Graphs and Combinatorics (2014) 30:549–563
DOI 10.1007/s00373-013-1297-9

ORIGINAL PAPER

On Group Choosability of Graphs, II

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Received: 6 October 2011 / Revised: 30 January 2013 / Published online: 6 March 2013

Abstract Given a group A and a directed graph G, let $F(G, A)$ denote the set of all maps $f : E(G) \to A$. Fix an orientation of G and a list assignment $L : V(G) \mapsto 2^A$. For an $f \in F(G, A)$, G is (A, L, f)-*colorable* if there exists a map $c : V(G) \mapsto \cup_{v \in V(G)} L(v)$ such that $c(v) \in L(v)$, $\forall v \in V(G)$ and $c(x) - c(y) \neq f(xy)$ for every edge $e = xy$ directed from x to y. If for any $f \in F(G, A)$, G has an (A, L, f)-coloring, then G is (A, L)-*colorable*. If G is (A, L)-colorable for any group A of order at least k and for any k-list assignment $L : V(G) \to 2^A$, then G is k-*group choosable*. The *group choice number*, denoted by $\chi_{gl}(G)$, is the minimum k such that G is k-group choosable. In this paper, we prove that every planar graph is 5-group choosable, and every planar graph with girth at least 5 is 3-group choosable. We also consider extensions of these results to graphs that do not have a K_5 or a $K_{3,3}$ as a minor, and discuss group choosability versions of Hadwiger's and Woodall's conjectures.

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Keywords Colorings · Group colorings · List coloring · Group list colorings · Group choice number · Group connectivity

1 Introduction

In this paper, we consider finite and simple graphs. Undefined terms and notations can be found in [1]. In particular, for a simple connected graph G, and for any $v \in V(G)$, $d_G(v)$, $\Delta(G)$, $\kappa(G)$, $c(G)$, and $\chi(G)$ denote the degree of vertex v, the maximum degree, the connectivity, the number of components of G and the chromatic number of G, respectively. When the graph G is understood from the context, we also use $d(v)$ for $d_G(v)$. If G is a directed graph, we again use $E(G)$ to denote the set of directed edges of G, and by $(u, v) \in E(G)$ we mean that a directed edge oriented from u to v is in G. A cycle of length n is referred as an *n-cycle*. If X is a vertex subset or an edge subset, then $G[X]$ is the subgraph of G induced by X. For a subset $S \subseteq V(G)$, let $N_G(S)$ denote the vertices in G that is adjacent to at least one vertex in S, and let $N_G[S] = N_G(S) \cup S$. Throughout this paper, $\mathbb{Z}$ denotes the set of integers, and for $m \in \mathbb{Z}$ with $m > 0$, $\mathbb{Z}_m$ denote the cyclic group of order m.

A *list assignment* of a graph G is a map L that assigns to each vertex $v \in V(G)$ a list $L(v)$ of colors. A proper vertex coloring c of G is an *L-coloring* of G if $\forall v \in V(G), c(v) \in L(v)$. For an integer k, a *k-list assignment* of G is a list assignment L with $|L(v)| = k$ for each vertex $v \in V(G)$; G is *k-choosable* if G has an L-coloring for every k-list assignment L of G. The *choice number*, $\chi_l(G)$, is the minimum k such that G is k-choosable.

Throughout this paper, A denotes a group with identity 0. We will use addition to denote the binary operation of A even when A is not Abelian. For a graph G, let $F(G, A) = \{f : E(G) \to A\}$. Fix an orientation of G. Given an $f \in F(G, A)$, a map $c : V(G) \to A$ such that $c(x) - c(y) \neq f(xy)$ for any $(x, y) \in E(G)$ is an (A, f)-coloring of G. If for any $f \in F(G, A)$, G has an (A, f)-coloring, then G is *A-colorable*. It is known [8] that whether G is A-colorable is independent of the orientation of G. The *group chromatic number* of G, $\chi_g(G)$, is the minimum k such that G is A-colorable for any group A of order at least k.

Given a group A and a graph G, let $F(G, A)$ denote the set of all maps $f : E(G) \to A$. Fix an orientation of G and a list assignment $L : V(G) \to 2^A$. For an $f \in F(G, A)$, G is *(A, L, f)-colorable* if there exists a map $c : V(G) \to \cup_{v \in V(G)} L(v)$ such that $c(v) \in L(v), \forall v \in V(G)$ and $c(x) - c(y) \neq f(xy)$ for any $(x, y) \in E(G)$. If for any $f \in F(G, A)$, G has an (A, L, f)-coloring, then G is *(A, L)-colorable*. If G is (A, L)-colorable for any group A of order at least k and for any k-list assignment $L : V(G) \to 2^A$, then G is *k-group choosable*. The *group choice number*, denoted by $\chi_{gl}(G)$, is the minimum k such that G is k-group choosable.

The concept of group choosability was first introduced in [13] and the basic properties of χ_{gl} were discussed in [3]. By definition,

$$|V(G)| \geq \chi_{gl}(G) \geq \max\{\chi_g(G), \chi_l(G)\}. \tag{1}$$

A graph H is a *minor* of a graph K if H is the contraction image of a subgraph of K. A graph G is **H-minor free** if G does not have H as a minor. Hadwiger [5] posed a well-known conjecture

Conjecture 1.1 (Hadwiger [5]) *For all $k \geq 1$, every k-chromatic graph has the complete graph K_k as a minor.*

Hadwiger's conjecture holds for $k \leq 4$ (see [4,5]), and for the case $k \in \{5, 6\}$, it is equivalent to the Four-color Theorem [5,19]. More results on Hadwiger's conjecture may be found in [10,11,20]. In [17], Mirzakhani constructed examples to show that there exists a planar graph G with $\chi_g(G) \geq \chi_l(G) \geq 5$. In [21], Thomasen proved that for a planar graph G, $\chi_l(G) \leq 5$. These results are later extended to graphs without K_5 minors or $K_{3,3}$-minors, as shown in [7,15,16]. In [11], Kawarabayashi and Mohar proposed a relaxed version of Hadwiger's conjecture.

Conjecture 1.2 (Kawarabayashi and Mohar [11]) *There exists a constant c such that every graph without K_k-minors is ck-choosable.*

This motivates a similar conjecture for group choice number of graphs.

Conjecture 1.3 *There exists a constant c such that every graph without K_k-minors is ck-group choosable.*

In [2], Chartrand, Geller and Hedetniemi proposed a conjecture, which was later corrected and reformulated by Woodall [25], as follows.

Conjecture 1.4 (Chartrand, Geller and Hedetniemi [2], Woodall [25]) *Let $k \geq 1$ be an integer, any graph G with $\chi(G) \geq k$ has either a complete graph K_k or a complete bipartite graph $K_{\lfloor \frac{k+1}{2} \rfloor, \lceil \frac{k+1}{2} \rceil}$ as a minor.*

Conjectures 1.1 and 1.4 might have motivated Woodall to propose a conjecture on choice number to forbid a complete bipartite minor.

Conjecture 1.5 (Woodall [26]) *Every graph with no $K_{r,s}$-minor is $(r + s - 1)$-choosable.*

Results evidencing Conjecture 1.5 can be found in the literature (see e.g. [9,26,27]). Here, we present a group choosability version of it.

Conjecture 1.6 *Every graph with no $K_{r,s}$-minor is $(r + s - 1)$-group choosable.*

In this paper, we investigate the group choice number for planar graphs, K_5-minor free graphs and $K_{r,s}$-minor free graphs with smaller values of r and s. In the next section, we prove that every simple planar graph is 5-group choosable. Then we extend this to K_5-minor free graphs and $K_{3,3}$-minor free graphs in consequent sections. In the last section, we prove that every $K_{3,3}$-minor free graph with large girth is 3-choosable.

2 Group Choosability of Planar Graphs

In [3], it is prove that

Lemma 2.1 *Let G be a graph, then* $\chi_{gl}(G) \leq \max_{H \subseteq G}\{\delta(H)\} + 1$.

Thus if G is a planar graph, then $\chi_{gl}(G) \leq 6$. In this section, we shall modified the methods used in [21] and [16] to prove that every planar graph is 5-group choosable. As in [1], a planar embedding of a planar graph G is referred as a *plane* graph, and the unique unbounded face of G is referred as the *outer face* of G. If F is a face of a plane graph G, then the edges of G incident with F induces a subgraph $\partial(F)$ of G, called the *boundary* of F. We shall use $Out(G)$ to denote the outer face boundary of G. A plane graph G is *near triangulation* if every face of G other than the outer face is a triangle. Also as in [1], if a plane graph G has a cycle C, then the simple curve C partitions the plane into two open sets, called the *interior* and *exterior* of C, respectively. The vertices of G contained in the interior of C together with $V(C)$ induces the subgraph $Int(C)$, and the vertices of G contained in the exterior of C together with $V(C)$ induces the subgraph $Ext(C)$. A cycle C of a plane graph G is *separating* if both $V(Int(C)) \neq V(C)$ and $V(Ext(C)) \neq V(C)$.

Theorem 2.2 *Suppose that G is a near triangulation plane graph with outer face C and that A is a group with* $|A| \geq 5$. *Let* $e = v_1v_2 \in E(C)$, $H = G[\{v_1, v_2\}]$, *and* $L : V(G) \mapsto 2^A$ *be a list assignment of G satisfying*

$$|L(v)| \begin{cases} \geq 5 & \text{if } v \notin V(C), \\ = 1 & \text{if } v \in \{v_1, v_2\} \\ \geq 3 & \text{if } v \in V(C) - \{v_1, v_2\} \end{cases}$$

If $f \in F(G, A)$ *and if H is* $(A, L|_{V(H)}, f|_{E(H)})$*-colorable. Then any* $(A, L|_{V(H)}, f|_{E(H)})$*-coloring of H can be extended to an* (A, L, f)*-coloring of G.*

Proof Let $n = |V(G)|$ and we argue by induction on n. By Lemma 2.1, the theorem holds trivially for $n \leq 5$, and so we assume that $n > 5$. If G has a cut vertex z, then G has two connected edge-disjoint subgraphs G_1 and G_2, such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{z\}$. We may assume that $v_1, v_2 \in V(G_1)$. By induction, any $(A, L|_{V(H)}, f|_{E(H)})$-coloring c_0 of H can be extended to an $(A, L|_{V(G_1)}, f|_{E(G_1)})$-coloring c_1 of G_1. Let $z' \in N_{G_2}(z) \cap V(G_2 - z)$ such that zz' is oriented from z to z'. Since $|L(z')| \geq 3$, $L(z') - \{-f(zz') + c_1(z)\} \neq \emptyset$ and so one can color z' with a color $c_1(z') \in L(z') - \{-f(zz') + c_1(z)\}$. By induction again, c_1 on $G_2[\{z, z'\}]$ can be extended to an $(A, L|_{V(G_2)}, f|_{E(G_2)})$-coloring c_2 of G_2. By definition, an (A, L, f)-coloring c of G extending c_0 can be obtained by combining c_1 and c_2. Hence we may assume that $\kappa(G) \geq 2$. Thus C is a cycle. We assume that C is so oriented that $C = v_1v_2 \cdots v_pv_1$ is a directed cycle.

Case 1 The cycle C has a chord.

Let $v_iv_j \neq v_1v_p$ with $1 \leq i \leq j \leq p$ and $i \leq p-2$ denote this chord of C, and let $C_1 = v_1v_2 \cdots v_iv_jv_{j+1} \cdots v_pv_1$ be the cycle contains v_1, v_p and the chord v_iv_j and $C_2 = v_iv_{i+1} \cdots v_jv_i$. By induction, any $(A, L|_{V(H)}, f|_{E(H)})$-coloring c_0 of H can be extended to an $(A, L|_{V(Int(C_1))}, f|_{E(Int(C_1))})$-coloring c_1 of $Ext(C_1)$. By induction again, c_1 on $Ext(C_2)[\{v_i, v_j\}]$ can be extended to an $(A, L|_{V(Int(C_2))}, f|_{E(Int(C_2))})$-coloring c_2 of $Ext(C_2)$. By definition, an (A, L, f)-coloring c of G extending c_0 can be obtained by combining c_1 and c_2.

Case 2 The cycle C has no chords.

Let $v_1, u_1, \ldots, u_m, v_{p-1}$ denote the neighbors of v_p in G. Let $G' = G - v_p$, $L(v_1) = \{a\}$. Since C has no chord and since G is a near triangulation, we may assume that $C' = v_1 v_2 \cdots v_{p-1} u_m \cdots, u_2 u_1 v_1$ is a directed cycle, which is also the outer cycle of G'. We further assume that for each $1 \le i \le m$, the edge $v_p u_i$ is directed from v_p to u_i. Since $|L(v_p)| \ge 3$, there are two distinct colors $x, y \in L(v_p) - \{f(v_p v_1) + a\}$. For $1 \le i \le m$, since C has no chords, $|L(u_i)| \ge 5$. Define

$$L'(z) = \begin{cases} L(u_i) - \{-f(v_p u_i) + x, -f(v_p u_i) + y\} & \text{if } z = u_i, 1 \le i \le m \\ L(z) & \text{otherwise.} \end{cases}$$

By induction, any $(A, L|_{V(H)}, f|_{E(H)})$-coloring c_0 of H can be extended to an $(A, L'|_{V(G')}, f|_{E(G')})$-coloring c' of G'. Extend c' to a coloring c on $V(G)$ by coloring v_p with $t \in \{x, y\} - \{-f(v_{p-1} v_p) + c'(v_{p-1})\}$. By the choices of x and y, c is an (A, L, f)-coloring of G extending c_0. □

Corollary 2.3 *Let G be a planar graph. Then $\chi_{gl}(G) \le 5$.*

By (1), Theorem 2.2 extends Theorem 2.1 in [16]. In [14], Král, Prangrác and Voss constructed a family of planar graphs G with $\chi_g(G) = 5$. By (1), the upper bound in Corollary 2.3 is sharp.

3 On Group Choosability Version of Hadwiger's Conjecture

In this section, we investigate the group choosability version of Hadwiger's conjecture, and provide some evidence for Conjecture 1.3 by showing that it holds for $k \le 5$ with $c = 1$. The cases when $k = 1, 2$ are trivial. It has been shown in [3] that $\chi_{gl}(G) \le 2$ if and only if G is a forest. As K_3-minor free graphs are precisely the forests, Conjecture 1.3 holds for $k = 3$ with $c = 1$ as well. We shall show that the same holds when $k \le 5$ and $c = 1$ in this section. The case when $k = 4$ follows immediately from the following Theorem of Direc and Lemma 2.1.

Theorem 3.1 (Dirac [4]) *If G is a simple K_4-minor free graph, then $\delta(G) \le 2$.*

Corollary 3.2 *If G is a simple K_4-minor free graph, then $\chi_{gl}(G) \le 3$.*

Proof By Theorem 3.1, $\max_{H \subseteq G}\{\delta(H)\} \le 2$ and so the corollary follows from Lemma 2.1. □

For the case when $k = 5$, we need more tools. Let G_1 and G_2 be two graphs whose intersection $G_1 \cap G_2$ is a complete graph on $k \le 3$ vertices. The graph obtained from the union $G_1 \cup G_2$ by deleting the edges of $G_1 \cap G_2$ is called the *k-sum* of G_1 and G_2. The Wagner graph, is the graph depicted below (Fig. 1).

Theorem 3.3 (Wagner [24]) *Let G be a connected K_5-minor free graph. One of the following must hold.*

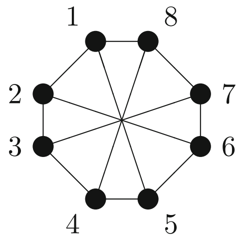

Fig. 1 The Wagner graph

(i) *G is a planar.*
(ii) *G is isomorphic to the Wagner graph.*
(iii) *G is isomorphic to* $K_{3,3}$.
(iv) *For some* $i \in \{1, 2, 3\}$, *G is the i-sum of two graphs* G_1 *and* G_2, *such that both* G_1 *and* G_2 *are proper minors of G.*

Note that if G is isomorphic to $K_{3,3}$ or to the Wagner graph, then $\max_{H \subseteq G}\{\delta(H)\} = 3 < 5$. Thus the next lemma can be routinely verified.

Lemma 3.4 *Let G be a* $K_{3,3}$ *or the Wagner graph, and H be a subgraph of G isomorphic to a* K_2, *A be a group of order at least 5, and* $L : V(G) \to 2^A$ *be a list assignment of G with* $|L(v)| \geq 5$ *for every* $v \in V(G)$. *If* $f \in F(G, A)$, *then any* $(A, L|_H, f|_H)$*-coloring* c_0 *of H can be extended to an* (A, L, f)*-coloring c of G.*

Theorem 3.5 *Let G be a* K_5*-minor free graph, H be a subgraph of G isomorphic to a* K_2 *or a* K_3, *A be a group of order at least 5, and* $L : V(G) \to 2^A$ *be a list assignment of G with* $|L(v)| \geq 5$ *for every* $v \in V(G)$. *If* $f \in F(G, A)$, *then any* $(A, L|_H, f|_H)$*-coloring* c_0 *of H can be extended to an* (A, L, f)*-coloring c of G.*

Proof We argue by contradiction and assume that

$$G \text{ is a counterexample with } |V(G)| \text{ minimized.} \tag{2}$$

It follows from (2) that $\kappa(G) \geq 2$. If G has two subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v_1, v_2\}$, then we may assume that the edge $e = v_1v_2$ is in both G_1 and G_2 with $f(e) \in A$, and that H is a subgraph of G_1. By (2), c_0 of H can be extended to an $(A, L|_{V(G_1)}, f|_{E(G_1)})$-coloring c_1 of G_1; and $c_1|_{\{v_1,v_2\}}$ can be extended to an $(A, L|_{V(G_2)}, f|_{E(G_2)})$-coloring c_2 of G_2. Thus an (A, L, f)-coloring c of G can be obtained by combining c_1 and c_2, contrary to (2). This proves Claim 1 below.

Claim 1 $\kappa(G) \geq 3$.

Claim 2 *G is non-planar.*

By contradiction, we assume that G is planar. If $H = K_2$ or if $H = K_3$ is not a separating cycle of G, then we may assume that G is a plane graph such that H is either on the outer face (when $H = K_2$ or H is the outer face (when $H = K_3$ is not separating). If $H = K_2$, then the theorem follows from Theorem 2.2. If H is the outer cycle, we denote $V(H) = \{u, v, w\}$ may assume that all edges incident with w in G

are directed from w under the orientation of G. Let $G' = G - w$ and $L' : V(G') \to 2^A$ be defined as follows

$$L'(x) = \begin{cases} L(x) - \{-f(wx) + c(w)\} & \text{if } xw \in E(G) \\ L(x) & \text{otherwise} \end{cases}$$

By Theorem 2.2, c_0 on $\{u, v\}$ can be extended to G'. This coloring of G', together with the original value of $c_0(w)$, is an extension of c_0, contrary to (2).

Hence $H = K_3$ is a separating cycle of G. By (2), c_0 can be extended to an $(A, L|_{V(Int(H))}, f|_{E(Int(H))})$-coloring c_1 and an $(A, L|_{V(Ext(H))}, f|_{E(Ext(H))})$-coloring c_2. It follows that an extension to an (A, L, f)-coloring of G is obtained by combining c_1 and c_2, contrary to (2). This proves Claim 2.

By Claims 1 and 2, by Lemma 3.4 and by Theorem 3.3, we may assume that G is the 3-sum of G_1 and G_2, such that each of G_1 and G_2 is a proper minor of G. Since G is K_5-minor free, G_1 and G_2 are also K_5-minor free. Let $C = G_1 \cap G_2$ be the 3-cycle. By the definition of 3-sums, $G = G_1 \cup G_2 - E(C)$. Hence we may assume that H is a subgraph of G_1, and that f is also defined on C arbitrarily. By (2), c_0 can be extended to an $(A, L|_{V(G_1)}, f|_{E(G_1)})$-coloring c_1. By (2) again, $c_1|_{V(C)}$ can be extended to an $(A, L|_{V(G_2)}, f|_{E(G_2)})$-coloring c_2. It follows that an extension to an (A, L, f)-coloring of G is obtained by combining c_1 and c_2, contrary to (2). □

The following is the direct consequence of Theorem 3.5.

Corollary 3.6 *Every K_5-minor free graph is 5-group choosable.*

4 On Group Choosability Version of Woodall's Conjecture

In this section, we investigate the group choosability version of Woodall's conjecture, and prove that Conjecture 1.6 holds for some values of r and s.

Theorem 4.1 *Let G be a $K_{r,s}$-minor free graph with $r \le s$. For $r = 1, 2$, G is $(r + s - 1)$-group choosable.*

Proof For $r = 1$, since the maximum degree of G is at most $s - 1$, by Lemma 2.1, $\chi_{gl}(G) \le s$. Now let $r = 2$, $S_1 = \{v_1\} \subseteq V(G)$, $G_1 = \langle N_G[S_1]\rangle$ and $N_G(v_1) = \{v_2, v_3, \ldots, v_{t_1-1}, v_{t_1}\}$. Without loss of generality, assume v_1v is a directed edge from v_1 to v for each $v \in N_G(v_1)$. Suppose that A is a group with $|A| \ge s + 1$, $L : V(G) \to 2^A$ with $|L(v)| = s + 1$ for each $v \in V(G)$ and $f \in F(G, A)$. Now let $a_1 \in L(v_1)$, $L_1 : V(G_1 - S_1) \to 2^A$ with $L_1(v) = L(v) - \{-f(v_1v) + a_1\}$ for each $v \in V(G_1 - S_1)$. Since $G_1 - S_1$ is $K_{1,s}$-minor free, there is an $(A, L_1, f|_{G_1 - S_1})$-coloring $\bar{c}_1$ for $G_1 - S_1$. By assigning a_1 to v_1, we extend $\bar{c}_1$ to an $(A, L|_{G_1}, f|_{G_1})$-coloring c_1 for G_1. If $G = G_1$ we are done otherwise let j_1 be the greatest integer in $[1, t_1]$ such that $N_G(v_{j_1}) - V(G_1) = \{v_{t_1+1}, \ldots, v_{t_2}\} \ne \emptyset$. Again assume $v_{j_1}v$ is a directed edge for each $v \in N_G(v_{j_1}) - V(G_1)$. Suppose $S_2 = S_1 \cup \{v_{j_1}\}$, $G_2 = \langle N_G[S_2]\rangle$. Now let $L_2 : V(G_2 - S_2) \to 2^A$ be a list assignment of G_2 with $L_2(v) = L(v) - \{-f(v_{j_1}v) + c_1(v_{j_1})\}$ for each $v \in N_G(v_{j_1}) - V(G_1)$ and $L_2(v) = \{c_1(v)\}$, otherwise. Since $G_2 - S_2$ is $K_{1,s}$-minor free, there is an

$(A, L_2, f|_{G_2-S_2})$-coloring $\bar{c}_2$ for $G_2 - S_2$. By assigning a_1 and $c_1(v_{j_1})$ to v_1 and v_{j_1}, respectively, we extend $\bar{c}_2$ to an $(A, L|_{G_2}, f|_{G_2})$-coloring c_2 for G_2. If $G = G_2$ we are done otherwise let j_2 be the greatest integer in $[1, t_2]$ such that $N_G(v_{j_2}) - V(G_2) = \{v_{t_2+1}, \ldots, v_{t_3}\} \neq \emptyset$, $S_3 = S_2 \cup \{v_{j_2}\}$ and $G_3 = \langle N_G[S_3] \rangle$ and repeat the same procedure. It is clear that for some natural number d, there is an (A, L, f)-coloring c_d for $G_d = G$.

Theorem 4.2 (Hall [6]) *Let G be a graph without $K_{3,3}$ minors. One of the followings must hold.*

(i) *G is a planar graph,*
(ii) *$G \cong K_5$,*
(iii) *G is a 1-sum or 2-sum of two graphs G_1 and G_2, such that both G_1 and G_2 are proper minors of G.*

Theorem 4.3 *Let G be a $K_{3,3}$-minor free graph, and let H be a subgraph of G such that $H \cong K_2$. Let A be a group, $L : V(G) \to 2^A$ be a 5-list assignment, and $f \in F(G, A)$ be a map. Then any $(A, L|_{V(H)}, f|_{E(H)})$-coloring c_0 of H can be extended to an (A, L, f)-coloring c of G.*

Proof We argue by contradiction and assume that

$$G \text{ is a counterexample with } |V(G)| \text{ minimized.} \tag{3}$$

By (1) and by Theorem 2.2, the theorem holds if G is a K_5 or is a planar graph. By (3), $G \ncong K_5$, G is not planar, and $\kappa(G) \geq 2$. Thus by Theorem 4.2, G is a 2-sum of G_1 and G_2 such that G_1 and G_2 are proper minors of G. We may assume that H is a subgraph of G_1. By the definition of a 2-sum, $K = G_1 \cap G_2 \cong K_2$. We may assume that f is also defined on $E(K)$ arbitrarily.

By (3), c_0 can be extended to an $(A, L|_{V(G_1)}, f|_{E(G_1)})$-coloring c_1. By (3) again, $c_1|_{V(K)}$ can also be extended to an $(A, L|_{V(G_2)}, f|_{E(G_2)})$-coloring c_2. Hence an (A, L, f)-coloring c of G can be obtained by combining c_1 and c_2, contrary to (3). □

A number of other upper bounds for χ_{gl} within some of the $K_{r,s}$-minor free graphs are in fact consequences of the next theorem and Lemma 2.1.

Theorem 4.4 *Let G be a graph. Each of the following holds.*

(i) (Kawarabayashi and Toft [12]) *If $\delta(G) \geq 6$, then G has a $K_{3,4}$-minor.*
(ii) (Kawarabayashi and Toft [12]) *If $\delta(G) \geq 8$, then G has a $K_{4,4}$-minor.*
(iii) (Kawarabayashi [7]) *Let G be a graph such that $|V(G)| \geq 2k+2$ and $|E(G)| \geq 2k(|V(G)| - k - 1) + 1$, where $k \geq 2$. Then G has a $K_{4,k}$-minor.*

Part (i) and (ii) of the next corollary follows from Theorem 4.4 and Lemma 2.1.

Corollary 4.5 *Let G be a graph. Each of the following holds.*

(i) *If G is a $K_{3,4}$-minor free, then $\chi_{gl}(G) \leq 6$.*
(ii) *If G is a $K_{4,4}$-minor free, then $\chi_{gl}(G) \leq 8$.*
(iii) *If G is a $K_{4,k}$-minor free, then $\chi_{gl}(G) \leq 4k$.*

Proof It suffices to prove Corollary 4.5(iii). Let G be a counterexample with $|V(G)|$ minimized. Then for some group A with $|A| \geq 4k$, for a $4k$-list assignment $L : V(G) \to 2^A$ and an $f \in F(G, A)$, G does not have an (A, L, f)-coloring. If G has a vertex v with $d_G(v) < 4k$, then by the minimality of G, $G - v$ has an $(A, L|_{V(G-v)}, f|_{E(G-v)})$-coloring c'. Since $d_G(v) < 4k$ and since $|L(v)| \geq 4k$, c' can be extended to an (A, L, f)-coloring by coloring v with a color in $L(v) - c'(N_G(v))$. Thus we must have $\delta(G) \geq 4k$. It follows that $|E(G)| \geq 2k|V(G)|$ and $|V(G)| \geq 4k + 1$. By Theorem 4.4 (iii), G must have a $K_{4,k}$-minor, contrary to the assumption that G is $K_{4,k}$-minor free. □

5 $K_{3,3}$-Minor Free Graphs with Girth At least 5

In this section, we shall modify the proof techniques of Thomassen in [22] and [23] to prove that every planar graph with girth at least 5 is 3-group choosable, and extend this result to $K_{3,3}$-minor free graphs.

Theorem 5.1 *Let G be a plane digraph with outer face boundary $Out(G)$ and with girth at least 5, A a group with $|A| \geq 3$, and $f \in F(G, A)$. Let P with $V(P) = \{v_1, v_2, \ldots, v_q\}$, $1 \leq q \leq 6$, be a path or cycle such that $V(P) \subseteq V(Out(G))$, and $c_0 : V(P) \to A$ be an $(A, f|_{E(G[V(P)])})$-coloring. Let $L : V(G) \to 2^A$ be a list assignment of G such that*

- (i) $\forall v \in V(P)$, $L(v) = \{c_0(v)\}$;
- (ii) $\forall v \in V(G) - V(Out(G))$, $|L(v)| = 3$;
- (iii) $\forall w \in V(Out(G)) - V(P)$, $|L(v)| \geq 2$; *and*
- (iv) *any edge in $E(G[\{v \in V(G) : |L(v)| \leq 2\}])$ is joining two vertices in P.*

Then c_0 can be extended to an (A, L, f)-coloring c of G.

Proof We argue by contradiction and assume that

$$G \text{ is a counterexample with } |V(G)| \text{ minimized.} \tag{4}$$

Let c_0 be an $(A, L|_{V(P)}, f|_{E(P)})$-coloring. By (4), G is connected. Suppose that G has a cut vertex. Then G has two subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{u\}$. Assume that if $u \notin V(P)$ and that $P \subset G_1$. By (4), c_0 can be extended to $(A, L|_{V(G_1)}, f|_{E(G_1)})$-coloring c_1 of G_1 and $(A, L|_{V(G_2)}, f|_{E(G_2)})$-coloring c_2 of G_2. Thus an (A, L, f)-coloring c of G extending c_0 is obtained by combining c_1 and c_2, contrary to (4). Hence

$$\kappa(G) \geq 2. \tag{5}$$

By (5), $Out(G)$ is a cycle C. Suppose that C has a chord e. Then $C \cup e$ has two cycles C_1 and C_2 such that $C_1 \cap C_2$ is the subgraph induced by the edge $e = uv$. We may assume that $v_1 \in V(C_1)$. Since G is a plane graph and since P is a path, $V(P) \cap V(C_1)$ will induce a path P' of C_1 which may contain the chord e. Let G_i be the plane subgraph of G with $Out(G_i) = C_i$. By (4), the restriction of c_0 on P' can be extended to an $(A, L|_{V(G_1)}, f|_{E(G_1)})$-coloring c_1, and by (4) again, the restriction of c_1 on $C[V(C_2) \cap (V(P) \cup \{u, v\})]$ can be extended to an $(A, L|_{V(G_2)}, f|_{E(G_2)})$-coloring c_2. Thus an (A, L, f)-coloring c of G can be obtained by combining c_1 and c_2, contrary to (4). Hence

$$C \text{ has no chords.} \tag{6}$$

In particular, C has no chords joining two vertices in $V(P)$. Hence we may choose the notation so that $C = v_1 \cdots v_q \cdots v_k v_1$ is a directed cycle with $(v_i, v_{i+1}) \in E(C)$ for each i (mod k). If $G[V(P)]$ is a cycle, then by (6), $P = C$, and so $q = k$. Let $G' = G - v_q$. For each $w \in N_G(v_q)$, we assume that $(v_q, w) \in E(G)$. Then update $L(w)$ for G' by deleting $c_0(v_q) - f(v_q w)$ from the original list $L(w)$. Since the girth of G is at least 5, the neighbors of v_q will be an independent set in $Out(G')$, and so G' also satisfies the hypothesis of Theorem 5.1. By (4), $c_0|_{V(P)-v_q}$ can be extended to an $(A, L|_{V(G')}, f|_{E(G')})$-coloring c of G'. Define $c(v_q) = c_0(v_q)$. Then c is in deed an (A, L, f)-coloring of G extending c_0, contrary to (4). Thus we may assume that

$$P \text{ is a path and } k \geq q + 1. \tag{7}$$

If $k \leq q+2$, then we extend c_0 by coloring $v_{q+1}, \ldots, v_k$ with $c_0(v_{q+1}) \in L(v_{q+1}) - \{c_0(v_q) - f(v_q, v_{q+1})\}, \ldots, c_0(v_k) \in L(v_k) - \{c_0(v_{k-1}) - f(v_{k-1}, v_k), f(v_k v_1) + c_0(v_1)\}$, respectively. Let $G'' = G - \{v_{q+1}, \ldots, v_k\}$ and update the list of the vertices of G'' by, for each $w \in (V(G) - V(C)) \cap N_G(v_j)$, (assuming $(v_j, w) \in E(G)$), resetting $L(w)$ as $L(w) - \{c_0(v_j) - f(v_j w)\}$, where $q + 1 \leq j \leq k$. Since girth of G is at least 5, $(V(G) - V(C)) \cap N_G(v_j)$ is an independent set. Thus by (4), c_0 can be extended first to an $(A, L|_{V(G'')}.f|_{E(G'')})$-coloring c'. Hence an (A, L, f)-coloring c of G is obtained by combining c' and c_0 on the vertices $v_{q+1}, \ldots, v_k$, contrary to (4). Thus, by the assumption of Theorem 5.1 (i), (iii) and (iv),

$$k \geq q + 3, |L(v_j)| \geq 2, q + 1 \leq j \leq k, \text{ and } |L(v_{q+1})| = |L(v_k)| = 3. \tag{8}$$

Claim 1 *If $q \geq 3$, then for any i with $2 \leq i \leq q - 1$, $d(v_i) \geq 3$. Moreover, for any v_i with $q < i \leq k$, if $d_G(v_i) = 2$, then $|L(v_i)| = 2$.*

By contradiction, we assume that $d_G(v_i) = 2$ for some i with $2 \leq i \leq q - 1$. Let $e_i = (v_{i-1}, v_{i+1})$ be a new edge oriented from v_{i-1} to v_{i+1} and let $G' = G - v_i + e_i$. Define $g \in F(G', A)$ by $g(e) = f(e)$ if $e \in E(G - v_i)$ and $g(e_i) \in A - \{c_0(v_{i-1}) - c_0(v_{i+1})\}$. As $V(G') \subseteq V(G)$, by (4), G' has an $(A, L|_{V(G')}, g)$-coloring c' extending c_0 on $V(P) - v_i$. Obtain c from c' by setting $c(v_i) = c_0(v_i)$. Then c is an (A, L, f)-coloring extending c_0, contrary to (4).

If for some i with $q < i \leq k$, both $d_G(v_i) = 2$ and $|L(v_i)| = 3$, then let $G'' = G - v_i + v_{i-1}v_{i+1}$ such that the new edge is so oriented that $(v_{i-1}, v_{i+1}) \in E(G'')$, and let f' be obtained from $f|_{E(G-v_i)}$ by defining $f'(v_{i-1}v_{i+1})$ arbitrarily. By (4), c_0 can be extended to an $(A, L|_{V(G-v_i)}, f')$-coloring c of G''. Since $|L(v_i)| \geq 3$, choose $c(v_i) \in L(v_i) - \{c(v_{i-1}) + f(v_{i-1}, v_i), c(v_{i+1} + f(v_i, v_{i+1})\}$ to obtain an (A, L, f)-coloring of G, extending c_0.

Claim 2 *G does not have a cycle C' with $|V(C')| \leq 6$ such that both $V(Int(C')) - V(C') \neq \emptyset$ and $V(Ext(C')) - V(C') \neq \emptyset$.*

If such C' exists, then by (4), c_0 can be extended first to an $(A, L|_{V(Ext(C'))}, f|_{E(Ext(C'))})$-coloring c', and then $c'|_{V(C')}$ can be extended to an $(A, L|_{V(Int(C'))}, f|_{E(Int(C'))})$-coloring c''. Hence an (A, L, f)-coloring c of G extending c_0 can be obtained by combining both c' and c''.

Claim 3 *G has no path of the form $v_i u v_j$, where $u \in V(Int(C) - V(C))$ and $1 \leq i < j \leq k$.*

Suppose that G has a path $P' = v_i u v_j$, such that $u \in V(Int(C) - V(C))$ and $1 \leq i < j \leq k$. Let $C_1 = v_1 v_2 \cdots v_i u v_j \cdots v_k v_1$ and $C_2 = u v_i v_{i+1} \cdots v_j u$ denote the two cycles in the subgraph induced by $E(C) \cup E(P')$ such that $|V(C_1) \cap V(P)| \geq |V(C_2) \cap V(P)|$. Assume that among all such paths P', we choose one so that

$$|V(Int(C_2))| \text{ is minimized.} \tag{9}$$

By (9), u is not adjacent to any v_t, $i + 1 \leq t \leq j - 1$. Since girth of G is at least 5, $i + 1 < j - 1$. If $|V(C)| \leq 6$, then by Claim 2, $V(Int(C_2)) = V(C_2)$. By (6) and (9), every vertex in $\{v_{i+1}, \ldots, v_{j-1}\}$ must be of degree 2 in G. By Claim 1, $i + 1 \geq q$. By Theorem 5.1 (iv), at least one vertex v in $\{v_{i+1}, \ldots, v_{j-1}\}$ satisfies $|L(v)| \geq 3$, contrary to Claim 1. Hence $|V(C)| \geq 7$. Let $X = \{x \in \{v_{i+1}, v_{j-1}\} - V(P) : |L(x)| \leq 2\}$.

Suppose that $X \neq \emptyset$. Assume that for each $x \in X$ and for any $w \in N_G(x) - V(P)$, $(x, w) \in E(G)$. Extend c_0 by defining $c_0(x) \in L(x)$ for each $x \in X$, and for each $w \in N_G(x) - V(P)$, update $L(w)$ as $L(w) - \{c_0(x) - f(xw)\}$. If $v_{i+1} \in X$, then by Theorem 5.1 (iv), both $|L(v_i)| \geq 3$ and $|L(v_{i+2})| \geq 3$, whence we can update $L(v_i)$ as $L(v_i) - \{c_0(x) + f(v_i v_{i+1})\}$ and $L(v_{i+2})$ as $L(v_{i+2}) - \{c_0(x) - f(v_{i+1} v_{i+2}\}$. Similarly, if $v_{j-1} \neq v_{i+1}$, then we update $L(v_{j-2})$ as $L(v_{j-2}) - \{c_0(x) + f(v_{j-2} v_{j-1})\}$ and $L(v_j)$ as $L(v_j) - \{c_0(x) - f(v_{j-1} v_j)\}$.

Now let $G' = G - X$, and let L' denote the updated list assignment of $V(G')$. Since the girth of G is at least 5, G' satisfies Theorem 5.1 (iv). By (4), the restriction of c_0 to $V(P) \cap V(G')$ can be extended to an $(A, L|_{V(G')}, f|_{E(G')})$-coloring c. Together with $c_0(v_{i+1})$ and $c_0(v_{j-1})$, c is indeed an (A, L, f)-coloring of G, extending c_0, contrary to (4).

Hence $X = \emptyset$. By (4), c_0 can be extended to an $(A, L|_{V(Int(C_1)}, f|_{E(Int(C_1))})$-coloring c_1. The restriction of c_1 on $(V(P) \cap V(C_2)) \cup \{v_i, u, v_j\}$ can be extended to an $(A, L|_{V(Int(C_2)}, f|_{E(Int(C_2))})$-coloring c_2. Thus an (A, L, f)-coloring of G extending c_0 can be obtained by combining c_1 and c_2, contrary to (4).

With similar arguments, Claim 4 below can also be obtained.

Claim 4 *G has no path of the form $v_i u u' v_j$*, where $u, u' \in V(Int(C) - V(C))$ and $1 \leq i < j \leq k$.

Suppose that G has a path $P' = v_i u u' v_j$, such that $u, u' \in V(Int(C) - V(C))$ and $1 \leq i < j \leq k$. Let $C_1 = v_1 v_2 \cdots v_i u v_j \cdots v_k v_1$ and $C_2 = u v_i v_{i+1} \cdots v_j u$ denote the two cycles in the subgraph induced by $E(C) \cup E(P')$ such that $|V(C_1) \cap V(P)| \geq |V(C_2) \cap V(P)|$. Assume that among all such paths P', we choose one so that (9) holds.

By Claim 3 and (9), u or u' is not adjacent to any v_t, $i+1 \le t \le j-1$. Since girth of G is at least 5, $i+1 \le j-1$. By Claim 2 and (9), we may further assume that $|V(C_2)| \ge 7$ and so $i+1 < j-1$.

Let $X = \{x \in \{v_{i+1}, v_{j-1}\} - V(P) : |L(x)| \le 2\}$. If $X \ne \emptyset$, then extend c_0 to include $c_0(x) \in L(x)$ for each $x \in X$, update $L(w)$ as in the proof of Claim 3, for all $w \in N_G(v_{i+1}) \cup N_G(v_{j-1}) - V(P)$, to obtain an updated list assignment L' of $G' = G - X$. Extend c_0 by defining By (4), G' has an $(A, L', f|_{E(G')})$-coloring c extending c_0. By including $c_0(x)$ for $x \in X$, we obtain an (A, L, f)-coloring c of G extending c_0, contrary to (4).

By (8), $|L(v_{q+2})| \ge 2$. As in [23], to complete the proof, we consider the following cases.

Case 1 $|L(v_{q+2})| = 3$.

Assume that for any $w \in N_G(v_q) - V(P)$, $(v_q, w) \in E(G)$. By (6), $N_G(v_q) \cap V(C) = \{v_{q-1}, v_{q+1}\}$. For each $w \in N_G(v_q) - V(P)$, reset $L(w)$ as $L(w) - \{c_0(v_q) - f(v_q w)\}$, and denote the updated list assignment as L'. As girth of G is at least 5, these $N_G(v_q) - V(C)$ is an independent set. By Claim 3, no edge in G joins a $w \in N_G(v_q) - V(C)$ to a vertex in $V(C)$. Since $|L(v_{q+2})| = 3$, in G', v_{q+1} is not adjacent to any vertex v with $|L'(v)| \le 2$. Thus $G' = G - v_q$ satisfies the hypothesis of Theorem 5.1 with G' replacing G. By (4), c_0 can be extended to an $(A, L', f|_{E(G')})$-coloring c of G'. Extending c by coloring v_q with $c_0(v_q)$, we obtain an (A, L, f)-coloring extending c_0, contrary to (4).

Case 2 $k = q+3$.

By (8), $|L(v_k)| = 3$. Assume that for any $w \in N_G(v_k) - V(C)$, $(v_k, w) \in E(G)$, and for any $w' \in N_G(v_{q+2}) - V(C)$, $(v_{q+2}, w') \in E(G)$. By (6), $N_G(v_k) \cap V(C) = \{v_{q+2}, v_1\}$, and $N_G(v_{q+2}) \cap V(C) = \{v_{q+1}, v_k\}$. Extend c_0 by setting $c_0(v_{q+2}) \in L(v_{q+2})$ and $c_o(v_k) \in L(v_k)$. For each $w \in N_G(v_k) - V(C)$, reset $L(w)$ as $L(w) - \{c_0(v_k) - f(v_k w)\}$; and for each $w' \in N_G(v_{q+2}) - V(C)$, reset $L(w')$ as $L(w') - \{c_0(v_{q+2}) - f(v_{q+2} w')\}$ and $L(v_{q+1})$ as $L(v_{q+1}) - \{c_0(v_{q+2}) + f(v_{q+1} v_{q+2})\}$. Denote the updated list assignment as L'. Let $G' = G - \{v_{q+2}, v_k\}$. As girth of G is at least 5, and by Claims 3 and 4, $(N_G(v_{q+2}) \cup N_G(v_k)) - V(C - \{v_{q+1}, v_1\})$ is an independent set of G'. Thus G' satisfies the hypothesis of Theorem 5.1. By (4), the restriction of c_0 in G' can be extended to an $(A, L', f|_{E(G')})$-coloring c of G'. Together with $c_0(v_{q+2})$ and $c_0(v_k)$, we obtain an (A, L, f)-coloring extending c_0, contrary to (4).

Case 3 $k \ge q+4$, $|L(v_{q+2})| = 2$ and $|L(v_{q+4})| = 3$.

By Theorem 5.1 (iv), $|L(v_{q+3})| = 3$. Assume that for any $w \in N_G(v_{q+1}) - V(C)$, $(v_{q+1}, w) \in E(G)$, and for any $w' \in N_G(v_{q+2}) - V(C)$, $(v_{q+2}, w') \in E(G)$. By (6), $N_G(v_{q+1}) \cap V(C) = \{v_q, v_{q+2}\}$, and $N_G(v_{q+2}) \cap V(C) = \{v_{q+1}, v_{q+3}\}$. Extend c_0 to $V(P) \cup \{v_{q+1}, v_{q+2}\}$ such that $c_0(v_{q+1}) \in L(v_{q+1}) - \{c_0(v_q) - f(v_q v_{q+1})\}$ and $c_0(v_{q+2}) \in L(v_{q+2}) - \{c_0(v_{q+1}) - f(v_{q+1} v_{q+2})\}$. For each $w \in N_G(v_{q+1}) - V(C)$, reset $L(w)$ as $L(w) - \{c_0(v_{q+1}) - f(v_{q+1} w)\}$; and for each $w' \in N_G(v_{q+2}) - V(C - v_{q+3})$, reset $L(w')$ as $L(w') - \{c_0(v_{q+2}) - f(v_{q+2} w')\}$. Denote the updated list assignment as L' and let $G' = G - \{v_{q+1}, v_{q+2}\}$. As girth of G is at least 5, and by Claims 3 and 4, $(N_G(v_{q+1}) \cup N_G(v_{q+2})) - V(C - \{v_q, v_{q+3}\})$ is an independent set of G'. Since $|L(v_{q+4})| = 3$, v_{q+2} is not adjacent to any vertex v with $|L'(v)| \le 2$. Thus G' satisfies the hypothesis of Theorem 5.1. By (4), the restriction of c_0 in G' can be extended to an $(A, L', f|_{E(G')})$-coloring c of G'. Together with $c_0(v_{q+1})$ and $c_0(v_{q+2})$, we obtain an (A, L, f)-coloring extending c_0, contrary to (4).

Case 4 $|L(v_{q+2})| = |L(v_{q+4})| = 2$.

Let $L(v_{q+4}) = \{a_1, a_2\}$. By Theorem 5.1 (iv), $|L(v_{q+3})| = 3$, and so there exists an $a \in L(v_{q+3})$ such that $a \notin \{f(v_{q+3}v_{q+4}) + a_1, f(v_{q+3}v_{q+4}) + a_2\}$. By (6), c_0 can be extended to $c_0 : V(G[V(P) \cup \{v_{q+1}, v_{q+2}, v_{q+3}\}]) \to A$ such that

$$c_0(v_{q+1}) \in L(v_{q+1}), c_0(v_{q+2}) \in L(v_{q+2}), \text{and } c(v_{q+3}) = a. \tag{10}$$

Assume that for any $w \in N_G(v_{q+1}) - V(C)$, $(v_{q+1}, w) \in E(G)$, for any $w' \in N_G(v_{q+2}) - V(C)$, $(v_{q+2}, w') \in E(G)$, and for any $w'' \in N_G(v_{q+3}) - V(C)$, $(v_{q+3}, w'') \in E(G)$.

For each $w \in N_G(v_{q+1}) - V(C)$, reset $L(w)$ as $L(w) - \{c_0(v_{q+1}) - f(v_{q+1}w)\}$, for each $w' \in N_G(v_{q+2}) - V(C)$, reset $L(w')$ as $L(w') - \{c_0(v_{q+2}) - f(v_{q+2}w')\}$, and for each $w'' \in N_G(v_{q+3}) - V(C)$, reset $L(w'')$ as $L(w'') - \{c_0(v_{q+3}) - f(v_{q+3}w'')\}$. Denote the updated list assignment as L' and let $G' = G - \{v_{q+1}, v_{q+2}, v_{q+3}\}$. As girth of G is at least 5, and by Claims 3 and 4, $(N_G(v_{q+1}) \cup N_G(v_{q+2})) \cup N_G(v_{q+3}) - V(C - \{v_q, v_{q+4}\})$ is an independent set of G'. Since $L'(v_{q+4}) = L(v_{q+4})$, any edge joining two vertices in $\{v \in V(G') : |L'(v)| \le 2\}$ are edges in P, and so G' satisfies the hypothesis of Theorem 5.1. By (4), the restriction of c_0 in G' can be extended to an $(A, L', f|_{E(G')})$-coloring c of G'. Together with $c_0(v_{q+1})$, $c_0(v_{q+2})$, and $c_0(v_{q+3})$, we obtain an (A, L, f)-coloring extending c_0, contrary to (4). □

The following corollary is the direct consequence of Theorem 5.1.

Corollary 5.2 *Every planar graph with girth at least* 5 *is* 3*-group choosable.*

In [18], the author conjectured if $H \in \{K_{3,3}, K_5\}$, then every H-minor free graph with girth at least 5 is 3-group choosable. Applying Theorem 4.2 and Corollary 5.2, we show that the conjecture holds for $H = K_{3,3}$.

Theorem 5.3 *Let G be a $K_{3,3}$-minor free graph with girth at least* 5*, then* $\chi_{gl}(G) \le 3$.

Proof Suppose that A is an abelian group of order at least 3, G is a $K_{3,3}$-minor free graph with girth at least 5, $L : V(G) \to 2^A$ is a 3-list assignment of G and $f \in F(G, A)$. We argue by induction on $|V(G)|$ to prove the conclusion. By Theorem 5.1, Theorem 5.3 holds if G is planar. Hence by Theorem 4.2 and by the assumption that G has girth at least 5, we may assume that G is connected and has two subgraphs G_1 and G_2 such that

$$G = G_1 \cup G_2, |V(G_1) \cap V(G_2)| = i \le 2, G_2 \text{ is planar with } |G_2| \text{ minimized.} \tag{11}$$

If $V(G_1) \cap V(G_2) = \{v\}$, then by induction, G_1 has an $(A, L|_{G_1}, f|_{E(G_1)})$-coloring c_1. By Theorem 5.3 with $P = \{v\}$, $c_1|_{\{v\}}$ can be extended to an $(A, L|_{G_2}, f|_{E(G_2)})$-coloring c_2. Hence, an (A, L, f)-coloring for G can be obtained by combining c_1 and c_2.

Thus we assume that G is 2-connected and $V(G_1) \cap V(G_2) = \{u, v\}$. By induction, G_1 has an $(A, L|_{G_1}, f|_{E(G_1)})$-coloring c_1. If $|V(G_2)| \le 5$, as the girth of G is at least 5, every vertex in $V(G_2) - \{u, v\}$ must have degree 2, and so $c_1|_{\{u,v\}}$ can be extended to an $(A, L|_{G_2}, f|_{G_2})$-coloring c_2 of G_2. Hence an (A, L, f)-coloring of G can be obtained by combining c_1 and c_2. Therefore, we assume that $|V(G_2)| \geq 6$.

If $uv \notin E(G)$, let $G'' = G \cup Q$ be a graph obtained from G by joining u and v with a new path $Q = uxyzv$ on 4-edges with $x, y, z \notin V(G)$. Let $S \subseteq A$ be a subset with $|S| = 3$, $h \in F(G'', A)$ and $L'' : V(G'') \to 2^A$ be given by

$$h(e) = \begin{cases} 0 & \text{if } e \in E(Q) \\ f(e) & \text{if } e \in E(G) \end{cases} \text{ and } L''(v) = \begin{cases} S & \text{if } v \in \{x, y, z\} \\ L(e) & \text{if } v \in V(G) \end{cases}$$

By induction, $G_1 \cup Q$ has an $(A, L''|_{V(G_1 \cup Q)}, h|_{E(G_1 \cup Q)})$-coloring c_1. Since G_2 is minimal, by Theorem 4.2, $G_2 + uv$ is planar and so $G_2 \cup Q$ is planar. Now in $G_2 \cup Q$ there are at most four colored vertices. So we can apply Theorem 5.1 to $G_2 \cup Q$ and find an $(A, L'|_{G_2 \cup Q}, f'|_{G_2 \cup Q})$-coloring, c_2, for $G_2 \cup Q$. Combining c_1 and c_2, we find an (A, L, f)-coloring for G. The proof for the case when $uv \in E(G)$ is a similar argument, and will be omitted. □

Acknowledgments The research of the third author is partially carried out in the IPM-Isfahan Branch and in part supported by a grant from IPM (No. 91050416).

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