_3\}$. This, together with $|I_Q| = q = 3$, forces $z_1' = z_2'$, and so I_Q has exactly three vertices z_1', z_3, z_4. As $z_4 \neq z_1'$, $z_4b \in ((I_Q - \{z_1'\}) \cup \{z_2'\}, B)_D$. By (2D), $z_1'z_4 \in A(D)$. So, $H \ominus Q + z_1Qz_1'Qz_2 + az_3z_1'z_4buPva$ is a spanning eulerian subdigraph of D, contrary to (3.2). This establishes Claim 2.

Define

$$A_0 = \{x \in A : N_D^+(x) \cap B = \emptyset\} \quad \text{and} \quad B_0 = \{x \in B : N_D^-(x) \cap A = \emptyset\}. \tag{3.8}$$

By (3.5), every (B, A)-dipath has increment at least q, and by Lemma 3.2, any two arc-disjoint (B, A)-dipath have disjoint increment, and so $\lambda_H(B, A) \leq t/q + 1$. As H is eulerian, $|\partial_H^+(U)| = |\partial_H^-(U)|$ for any $U \subseteq V(D)$. It follows from Menger's Theorem (Page 170, Theorem 7.16 of [3]) that $\lambda_H(A, B) = \lambda_H(B, A) \leq t/q + 1$. By the definition of A_0 and B_0 and by (3.7),

$$\max\{|A - A_0|, |B - B_0|\} \leq \lambda_H(A, B) \leq t/q + 1 \leq 3. \tag{3.9}$$

By Claim 2,

$$N_D^+(A) \cap I_Q \subseteq \{z_2'\} \quad \text{and} \quad N_D^-(B) \cap I_Q \subseteq \{z_1'\}. \tag{3.10}$$

Claim 3. *There exist vertices $a \in A_0$ and $b \in B_0$ such that $az_2', z_1'b \notin A(D)$.*

By symmetry, it suffices to prove the existence of a. We shall show the following statements.

(3A) $A_0 \neq \emptyset$ and $B_0 \neq \emptyset$.

By contradiction, we assume that $A_0 = \emptyset$. By (3.9) and Claim 1(iii), $\delta^+ \leq |A| + p - 1 \leq p + t/q \leq p + t/p$. If $\lambda_H(B, A) \geq 2$, then there is a (B, A)-dipath Q' disjoint with I_Q. Then $p \leq q \leq |I_{Q'}| \leq |T| = t \leq 2$, and so $\delta^+ \leq 3$, contrary to the fact that $\delta^+ \geq 4$.

Hence $\lambda_H(B, A) = 1$ and so $|A| = |A - A_0| = 1$. Then $A = \{z_2\}$. Also by $\lambda_H(A, B) = 1$, $|N_D^+(z_2) \cap B| \le 1$. By (3.3), $N_D^+(z_2) \cap V(P) = \emptyset$ and by Claim 2, $|N_D^+(z_2) \cap I_Q| \le 1$. Hence $|R \cup T| \ge |N_D^+(z_2) \cap (R \cup T)| \ge d_D^+(z_2) - |N_D^+(z_2) \cap (B \cup I_Q)| \ge \delta^+ - 2 \ge 2$. This, together with (3.7), forces $|R \cup T| = 2$ and $|N_D^+(z_2) \cap I_Q| = 1$. Then by Claim 2, $z_2z_2' \in A(D)$. It follows that $H \ominus Q + z_1uPvz_2z_2'\bar{Q}z_2$ is an eulerian subdigraph of D with order $|V(H)| - q + p + 1$. By (3.3), $q \ge p + 1$. Also by (3.7), $r + t \le p + 2 - q \le 1$, a contradiction to the deduced fact $|R \cup T| = 2$. This proves $A_0 \neq \emptyset$. The proof for $B_0 \neq \emptyset$ is similar and so (3A) holds.

(3B) There exists a vertex $a \in A_0$ such that $az_2' \notin A(D)$.

Assume that for every $a' \in A_0$, $a'z_2' \in A(D)$. By (3A), pick a vertex $a' \in A_0$. Then $H \ominus Q + z_1uPva'z_2'\bar{Q}z_2$ is an eulerian subdigraph of order at least $|V(H)| - q + p + 1$. By (3.3), $|V(H)| - q + p + 1 \le |V(H)|$ and so $q \ge p + 1 \ge 2$. Hence by (3.7), we have $r + t \le 1$.

For any $a'' \in A_0 - \{z_2\}$, by the assumption and by Lemma 3.2, $a''z_2' \in A(D) - A(H)$. So, by (3.4), $z_2'a'' \notin A(D)$. Furthermore, by Claim 2, $N_D^+(z_2') \cap B = \emptyset$. Also, by (3.3), $N_D^+(z_2') \cap V(P) = \emptyset$. So, $N_D^+(z_2') \subseteq [(R \cup T \cup I_Q) \cap N_D^+(z_2')] \cup (A - A_0) \cup \{z_2\}$. This, together with $\delta^+ \ge 4$ and (3.9), implies that

$$|N_D^+(z_2') \cap I_Q| \ge d_D^+(z_2') - |A - A_0| - (r + t) - 1 \ge 2 - (r + t) - t/q. \tag{3.11}$$

As $q \ge 2$ and $r + t \le 1$, from (3.11), $|N_D^+(z_2') \cap I_Q| \ge 1$. Let $z_3 \in N_D^+(z_2') \cap I_Q$ and $a' \in N_D^-(z_2') \cap A$. Then $H \ominus Q + z_1uPva'z_2'z_3\bar{Q}z_2$ is an eulerian subdigraph of order at least $|V(H)| - q + p + 2$. This, together with (3.7), implies $q = p + 2$ and $r = t = 0$, and so $|V(\bar{Q}[z_3, z_2'])| = 2$. Thus $V(H) = A \cup B \cup I_Q$. Again by (3.11), $|N_D^+(z_2') \cap I_Q| \ge 2$, and so there is a vertex $z_3' \in I_Q - \{z_2', z_3\}$ such that $z_2'z_3' \in A(D)$. Since $|V(\bar{Q}[z_3', z_2'])| \neq |V(\bar{Q}[z_3, z_2'])| = 2$, $|V(\bar{Q}[z_3', z_2'])| \ge 3$. Thus $H \ominus Q + z_1uPva'z_2'z_3'\bar{Q}z_2$ is an eulerian subdigraph of order at least $|V(H)| - q + p + 3 > |V(H)|$, contrary to (3.3). This proves Claim 3.

Claim 4. *Let $a \in A_0$, $b \in B_0$. Each of the following holds.*
(i) *$N_D^+(a) \cap (B \cup I_Q) = \emptyset$ and $N_D^-(b) \cap (A \cup I_Q) = \emptyset$.*
(ii) *$N_D^+(a) \cap V(P) = \emptyset$ and $N_D^-(b) \cap V(P) = \emptyset$.*
(iii) *$N_D^+(a) \subseteq R \cup T \cup (A - \{a\})$ and $N_D^-(b) \subseteq R \cup T \cup (B - \{b\})$.*
(iv) *$(\{a\}, R)_D \cup (R, \{b\})_D \subseteq A(D) - A(H)$.*
(v) *For any $x \in R \cup I_Q$, $x \notin N_D^+(a) \cap N_D^-(b)$.*

By (3.8), $N_D^+(a) \cap B = \emptyset$ and $N_D^-(b) \cap A = \emptyset$. By Claims 2 and 3, $N_D^+(a) \cap I_Q = \emptyset$ and $N_D^-(b) \cap I_Q = \emptyset$. This proves (i). Claim 4(ii) follows (3.3), (iv) follows from the definition of H and R, and (iii) follows from Claim 3(i)–(ii).

For (v), if for some $x \in R \cup I_Q$, $x \in N_D^+(a) \cap N_D^-(b)$, by (i), $x \notin I_Q$, and so $x \in R$. By (iv), $ax, xb \notin A(H)$, and so $H + axbuPva$ is an eulerian subdigraph of D with $|V(H)| + p + 1 > |V(H)|$ vertices, contrary to (3.3). This proves (v), and completes the proof for Claim 4.

Claim 4 (v) suggests that each vertex in $R \cup I_Q$ contributes at most 1 to $d_D^+(a) + d_D^-(b)$; and each vertex in T contributes at most 2 to $d_D^+(a) + d_D^-(b)$. It follows from Claim 4 that $d_D^+(a) + d_D^-(b) \le |A| - 1 + |B| - 1 + r + 2t = |A| + |B| + r + 2t - 2$, and so

$$\begin{aligned} n &= |A| + |B| + p + q + r + t \\ &\ge \delta^+ + \delta^- + 2 - r - 2t + p + q + r + t \ge n - 2 + p + q - t. \end{aligned}$$

This, together with (3.7), implies

$$p + q \le t + 2, \qquad 2q + r \le 4 \quad \text{and} \quad p \le q \le 2. \tag{3.12}$$

Claim 5. $\lambda_H(B, A) = 1$.

Suppose, that $\lambda_H(B, A) \ge 2$. Then, by the definition of I_Q there exists a (B, A)-dipath Q' in $H - I_Q$. So, $I_Q \cap I_{Q'} = \emptyset$. Assume Q' is from $z_3 \in B$ to $z_4 \in A$ and z_3', z_4' are the first vertex and the last vertex in $I_{Q'}$ of $\bar{Q}'$, respectively. Then, similar to Claim 2, we also have

$$(A, (I_{Q'} - \{z_4'\}) \cup \{z_3'\})_D = ((I_{Q'} - \{z_3'\}) \cup \{z_4'\}, B)_D = \emptyset. \tag{3.13}$$

This, together with $I_{Q'} \subseteq T$, implies $T \nsubseteq N_D^+(A)$ and $T \nsubseteq N_D^-(B)$. So, $|N_D^+(a) \cap T|, |N_D^-(b) \cap T| \le t - 1$. It follows that $d_D^+(a) + d_D^-(b) \le |A| - 1 + |B| - 1 + r + 2(t - 1) = n - p - q + t - 4 \le \delta^+ + \delta^- - p - q + t$. Thus $p + q \le t$. Together this with (3.7), the equation holds, which implies $p = q = 1$, $t = 2$, $r = 0$ and $|A| = \delta^+$, $|B| = \delta^-$.

If $|I_{Q'}| \ge 2$, then $T = I_{Q'}$ as $I_{Q'} \subseteq T$. By the fact $a \in A_0$ and $b \in B_0$, $|N_D^+(a) \cap I_{Q'}| \ge d_D^+(a) - |A| + 1 \ge 1$ and $|N_D^-(b) \cap I_{Q'}| \ge d_D^-(b) - |B| + 1 \ge 1$. Combining these with (3.13), $az_4', z_3'b \in A(D)$ and $z_3' \neq z_4'$. Thus, $H - A(Q') - I_{Q'} + z_3\bar{Q}'z_3'buPvaz_4'\bar{Q}'z_4$ is an eulerian subdigraph violating (3.3).

So, $|I_{Q'}| = 1$ and let $T = I_{Q'} \cup \{w\}$. By Claim 4(ii), Claim 2 and by (3.13), $a'w, wb' \in A(D)$ for any $a' \in A_0$ and any $b' \in B_0$. Furthermore, if there exist vertices $a'' \in A_0$ and $b'' \in B_0$ such that $a''w, wb'' \notin A(H)$, then $H + a''wb''uPva''$ is an eulerian subdigraph violating (3.3). Hence, without loss of generality, we may assume $a'w \in A(H)$ for any $a' \in A_0$. Thus $d_H^-(w) \ge |A_0|$. As H is eulerian, $d_H^+(w) = d_H^-(w) \ge |A_0|$. Moreover, since no arc in $(A, \{w\})_D$ lies in any (B, A)-dipath, by

Lemma 3.2, w cannot lie in any increment of (B, A)-dipath. It follows that $\lambda_H(B, A) = 2$. If, for every $b' \in B_0$, $wb' \in A(H)$, then $\lambda_H(B, A) = \lambda_H(A, B) \geq \max\{|A - A_0|, |B - B_0|\} + \min\{|A_0|, |B_0|\} \geq \min\{|A|, |B|\} \geq 4$, a contradiction. Hence, there exists $b_0 \in B_0$ such that $wb_0 \notin A(H)$. Also, by $\lambda_H(A, B) \leq 2$, we see that $|N_H^+(w) \cap B| \leq 2 - |A - A_0|$. Thus $|N_H^+(w) \cap A| = d_H^+(w) - |N_H^+(w) \cap B| \geq |A_0| - 2 + |A - A_0| > 1$, which implies there exists a vertex $a_0 \in A$ such that $wa_0 \in A(H)$. Thus, $H - wa_0 + wb_0uPva_0$ is an eulerian subdigraph violating (3.3), a contradiction which completes the proof of this claim.

Claim 6. $p = 1$.

Suppose, $p \geq 2$. By (3.12), $p = q = t = 2$ and $r = 0$. Then by Claim 1(iii), $|A| \geq \delta^+ - 1$, $|B| \geq \delta^- - 1$ and thus $n = |A| + |B| + p + q + r + t \geq \delta^+ + \delta^- + 4 \geq n$, which implies $|A| = \delta^+ - 1$, $|B| = \delta^- - 1$.

Let $T = \{w_1, w_2\}$. For any vertex $a' \in A_0$, if $a'z_2' \in A(D)$ then $H \ominus Q + z_1uPva'z_2'\bar{Q}z_2$ is an eulerian subdigraph of order $|V(H)| - q + p + 1 > |V(H)|$, contrary to (3.3). Hence $N_D^+(a') \subseteq (A - \{a'\}) \cup T$ and $|N_D^+(a') \cap T| \geq d_D^+(a') - |A - \{a'\}| \geq \delta^+ - |A| + 1 = 2$, which implies $a'w_1, a'w_2 \in A(D)$. Similarly, for any $b' \in B_0$, we also have $w_1b', w_2b' \in A(D)$.

Since $\lambda_H(A, B) = 1$, $\max\{|A - A_0|, |B - B_0|\} + \min\{|N_H^-(w_1) \cap A|, |N_H^+(w_1) \cap B|\} \leq 1$. Without loss of generality, we may assume $|N_H^-(w_1) \cap A| \leq |N_H^+(w_1) \cap B|$. Then

$$\max\{|A - A_0|, |B - B_0|\} + |N_H^-(w_1) \cap A| \leq 1. \tag{3.14}$$

Since $|A_0| = |A| - |A - A_0| \geq \delta^+ - 1 - 1 \geq 2$, there exists a vertex $a_0 \in A_0$ such that $a_0w_1 \notin A(H)$. By (3.3), $(\{w_1\}, B)_D \subseteq A(H)$. So, $|N_H^+(w_1)| \geq |N_H^+(w_1) \cap B| \geq |B_0|$. On the other hand, if $N_H^-(w_1) \cap B \neq \emptyset$, say $b' \in N_H^-(w_1) \cap B$, then $H - b'w_1 + b'uPva_0w_1$ is an eulerian subdigraph violating (3.3). Hence $N_H^-(w_1) \cap B = \emptyset$, and so $|N_H^-(w_1)| = |N_H^-(w_1) \cap A| + |N_H^-(w_1) \cap T| \leq |N_H^-(w_1) \cap A| + 1$. By (3.14), $|N_H^-(w_1)| \leq 1 + |N_H^-(w_1) \cap A| \leq 2 - \max\{|A - A_0|, |B - B_0|\} \leq 2 - |B - B_0|$. Combining this with $|N_H^+(w_1)| \geq |B_0|$, $\delta^- - 1 = |B| = |B_0| + |B - B_0| \leq 2$, contrary to the fact that $\delta^- \geq 4$. The proof of this claim is completed.

By Claim 5, and since H is eulerian, $\lambda_H(B, A) = 1$ and so $\lambda_H(A, B) = 1$. By Menger's Theorem, $H \ominus Q$ has a partition, say $\{A', B'\}$, such that $A \subseteq A'$, $B \subseteq B'$ and $|\partial^+_{H-A(Q)}(A')| = 1$. As Q is the only (B, A)-dipath in H, $(B', A')_{H-A(Q)} = \emptyset$. Choose such a partition $\{A', B'\}$ such that

$$\mu := \min\{|A'| - |A|, |B'| - |B|\} \text{ is minimized.} \tag{3.15}$$

Denote $\partial_H^+(A') = \{a'b'\}$, where $a' \in A'$ and $b' \in B'$. Then $|A'| \geq |A| \geq \delta^+$, $|B'| \geq |B| \geq \delta^-$. For vertices x and y, define

$$\delta_{x=y} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

As $|A' - A| + |B' - B| = t \leq 2$, from (3.15) we see that $\mu \leq 1$. Next, we show $D \in \mathcal{F}_0(k_1, k_2, 2)$ for some k_1, k_2, by discussing two cases according to the value of μ.

Case 1. $\mu = 0$.

In this case, either $A' = A$ or $B' = B$. Without loss of generality, we may assume $A' = A$. First, we give the following claim.

Claim 7. $N_D^-(z_2') \cap A \neq \emptyset$.

Suppose, to the contrary, that $N_D^-(z_2') \cap A = \emptyset$. Then by Claim 2, $(A, I_Q)_D = \emptyset$. For any $x \in A - \{a'\}$, by Claim 1(ii), $|N_D^+(x) \cap K| = |N_D^+(x) \cap (K \cup B)| \geq d_D^+(x) - (|A| - 1) \geq 1$, where $K = (B' - B) \cup R$. Furthermore, we have the following claim.

(7A) For any $x \in A - \{a'\}$, $|N_D^+(x) \cap K| \geq 1$ and for any $y \in K$ with $N_D^-(y) \cap (A - \{a'\}) \neq \emptyset$, $|N_D^+(y) \cap A| \geq 1$.

The first part of (7A) is true clearly. For the last part, let $y \in K$ and $a_1 \in A - \{a'\}$ such that $a_1y \in A(D)$. If there exists $b_1 \in N_H^-(y) \cap B$, then $H - b_1y + b_1ua_1y$ is an eulerian subdigraph with $|V(H)| + 1$ vertices, contrary to (3.3). So, $N_H^-(y) \cap B = \emptyset$. Hence $d_H^+(y) = d_H^-(y) = |N_H^-(y) \cap K| + \delta_{y=b'} \leq 2$, since $|K| = n - |A| - |B| - p - q \leq n - \delta^+ - \delta^- - 2 \leq 2$. Also, we have $N^+_{D-A(H)}(y) \cap B = \emptyset$, as otherwise, assuming $b_2 \in N^+_{D-A(H)}(y) \cap B$, $H + yb_2uPva_1y$ is a bigger eulerian subdigraph, a contradiction to (3.3). Hence, $|N_D^+(y) \cap A| = d_D^+(y) - |N_D^+(y) \cap B| - |N_D^+(y) \cap K| \geq \delta^+ - d_H^+(y) - 1 \geq 1$. This proves (7A).

As $a'b'$ is the only arc from A to B', there exists $x_1 \in A$ such that either $x_1a' \in A(H)$ or $a'x_1 \in A(D) - A(H)$, since otherwise, for any $x' \in A - \{a'\}$, $x'a' \notin A(H)$ and $a'x' \notin A(D) - A(H)$, thus $d_H^-(a') = |N_H^-(a') \cap I_Q| \leq 1$ and $d_H^+(a') = d_D^+(a') - d^+_{D-A(H)}(a') \geq \delta^+ - |N_D^+(a') \cap K| \geq 4 - |K| \geq 2$, a contradiction. Next, for each $i \geq 1$, we pick $y_i \in N_D^+(x_i) \cap K$ and $x_{i+1} \in N_D^+(y_i) \cap (A - \{a'\})$. The existence of such y_i is assured by (7A). If for some i, such an x_i does not exist, then by (7A), $N_D^+(y_{i-1}) \cap A = \{a'\}$. Thus, if $x_1a' \in A(H)$ then let $H' = H - x_1a' + x_1y_1x_2 \ldots y_{i-1}a'$, and if $a'x_1 \in A(D) - A(H)$ then let $H' = H + a'x_1y_1x_2 \ldots y_{i-1}a'$. Then H' is an eulerian subdigraph with at least $|A(H)| + 1$ arcs, contrary to (3.4). Hence, we can form sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$. Then there is a dicycle C whose arcs are in $\{x_iy_i, y_ix_{i+1} \mid i = 1, 2 \ldots\} \subseteq A(D) - A(H)$. Thus $H + A(C)$ is an eulerian subdigraph, contrary to (3.4). This finish the proof of Claim 7.

Assume $a_1z_2' \in A(D)$ for some $a_1 \in A$. Then $z_2' \neq z_1'$ by Claim 2 and $I_Q = \{z_1', z_2'\}$ by (3.12). Thus $\bar{Q} = Q$ and $|B' - B| = n - |A| - |B| - p - q - r \leq n - \delta^+ - \delta^- - 3 \leq 1$.

Note that Q is a (B, A)-dipath. If $V(Q[z_1, z_2']) \cap (B' - B) = \emptyset$, then $Q[z_1, z_2'] = z_1z_1'z_2'$ and thus $H - z_1z_1'z_2' + z_1uPva_1z_2'$ is an eulerian subdigraph with exactly $|V(H)|$ vertices and $|A(H)| + 1$ arcs, contrary to (3.4). Hence, $V(Q[z_1, z_2']) \cap (B' - B) \neq \emptyset$. Together with the fact $|B' - B| \leq 1$, we see that $|B' - B| = 1$. Let $B' - B = \{w\}$. Then either $Q = z_1wz_1'z_2'z_2$ or $Q = z_1z_1'wz_2'z_2$.

In fact, we will show $Q = z_1wz_1'z_2'z_2$. Suppose $Q = z_1z_1'wz_2'z_2$. If there exists a vertex $b_2 \in B - \{b'\}$ such that $b_2w \in A(H)$, then $H - b_2wz_2' + b_2uPva_1z_2'$ is a spanning eulerian subdigraph, contrary to (3.2). So, $N_H^-(w) \cap B = \emptyset$. This, together with $w \notin I_Q$, forces $w = b'$ and $N^-_{H-A(Q)}(w) = \{a'\}$. It follows that $|N^+_{H-A(Q)}(w)| = 1$ and thus $\{A \cup \{w\}, B\}$ is also a candidate partition of $\{A', B'\}$, in which the value of μ is also 0. Then similar to Claim 7, we also have $N_D^+(z_1') \cap B \neq \emptyset$. Let $b_1 \in B$ such that $z_1'b_1 \in A(D)$. Then $H - A(Q) + z_1z_1'b_1uPva_1z_2'z_2$ is a spanning eulerian subdigraph, contrary to (3.2). Hence, $Q = z_1wz_1'z_2'z_2$.

Now, we show that $D \in \mathcal{F}_0(k_1, k_2, 2)$, in which $\{z_1', u\}$ plays the role of U in the definition. To this end, it suffices to show $(A \cup \{z_2'\}, \{z_1'\})_D = (\{z_1'\}, B')_D = \emptyset$ and $|(A \cup \{z_2'\}, B')_D| = 1$. In fact, by Claim 2, $(A, \{z_1'\})_D = \emptyset$. If $z_2'z_1' \in A(D)$, then $H - z_1Qz_2 + z_1uPva_1z_2'z_1'z_2'z_2$ is a spanning eulerian subdigraph, contrary to (3.2). Thus, $(A \cup \{z_2'\}, \{z_1'\})_D = \emptyset$. If there exists $z_1'b_3 \in (\{z_1'\}, B')_D$ for some $b_3 \in B'$, then if $b_3 \in B$ then $H - z_1Qz_2 + z_1z_1'b_3uPva_1z_2'z_2$ is a spanning eulerian subdigraph, contrary to (3.2), and if $b_3 = w$ then $H - z_1Qz_2 + z_1uPva_1z_2'z_2 + wz_1'w$ is a spanning eulerian subdigraph, contrary to (3.2) again. So, $(\{z_1'\}, B')_D = \emptyset$. Finally, by Claim 1(ii) and Claim 2, $(A \cup \{z_2'\}, B)_D \subseteq A(H)$. If there exists $a_2w \in (A \cup \{z_2'\}, \{w\})_D \cap (A(D) - A(H))$, then $H - z_1Qw + z_1uPv(a_1)a_2w$ is an eulerian subdigraph with at least $|V(H)|$ vertices and with at least $|A(H)| + 1$ arcs, contrary to (3.3) or (3.4). So, $|(A \cup \{z_2'\}, B')_D| = |(A \cup \{z_2'\}, B')_H| = |\{a'b'\}| = 1$. Hence, $D \in \mathcal{F}_0(k_1, k_2, 2)$, where $k_1 = |A|$ and $k_2 = |B'| - 1 = |B|$.

Case 2. $\mu = 1$.

In this case, $|A'| = |A| + 1 = \delta^+ + 1$, $|B'| = |B| + 1 = \delta^- + 1$ and $q = 1$. In order to show $D \in \mathcal{F}_0(k_1, k_2, 2)$ for some k_1, k_2, we give the following claim firstly.

Claim 8. $|(A', B')_D| = 1$.

Since $|\partial_H^+(A')| = 1$, it suffices to show that $(A', B')_{D-A(H)} = \emptyset$. Suppose there exists an arc $xy \in (A', B')_{D-A(H)}$. First, we assume $x \in A$. Then, by Claim 1(ii), $y \in B - B'$. Furthermore, if there exists a vertex $y' \in B$ such that $y'y \in A(H)$ then $H - y'y + y'uPvxy$ is an eulerian subdigraph with at least $|V(H)| + 1$ vertices, contrary to (3.3). Hence $N_H^-(y) \cap B = \emptyset$. This, together with $N^+_{H-A(Q)}(y) \cap A' = \emptyset$, implies that $|N^+_{H-A(Q)}(y) \cap B| = d^+_{H-A(Q)}(y) = d^-_{H-A(Q)}(y) = |N^-_{H-A(Q)}(y) \cap A'|$. It follows that $|\partial^+_{H-A(Q)}(A' \cup \{y\})| = |\partial^+_{H-A(Q)}(A)| = 1$, which implies $\{A' \cup \{y\}, B\}$ is a candidate of partition (A', B') such that $\mu = 0$, contrary to the assumption of this case. Hence, $x \notin A$. Similarly, we also have $y \notin B$. Thus $A' = A \cup \{x\}$ and $B' = B \cup \{y\}$ and $|\partial_D^+(A')| = 2$. For any $x' \in A - \{a'\}$, $d_D^+(x') = |N_D^+(x') \cap (A' - \{x'\})| \le |A' - \{x'\}| = \delta^+$, which implies $A' - \{x'\} \subseteq N_D^+(x')$. In particular, $x'x \in A(D)$. Thus $A - \{a'\} \subseteq N_D^-(x)$. Similarly, $B - \{b'\} \subseteq N_D^+(y)$.

If there exists a vertex $x' \in A - \{a'\}$ such that $x'x \notin A(H)$, then $(\{y\}, B)_D \subseteq A(H)$, as otherwise, say $yb_1 \in A(D) - A(H)$ for some $b_1 \in B$, then $H + yb_1uPvx'xy$ is an eulerian subdigraph violating (3.3). Thus, $d_H^+(y) \ge |B - \{b'\}| = \delta^- - 1 \ge 3$. On the other hand, if there exists $b_2 \in B$ such that $b_2y \in A(H)$ then $H - b_2y + b_2uPvx'xy$ is an eulerian subdigraph violating (3.3). Hence, $d_H^-(y) = |N_H^-(y) \cap A'| \le 1$, a contradiction to the fact $d_H^+(y) = d_H^-(y)$. Therefore, $A - \{a'\} \subseteq N_H^-(x)$. Then $d_H^-(x) \ge \delta^+ - 1 \ge 3$. Thus, $|N_H^+(x) \cap A| = |\partial_H^+(x) - \{a'b'\}| \ge d_H^+(x) - 1 = d_H^-(x) - 1 \ge 2$. So there is a vertex $x_1 \in A - \{a'\}$ such that $xx_1, x_1x \in A(H)$. Similarly, there exists a vertex $y_1 \in B - \{b'\}$ such that $yy_1, y_1y \in A(H)$. Then $H - xx_1 - y_1y + xy + y_1uPvx_1$ is an eulerian subdigraph with at least $|V(H)| + 1$ vertices, contrary to (3.3). This proves Claim 8.

By Claim 8, let $(A', B')_D = \{u_1v_1\}$, where $u_1 \in A'$, $v_1 \in B'$. By the assumption of this case, assume $A' = A \cup \{u_2\}$ and $B' = B \cup \{v_2\}$. As $|A'| = \delta^+ + 1$, $u_2 \in N_D^+(x)$ for $x \in A - \{u_1\}$. Thus $A - \{u_1\} \subseteq N_D^-(u_2)$.

By Claims 2 and 8, in order to show $D \in \mathcal{F}_0(k_1, k_2, 2)$ for some k_1, k_2, it suffices to show that $u_2z_1' \notin A(D)$ and $z_1'v_2 \notin A(D)$. Suppose, without loss of generality, that $u_2z_1' \in A(D)$. If there exists $u_3 \in N_D^+(z_1') \cap A$ such that $u_2u_3 \in A(H)$, then $H - u_2u_3 + u_2z_1'u_3$ is an eulerian subdigraph of D with $|A(H)| + 1$ arcs, a contradiction to (3.4). So, $N_H^+(u_2) \cap N_D^+(z_1') \cap A = \emptyset$. Also, by Claim 2, $|N_D^+(z_1') \cap A| = d_D^+(z_1') - |N_D^+(z_1') \cap B'| \ge \delta^+ - 1$. Thus, by Claim 8, $d_H^+(u_2) \le |N_H^+(u_2) \cap A| + 1 \le |A| - |N_D^+(z_1') \cap A| + 1 \le 2$. It follows that $|N^-_{D-A(H)}(u_2) \cap A - \{u_1\}| \ge |N_D^-(u_2) \cap A - \{u_1\}| - d_H^+(u_2) \ge |A - \{u_1\}| - d_H^-(u_2) \ge 1$. Let $u_4 \in N^-_{D-A(H)}(u_2) \cap A - \{u_1\}$. Then $H \ominus Q + z_1uPvx_2u_4z_1'z_2$ is a spanning eulerian subdigraph, a contradiction. Similarly, $z_1'v_2 \notin A(D)$. So, $D \in \mathcal{F}_0(\delta^+, \delta^-, 2)$, which completes the proof. ■

If we focus on the minimum degree condition, the following corollary can be obtained easily from Theorem 3.4.

Corollary 3.5. *Let D be a digraph of order $n \ge 11$ and minimum degree $\min\{\delta^+(D), \delta^-(D)\} \ge n/2 - 2$. Then D is not supereulerian if and only if n is even and $D \in \mathcal{F}_0(n/2 - 2, n/2 - 2, 2)$.*

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