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ORIGINAL PAPER

Collapsible Graphs and Hamiltonicity of Line Graphs

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Abstract Thomassen conjectured that every 4-connected line graph is Hamiltonian. Chen and Lai (Combinatorics and Graph Theory, vol 95, World Scientific, Singapore, pp 53–69; Conjecture 8.6 of 1995) conjectured that every 3-edge connected and essentially 6-edge connected graph is collapsible. Denote $D_3(G)$ the set of vertices of degree 3 of graph G. For $e = uv \in E(G)$, define $d(e) = d(u) + d(v) - 2$ the *edge degree* of e, and $\xi(G) = \min\{d(e) : e \in E(G)\}$. Denote by $\lambda^m(G)$ the m-restricted edge-connectivity of G. In this paper, we prove that a 3-edge-connected graph with $\xi(G) \geq 7$, and $\lambda^3(G) \geq 7$ is collapsible; a 3-edge-connected simple graph with $\xi(G) \geq 7$, and $\lambda^3(G) \geq 6$ is collapsible; a 3-edge-connected graph with $\xi(G) \geq 6$, $\lambda^2(G) \geq 4$, and $\lambda^3(G) \geq 6$ with at most 24 vertices of degree 3 is collapsible; a 3-edge-connected simple graph with $\xi(G) \geq 6$, and $\lambda^3(G) \geq 5$ with at most 24 vertices of degree 3 is collapsible; a 3-edge-connected graph with $\xi(G) \geq 5$, and $\lambda^2(G) \geq 4$ with at most 9 vertices of degree 3 is collapsible. As a corollary, we show that a 4-connected line graph $L(G)$ with minimum degree at least 5 and $|D_3(G)| \leq 9$ is Hamiltonian.

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1 Introduction

Unless stated otherwise, we follow [1] for terminology and notations, and we consider finite connected graphs without loop. In particular, we use $\kappa(G)$ and $\lambda(G)$ to represent the *connectivity* and *edge-connectivity* of a graph G. A graph is *trivial* if it contains no edges. A vertex (edge) cut X of G is *essential* if $G - X$ has at least two non-trivial components. For an integer $k > 0$, a graph G is *essentially k-(edge)-connected* if G does not have an *essential (edge-)cut X* with $|X| < k$. In particular, the *essential edge-connectivity* of G is the size of a minimum essential edge-cut. For $u \in V(G)$, let $d_G(u)$ be the degree of u, or simply $d(u)$ if no confusion arises. For $e = uv \in E(G)$, define $d(e) = d(u) + d(v) - 2$ the *edge degree* of e, and $\xi(G) = \min\{d(e) : e \in E(G)\}$.

An edge set F is said to be an m-restricted edge-cut of a connected graph G if $G - F$ is disconnected and each component of $G - F$ contains at least m vertices. Let *m-restricted edge-connectivity* ($\lambda^m(G)$) be the minimum size of all m-restricted edge-cut. Clearly, a minimal essential edge-cut is 2-restricted edge cut, and a 2-restricted edge cut is an essential edge-cut. So the essential edge-connectivity equals the 2-restricted edge-connectivity for a graph G. Esfahanian [6] proved that if a connected graph G with $|V(G)| \geq 4$ is not a star $K_{1,n-1}$, then $\lambda^2(G)$ exists and $\lambda^2(G) \leq \xi(G)$. Thus, an essentially k-edge connected graph has edge-degree at least k.

Corresponding to the 3-restricted edge-cut, we define P_2-edge-cuts. An edge cut F of G is a P_2-edge-cut of G if at least two components of $G - F$ contain P_2, where P_2 denote a path with three vertices. Clearly, a minimal P_2-edge-cut of G is a 3-restricted edge-cut of G, and a 3-restricted edge-cut of G is a P_2-edge-cut of G. It is not difficult to see that a P_2-edge-cut of G implies a 3-restricted edge-cut. Thus, the size of a P_2-edge-cut of G is not less than the 3-restricted edge-connectivity of G.

Denote $D_i(G)$ the set of vertices of degree i and let $d_i(G) = |D_i(G)|$, respectively. If there is no confusion, we use D_i and d_i for $D_i(G)$ and $d_i(G)$, respectively. For a subgraph $A \subseteq G$, $v \in V(G)$, $N_G(v)$ denotes the set of the neighbors of v in G and $N_G(A)$ denotes the set $(\bigcup_{v \in V(A)} N_G(v)) \setminus V(A)$. If no confusion, we use an edge uv for a subgraph whose vertex set is $\{u, v\}$ and edge set $\{uv\}$. Denote $G[X]$ the subgraph induced by the vertex set X of $V(G)$.

The line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common. From the definition of a line graph, if $L(G)$ is not a complete graph, then a subset $X \subseteq V(L(G))$ is a vertex cut of $L(G)$ if and only if X is an essential edge cut of G. Thomassen in 1986 posed the following conjecture:

Conjecture 1.1 (Thomassen [16]) *Every 4-connected line graph is Hamiltonian.*

Theorem 1.2 (Zhan [18]) *Every 7-connected line graph is Hamiltonian.*

Very recently, an important progress towards Thomassen's Conjecture was submitted by Kaiser and Vrána [9] in which the following theorem is listed:

Theorem 1.3 ([9]) *5-connected line graph with minimum degree at least 6 is Hamiltonian.*

So we clearly have:

Corollary 1.4 *6-connected line graph is Hamiltonian.*

For the known results on Hamiltonicity of line graphs and claw-free graphs, the reader is suggested to refer to [7,8,10,12,14,19]. The next conjecture is posed by Chen and Lai [4]:

Conjecture 1.5 (Chen and Lai Conjecture 8.6 of [4]) *Every 3-edge-connected and essentially 6-edge connected graph G is collapsible.*

In this paper, we prove that a 3-edge-connected graph with $\xi(G) \geq 7$, and $\lambda^3(G) \geq 7$ is collapsible; a 3-edge-connected simple graph with $\xi(G) \geq 7$, and $\lambda^3(G) \geq 6$ is collapsible; a 3-edge-connected graph with $\xi(G) \geq 6$, $\lambda^2(G) \geq 4$, and $\lambda^3(G) \geq 6$ with at most 24 vertices of degree 3 is collapsible; a 3-edge-connected simple graph with $\xi(G) \geq 6$, and $\lambda^3(G) \geq 5$ with at most 24 vertices of degree 3 is collapsible. a 3-edge-connected graph with $\xi(G) \geq 5$, and $\lambda^2(G) \geq 4$ with at most 9 vertices of degree 3 is collapsible. As a corollary, we show that a 4-connected line graph $L(G)$ with minimum degree at least 5 and $|D_3(G)| \leq 9$ is Hamiltonian.

2 Reductions

Catlin [2] introduced collapsible graphs. For a graph G, let $O(G)$ denote the set of odd degree vertices of G. A graph G is *eulerian* if G is connected with $O(G) = \emptyset$, and G is *supereulerian* if G has a spanning eulerian subgraph. A graph G is *collapsible* if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 (\mod 2)$, G has a spanning connected subgraph H_R such that $O(H_R) = R$. Note that when $R = \emptyset$, a spanning connected subgraph H with $O(H) = \emptyset$ is a spanning eulerian subgraph of G. Thus every collapsible graph is supereulerian. Catlin [2] showed that any graph G has a unique subgraph H such that every component of H is a maximally collapsible subgraph of G and every non-trivial collapsible subgraph of G is contained in a component of H. For a subgraph H of G, the graph G/H is obtained from G by identifying the two ends of each edge in H and then deleting the resulting loops. The contraction G/H is called the *reduction* of G if H is the maximal collapsible subgraph of G. A graph G is *reduced* if it is the reduction of itself. Let $F(G)$ denote the minimum number of edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. The following summarizes some of the former results concerning collapsible graphs.

Theorem 2.1 *Let G be a connected graph. Each of the following holds.*

(i) (*Catlin* [2]) *If H is a collapsible subgraph of G, then G is collapsible if and only if G/H is collapsible; G is supereulerian if and only if G/H is supereulerian.*
(ii) (*Catlin, Theorem 5 of* [2]) *A graph G is reduced if and only if G contains no non-trivial collapsible subgraphs. As cycles of length less than 4 are collapsible, a reduced graph does not have a cycle of length less than 4.*

(iii) (*Catlin, Theorem 8 of* [2]) *If* G *is reduced and if* $|E(G)| \geq 3$, *then* $\delta(G) \leq 3$, *and* $2|V(G)| - |E(G)| \geq 4$.
(iv) (*Catlin* [2]) *If* G *is reduced and if* $|E(G)| \geq 3$, *then* $\delta(G) \leq 3$ *and* $F(G) = 2|V(G)| - |E(G)| - 2$.
(v) (*Catlin et al.* [3]) *Let* G *be a connected reduced graph. If* $F(G) \leq 2$, *then* $G \in \{K_1, K_2, K_{2,t}\}(t \geq 1)$.

Let G be a connected and essentially 3-edge-connected graph such that $L(G)$ is not a complete graph. The *core* of this graph G, denoted by G_0, is obtained by deleting all the vertices of degree 1 and contracting exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$.

Lemma 2.2 (Shao [15]) *Let* G *be an essentially 3-edge-connected graph* G.

(i) G_0 *is uniquely defined, and* $\lambda(G_0) \geq 3$.
(ii) *If* G_0 *is supereulerian, then* $L(G)$ *is Hamiltonian.*

3 The Lower Bound of the Number of Edges in a Graph Dependent on Edge Degree

In the following lemma, the graph considered may have loops. Note that a loop is an edge with two same endpoints. For a graph G and $u \in V(G)$, denote $E_G(u)$ the set of edges incident with u in G. When the graph G is understood from the context, we write E_u for $E_G(u)$ simply. When a graph G is understood from the context, we use δ and n for $\delta(G)$ and $|V(G)|$, respectively.

Lemma 3.1 *Let* G *be a graph with minimum degree* $\delta \geq 3$, $\xi(G) \geq 2\delta + k - 2$ *and* $k \geq 1$. *Then* $|E(G)| \geq 2n + \frac{\delta^2 + (k-4)\delta - 2k}{\delta + k} d_\delta$.

Proof Let $N(G) = N_G(D_\delta)$, $T(G) = V(G) \setminus (N \cup D_\delta)$ (or simply, we use N and T for $N(G)$ and $T(G)$). Note that G is a graph with $\xi(G) \geq 2\delta - 1$, then D_δ is an independent set of G and the degree of the vertices in N is at least $\delta + k$, the vertices in T is at least $\delta + 1$. We prove this claim by induction on $|T|$.

We first let $|T| = \emptyset$. The degree of the vertex in N is at least $\delta + k$. If $|N| > \frac{\delta}{\delta + k} d_\delta$, we have

$$\begin{aligned} |E(G)| = \frac{\sum i d_i}{2} &\geq \frac{\delta d_\delta}{2} + \frac{\delta + k}{2}|N| = \frac{\delta + k}{2} n - \frac{k}{2} d_\delta \\ &= 2n + \frac{\delta + k - 4}{2} n - \frac{k}{2} d_\delta \\ &= 2n + \frac{\delta + k - 4}{2}(d_\delta + |N|) - \frac{k}{2} d_\delta \\ &= 2n + \frac{\delta - 4}{2} d_\delta + \frac{\delta + k - 4}{2}|N| \\ &\geq 2n + \frac{\delta - 4}{2} d_\delta + \frac{\delta + k - 4}{2} \frac{\delta}{\delta + k} d_\delta \end{aligned}$$

$$
\begin{aligned}
&= 2n + \frac{(\delta-4)(\delta+k)+\delta(\delta+k-4)}{2(\delta+k)} d_\delta \\
&= 2n + \frac{\delta^2+(k-4)\delta-2k}{\delta+k} d_\delta. \qquad (1)
\end{aligned}
$$

If $|N| \leq \frac{\delta}{\delta+k} d_\delta$, we have

$$
\begin{aligned}
|E(G)| &\geq \delta d_\delta = 2n + \delta d_\delta - 2n \\
&= 2n + \delta d_\delta - 2(\delta + |N|) = 2n + \delta d_\delta - 2|N| \\
&= 2n + (\delta+2) d_\delta - \frac{2\delta}{\delta+k} d_\delta \\
&= 2n + \frac{\delta^2+(k-4)\delta-2k}{\delta+k} d_\delta. \qquad (2)
\end{aligned}
$$

Now, we assume $|T| = 1$ and $T = \{u\}$. Clearly, $d(u) \geq \delta+1 \geq 4$. We first suppose $d(u) = 2s$ for some $s \geq 2$. Assume that there is l loops on u and let $2s = 2l + 2t$. Now, we delete the l loops of u and label the $2t$ neighbors corresponding the $2t$ edges naturally. Denote the $2t$ neighbors by $N'(u) = \{u_1, u_2, \ldots, u_{2t}\}$ (it is not a set if $G[\{u\} \cup N(u)]$ contains some multi-edges), that is, $N'(u)$ contains v k times if there is k edges between u and v. We construct a graph G' by (i) : deleting vertex u and edges $uu_i, i = 1, 2, \ldots, 2t$; (ii) : adding new edges $u_1u_2, u_3u_4, \ldots, u_{2t-1}u_{2t}$. It can be seen that $D_\delta(G) = D_\delta(G')$, $V(G') = V(G) \setminus \{u\}$, $E(G') = (E(G) \setminus E_u) \cup \{u_1u_2, u_3u_4, \ldots, u_{2t-1}u_{2t}\}$. Hence, $|V(G')| = |V(G)| - 1$, $|E(G')| = |E(G)| - \frac{d(u)}{2}$, $\xi(G') \geq 2\delta + k - 2$. Note that the set $T(G')$ is $\emptyset$, then we have $|E(G')| \geq 2(n-1) + \frac{\delta^2+(k-4)\delta-2k}{\delta+k} d_\delta$. Therefore,

$$
\begin{aligned}
|E(G)| &= |E(G')| + \frac{d(u)}{2} \\
&\geq 2|V(G')| + \frac{\delta^2+(k-4)\delta-2k}{\delta+k} d_\delta + \frac{d(u)}{2} \\
&= 2(|V(G)|-1) + \frac{\delta^2+(k-4)\delta-2k}{\delta+k} d_\delta + \frac{d(u)}{2} \\
&= 2|V(G)| + \frac{\delta^2+(k-4)\delta-2k}{\delta+k} d_\delta + \left(\frac{d(u)}{2} - 2\right) \\
&\geq 2|V(G)| + \frac{\delta^2+(k-4)\delta-2k}{\delta+k} d_\delta. \qquad (3)
\end{aligned}
$$

Next, we suppose $u \in T$ with l loops, $d(u) = 2s+1$ and $2s+1 = 2l+2t+1$ for some $s \geq 2$ and $N(u) = \{u_1, u_2, \ldots, u_{2t+1}\}$. Let $u' \in N$, we first construct G' by adding a new edge uu'. Now, u is in the $T(G')$ and $d_{G'}(u) \geq 6$ is even. Similarly, we construct a new graph G'' such that $T(G'')$ is empty. Note that $\frac{d_{G'}(u)}{2} \geq 3$, then

$$
\begin{aligned}
|E(G')| &= |E(G'')| + \frac{d_{G'}(u)}{2}\\
&\geq 2|V(G'')| + \frac{\delta^2+(k-4)\delta-2k}{\delta+k}d_\delta + \frac{d_{G'}(u)}{2}\\
&= 2(|V(G')|-1) + \frac{\delta^2+(k-4)\delta-2k}{\delta+k}d_\delta + \frac{d_{G'}(u)}{2}\\
&= 2|V(G)| + \frac{\delta^2+(k-4)\delta-2k}{\delta+k}d_\delta + \left(\frac{d_{G'}(u)}{2}-2\right)\\
&\geq 2|V(G')| + \frac{\delta^2+(k-4)\delta-2k}{\delta+k}d_\delta + 1. \qquad (4)
\end{aligned}
$$

Thus, $|E(G)| = |E(G')| - 1 \geq 2|V(G)| + \frac{\delta^2+(k-4)\delta-2k}{\delta+k}d_\delta$.

Assume that the claim holds for $1 \leq |T| < m$ and $|T| = m \geq 2$ in the following. Take a vertex $u \in T$ such that $d(u) = \min\{d(v)|v \in T\}$. Clearly, by the argument above, if $d(u)$ is even, then, the claim holds by constructing a new graph G' (similar to the case when $|T| = 1$, i.e. G' is constructed by deleting the vertex u, $l+t$ edges, and adding t new edges) with $|T| = m-1$ and then by induction. Assume $d(u)$ is odd. Similar to the case when $|T| = 1$. We first construct a new graph G' by adding a new edge as the case $|T| = 1$. It can be seen that $d_{G'}(u)$ is even and $d_{G'}(u) \geq 6$. Then we construct a new graph G'' similar to that of $|T| = 1$, by induction and the argument similar to that of (4), the claim holds. We complete the proof of the claim. □

In this paper, we only need the following three special cases of Lemma 3.1:

Corollary 3.2 *Let G is a graph with $\delta(G) \geq 3$, $\xi(G) \geq 7$. Then $|E(G)| \geq 2|V(G)|$.*

Corollary 3.3 *Let G be a graph with $\delta(G) \geq 3$, $\xi(G) \geq 6$. Then $|E(G)| \geq 2|V(G)| - \frac{d_3}{5}$.*

Corollary 3.4 *Let G be a graph with $\delta(G) \geq 3$, $\xi(G) \geq 5$. Then $|E(G)| \geq 2|V(G)| - \frac{d_3}{2}$.*

4 Collapsible graphs and Hamiltonicity of line graphs

Let G' be the reduction of G. Note that contraction do not decrease the edge connectivity of G, then G' is either a k-edge connected graph or a trivial graph if G is k-edge connected. Assume that G' is the reduction of a 3-edge-connected graph and non-trivial. It follows from Theorem 2.1 (v) and G' is 3-edge connected that $F(G') \geq 3$. Then by Theorem 2.1 (iv), we have $|E(G')| \leq 2|V(G')| - 5$.

A subgraph of G is called a 2-path or a P_2 subgraph of G if it is isomorphic to a $K_{1,2}$ or a 2-cycle. An edge cut X of G is a *2-path-edge-cut* of G if at least two components of $G - X$ contain 2-paths. Clearly, a P_2-edge-cut of a graph G is also a 2-path-edge-cut of G. By the definition of a line graph, for a graph G, if $L(G)$ is not a complete graph, then $L(G)$ is essentially k-connected if and only if G does not have a 2-path-edge-cut with size less than k. Since G_0 is a contraction of G, every P_2-edge-cut of G_0 is also a P_2-edge-cut of G.

Lemma 4.1 (Lai et al. Lemma 2.3 of [10]) *Let $k > 2$ be an integer, and let G be a connected and essentially 3-edge-connected graph. If $L(G)$ is essentially k-connected, then every 2-path-edge-cut of G_0 has size at least k.*

We call a vertex of G' *non-trivial* if the vertex is obtained by contracting a collapsible subgraph of G_0, and *trivial*, otherwise. Assume that $k \geq 3$ is an integer, and G is a 3-edge-connected and essentially k-edge-connected graph. Thus G_0 has no non-trivial vertex of degree i such that $3 \leq i < k$.

Lemma 4.2 *Let G be a reduced 3-edge-connected non-trivial graph. Then $d_3 \geq 10$.*

Proof Since $F(G') \geq 3$, we have

$$4|V(G)| - 10 \geq 2|E(G)| = \sum id_i \geq 3d_3 + 4(|V(G)| - d_3) = 4|V(G)| - d_3.$$

Thus, $d_3 \geq 10$. □

If V_1 and V_2 are two disjoint subsets of $V(G)$, then $[V_1, V_2]_G$ denotes the set of edges in G with one end in V_1 and the other end in V_2. When the graph G is understood from the context, we also omit the subscript G and write $[V_1, V_2]$ for $[V_1, V_2]_G$. If H_1 and H_2 are two vertex disjoint subgraphs of G, then we write $[H_1, H_2]$ for $[V(H_1), V(H_2)]$. Assume that u is a non-trivial vertex of G', and it is obtained by contracting a maximal connected collapsible subgraph H of G. We call H the *preimage* of u and let $PM(u) = H$. If a subgraph X of G' is obtained by contracting some maximal connected collapsible subgraph U of G. We call U the *preimage* of X and let $PM(X) = U$. In particular, we call X non-trivial if $X \ncong U$.

Theorem 4.3 *A 3-edge-connected graph with $\xi(G) \geq 7$, and $\lambda^3(G) \geq 7$ is collapsible.*

Proof Let G be a 3-edge-connected graph with $\xi(G) \geq 7$, and $\lambda^3(G) \geq 7$ and G' be the reduction of G. By way of contradiction, suppose that G' is non-trivial. Note that $F(G') \geq 3$ and thus $|E(G')| \leq 2|V(G')| - 5$, then we can obtain a contradiction by Corollary 3.2 if $\xi(G') \geq 7$. So we next show that the edge degree of G' is at least 7.

Suppose that there is an edge $e = uv$ with $d(e) < 7$ in G'. By Theorem 2.1 (ii) and Lemma 4.2, it is easy to see that $G' - \{u, v\}$ contains a component having at least three vertices. Note that the edge degree of uv is less than 7, then uv is clearly non-trivial. Thus, $[PM(uv), PM(G' - \{u, v\})]_G$ is a P_2-edge-cut of G, but its size is less than 7, a contradiction. We complete the proof. □

Note that a simple graph contains no 2-cycle, then each non-trivial collapsible connected subgraph of a graph having at least three vertices. If we consider the simple graph, the condition of Theorem 4.3 can be weaken slightly.

Theorem 4.4 *A 3-edge-connected simple graph with $\xi(G) \geq 7$, and $\lambda^3(G) \geq 6$ is collapsible.*

Proof Let G be a 3-edge-connected simple graph with $\xi(G) \geq 7$, and $\lambda^3(G) \geq 6$ and G' is the reduction of G. By way of contradiction, suppose that G' is non-trivial. Note that $F(G') \geq 3$ and thus $|E(G')| \leq 2|V(G')| - 5$, then we can obtain a contradiction by Corollary 3.2 if $\xi(G') \geq 7$. So we show that the edge degree of G' is at least 7. Not that G is 3-edge connected and so is G', then it is sufficient to show that the contraction does not product new vertices of degree less than 6.

By Theorem 2.1 (ii), $G' - \{u\}$ contains a component with at least three vertices, for any vertex $u \in V(G')$. Suppose that $u \in V(G')$ is a vertex obtained by contracting a maximal connected collapsible subgraph H of G. If u is non-trivial, then $|V(PM(u))| \geq 3$ sine G is simple graph. Then $[PM(u), PM(G' - \{u\})]$ is a P_2-edge-cut of G. If $d_{G'}(u) < 6$, then we get a P_2-edge-cut whose size is less than 6, a contradiction. That is, the edge degree of G' is at least 7. We complete the proof. □

By Lemma 2.2 and Theorem 4.3, we have

Corollary 4.5 (Zhan [18]) *A 7-connected line graph is Hamiltonian.*

By a very similar proof to that of Theorems 4.3 and 4.4, we obtain the following theorem.

Theorem 4.6 *A 3-edge-connected graph with $\xi(G) \geq 6$, $\lambda^2(G) \geq 4$, and $\lambda^3(G) \geq 6$ with at most 24 vertices of degree 3 is collapsible.*

Proof Let G be a 3-edge-connected graph with $\xi(G) \geq 6$, $\lambda^2(G) \geq 4$, and $\lambda^3(G) \geq 6$ and at most 24 vertices of degree 3, and G' be the reduction of G.

By an argument similar to that of Theorem 4.3, one can see that the edge degree of G' is at least 6. In fact, suppose that there is an edge $e = uv$ with $d(e) < 6$ in G'. By Theorem 2.1 (ii) and Lemma 4.2, it is easy to see that $G' - \{u, v\}$ contains a component having at least three vertices. Note that the edge degree of uv is less that 6, then uv is clearly non-trivial. Thus, $[PM(uv), PM(G' - \{u, v\})]_G$ is a P_2-edge-cut of G, but its size is less that 6, a contradiction.

Note that $\lambda^2(G) \geq 4$, then G' clearly contains no non-trivial vertex of degree 3, that is, $|D_3(G')| \leq |D_3(G)|$. By Corollary 3.3, we have $|E(G')| \geq 2|V(G')| - \frac{|D_3(G')|}{5}$. If $|D_3(G')| \leq |D_3(G)| \leq 24$, then $|E(G')| \geq 2|V(G')| - \frac{|D_3(G')|}{5} \geq 2|V(G)| - 4$ (note that the number of edges is an integer) which contradicts $|E(G')| \leq 2|V(G')| - 5$. Thus, the claim holds. □

Similar to Theorem 4.4, we have the following theorem:

Theorem 4.7 *A 3-edge-connected simple graph with $\xi(G) \geq 6$, and $\lambda^3(G) \geq 5$ with at most 24 vertices of degree 3 is collapsible.*

Proof The proof is similar to that of Theorem 4.4. Let G be a 3-edge-connected simple graph with $\xi(G) \geq 6$, and $\lambda^3(G) \geq 5$, at most 24 vertices of degree 3, and G' is the reduction of G. By way of contradiction, suppose that G' is non-trivial. Note that $F(G') \geq 3$ and thus $|E(G')| \leq 2|V(G')| - 5$, then we obtain a contradiction by by an argument similar to that of Theorem 4.7 if $\xi(G') \geq 6$. So we show that the edge degree of G' is at least 6. Not that G is 3-edge-connected and so is G', then it is sufficient to show that the contraction does not product a new vertex of degree less than 5.

By Theorem 2.1 (ii), $G' - \{u\}$ contains a component with at least three vertices, for any vertex $u \in V(G')$. Suppose that $u \in V(G')$ is a vertex obtained by contracting a maximal connected collapsible subgraph H of G. If u is non-trivial, then $|V(PM(u))| \geq 3$ since G is simple graph. Then $[PM(u), PM(G' - \{u\})]$ is a P_2-edge-cut of G. If $d_{G'}(u) < 6$, then we get a is a P_2-edge-cut less than 6, a contradiction. Thus, the edge degree of G' is at least 6. We complete the proof. □

By Theorem 4.7, we have the following corollary:

Corollary 4.8 (Yang et al. [17]) *For a 5-connected line graph $L(G)$ with minimum degree at least 6, if G is simple and $|D_3(G)| \leq 24$, them $L(G)$ is Hamiltonian.*

Similarly as above, we list the following results without proof.

Theorem 4.9 *A 3-edge-connected graph with $\xi(G) \geq 5$, and $\lambda^2(G) \geq 4$ with at most 9 vertices of degree 3 is collapsible.*

Theorem 4.10 *A 3-edge-connected simple graph with $\xi(G) \geq 5$, and $\lambda^3(G) \geq 4$ with at most 9 vertices of degree 3 is collapsible.*

Corollary 4.11 *A 4-connected line graph $L(G)$ with minimum degree at least 5 and $|D_3(G)| \leq 9$ is Hamiltonian.*

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References

1. Bondy, J.A.U.S.R. Murty: Graph theory with application. Macmillan, London (1976)
2. Catlin, P.A.: A reduction method to find spanning Eulerian subgraphs, J. Graph Theory **12**, 29–45 (1988)
3. Catlin, P.A., Han, Z., Lai, H.-J.: Graphs without spanning eulerian subgraphs. Discrete Math. **160**, 81–91 (1996)
4. Chen, Z.-H., Lai, H.-J.: Reduction techniques for supereulerian graphs and related topics: an update. In: Tung-Hsin, K. (ed.) Combinatorics and Graph Theory, vol. 95, pp. 53–69. World Scientific, Singapore (1995)
5. Chen, Z.-H., Lai, H.-Y., Lai, H.-J., Weng, G.Q.: Jacson's conjecture on wularian subgraphs. In: Alavi, Y., Lick, D.R., Liu, J. (eds.) Combinatorics, Graph Theory, Algorithms and Applications, pp. 53–58. The Proceeding of the Third China-USA Conference, Beijing, P.R. China (1993)
6. Esfahanian, A.H., Hakimi, S.L.: On computing a conditional edge-connectivity of a graph. Inf. Process. Lett. **27**, 195–199 (1988)
7. Hu, Z., Tian, F., Wei, B.: Hamilton connectivity of line graphs and claw-free graphs. J. Graph Theory **50**, 130–141 (2005)
8. Hu, Z., Tian, F., Wei, B.: Some results on paths and cycles in claw-free graphs. Preprint
9. Kaiser, T., Vrána, P.: Hamilton cycles in 5-connected line graphs. Submitted on 20 Sep 2010. arXiv:1009.3754v1
10. Lai, H.-J., Shao, Y., Wu, H., Zhou J.: Every 3-connected, essentially 11-connected line graph is Hamiltonian. J. Combin. Theory Ser. B **96**, 571–576 (2006)
11. Lai, H.-J., Shao, Y., Yu, G., Zhan, M.: Hamiltonian connectedness in 3-connected line graphs. Discrete Appl. Math. **157**(5), 982–990 (2009)
12. Li, H.: A note on Hamiltonian claw-free graphs. Rapport de recherche no. 1022, LRI, UMR 8623 CNRS-UPS, Bat. 490, Université de Paris sud, 91405 Orsay, France (1996)
13. Matthews, M.M., Sumner, D.P.: Hamiltonian results in $K_{1,3}$-free graphs. J. Graph Theory **8**, 139–146 (1984)
14. Ryjáček, Z.: On a closure concept in claw-free graphs. J. Combin. Theory Ser. B **70**, 217–224 (1997)
15. Shao, Y.: Claw-free graphs and line graphs. Ph.D dissertation, West Virginia University (2005)

16. Thomassen, C.: Reflections on graph theory. J. Graph Theory **10**, 309–324 (1986)
17. Yang, W., Lai, H., Li, H., Xiao, G.: Collapsible graphs and hamiltonicity of line graphs. Submitted
18. Zhan, S.: On Hamiltonian line graphs and connectivity. Discrete Math. **89**, 89–95 (1991)
19. Zhan, M.: Hamiltonicity of 6-connected line graphs. Discrete Appl. Math. **158**, 1971–1975 (2010)