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Supereulerian Graphs and the Petersen Graph

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Abstract A graph G is supereulerian if G has a spanning eulerian subgraph. Boesch et al. [*J. Graph Theory*, **1**, 79–84 (1977)] proposed the problem of characterizing supereulerian graphs. In this paper, we prove that any 3-edge-connected graph with at most 11 edge-cuts of size 3 is supereulerian if and only if it cannot be contractible to the Petersen graph. This extends a former result of Catlin and Lai [*J. Combin. Theory, Ser. B*, **66**, 123–139 (1996)].

Keywords Supereulerian graphs, petersen graph, edge-cut, reduction, contraction

MR(2010) Subject Classification O5C25, O5C38, O5C45

1 Introduction

We consider finite, undirected and loopless graphs. Undefined terms and notaions will follow Bondy and Murty [4]. In particular, $\kappa(G)$ and $\kappa'(G)$ denote the connectivity and the edge-connectivity of a graph G, respectively. A graph G is nontrivial if $|E(G)| > 0$, and we write $H \subseteq G$ to mean that H is a subgraph of G. Let $O(G)$ denote the set of all odd degree vertices of a graph G, and $g(G)$ (called the *girth* of G) be the length of a shortest cycle in G. A graph G is *even* if $O(G) = \emptyset$, and is *eulerian* if it is both even and connected. If G has a spanning eulerian subgraph, then G is *supereulerian*. The supereulerian graph problem, raised by Boesch et al. [3], seeks to characterize supereulerian graphs. Pulleyblank [16] showed that determining if a graph is supereulerian, even when restricted to planar graphs, is NP-complete. For more in the literature on supereulerian graphs, see Catlin's survey [6] and its update by Chen and Lai [11].

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For $X \subseteq E(G)$, the *contraction* G/X is obtained from G by contracting each edge of X and deleting the resulting loops. If $H \subseteq G$, we write G/H for $G/E(H)$. If H is connected, let v_H denote the vertex in G/H to which H is contracted, in this case, H is called the *preimage* of v_H.

A graph G is *collapsible* if for every even subset $R \subseteq V(G)$, G has a spanning connected subgraph H_R with $O(H_R) = R$. In particular, K_1 is both supereulerian and collapsible, and any collapsible graph G is supereulerian.

In [5], Catlin showed that every graph G has a unique collection of pairwise disjoint maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$. The graph obtained from G by contracting each H_i into a single vertex $(1 \le i \le c)$, is called the *reduction* of G. A graph is *reduced* if it is the reduction of some other graph.

Since every 4-edge-connected graph is collapsible [5], and so supereulerian [15], efforts to characterize supereulerian graphs have been within families of 3-edge-connected graphs. Chen et al. [10, 12, 13] investigated conditions under which a 3-edge-connected graph G is supereulerian if and only if G cannot be contracted to the Petersen graph. These settled the 3-edge-connected case of a conjecture by Benhocine et al. [1]. Caltin et al. considered 3-edge-connected graphs with limited number of 3-edge cuts. They proved the following:

Theorem 1.1 ([9, Theorem 3.12]) *Let G be a 3-edge-connected graph. If G has at most 10 edge-cuts of size 3, then exactly one of these holds*:

(i) *G is supereulerian*;

(ii) *The reduction of G is the Petersen graph.*

Theorem 1.2 ([9, Theorem 3.14]) *Let G be a 3-edge-connected graph. If G has at most 11 edge-cuts of size 3, then exactly one of these holds*:

(i) *G is supereulerian*;

(ii) *The reduction of G is the Petersen graph*;

(iii) *The reduction of G is a nonsupereulerian graph of order between 17 and 19, with girth at least 5, with exactly 11 vertices of degree 3 and 1 vertex of degree 5, and with the remaining vertices independent and of degree 4.*

It has been a question whether graphs stated in Theorem 1.2 (iii) exist or not. In this paper, we settle this problem by showing that no such graphs exist.

Theorem 1.3 *Let G be a 3-edge-connected graph. If G has at most 11 edge-cuts of size 3, then the following are equivalent*:

(i) *G is supereulerian*;

(ii) *The reduction of G is not the Petersen graph.*

The following notations will be used throughout this paper. For a graph G and integer $i \ge 1$, let $D_i(G) = \{v | d_G(v) = i,\ v \in V(G)\}$ and $d_i(G) = |D_i(G)|$. When G is understood, we write d_i for $d_i(G)$. Let $F(G)$ denote the minimum number of extra edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. Let $E_G(v) = \{uv | uv \in E(G), u \in V(G)\}$ and $N_G(v) = \{u | uv \in E(G), u \in V(G)\}$. When G is understood, we write $N(v)$ for $N_G(v)$ and $E(v)$ for $E_G(v)$.

Our proof depends on a new sufficient condition for a graph to be supereulerian. Let $\mathcal{F}$

denote the collection of all connected graphs satisfying each of the following:

(F1) $d_5(G) = 1$, $d_3(G) = 11$,

(F2) $3 \le \delta(G) \le \Delta(G) \le 5$,

(F3) $g(G) \ge 5$, and

(F4) no edge of G joins two vertices of even degree in G.

The following associate result plays an important role in our proof of Theorem 1.3.

Theorem 1.4 *Let $G \in \mathcal{F}$ be a graph. Then G is supereulerian.*

The paper will be organized as follows. In the next section, we present the preliminaries of Catlin's reduction method and the related theory that will be used in the proofs. We then prove Theorem 1.3 assuming the validity of Theorem 1.4. The last section will be devoted to the proof of Theorem 1.4.

2 Prerequisites

In this section, we present Catlin's reduction method to be used in our proofs.

Theorem 2.1 ([5, Theorems 3, 5 and 8]) *Let G be a connected graph.*

(i) *Let H be a collapsible subgraph of G. Then G is supereulerian if and only if G/H is supereulerian.*

(ii) *G is reduced if and only if G has no nontrivial collapsible subgraphs.*

(iii) *Let G' be the reduction of G. Then G is supereulerian if and only if G' is supereulerian, and G is collapsible if and only if $G' = K_1$.*

(iv) *If G is reduced, then every subgraph of G is also reduced.*

Theorem 2.2 ([8, Theorem 1.5]) *Let G be a reduced graph. If $F(G) \le 2$, then $G \in \{K_1, K_2, K_{2,t} (t \ge 1)\}$.*

Theorem 2.3 ([7, Theorem 7]) *If G is a connected reduced graph, then $F(G) = 2|V(G)| - |E(G)| - 2$.*

Corollary 2.4 *If G is a connected reduced graph, then $2F(G) = 3d_1 + 2d_2 + d_3 - \sum_{j \ge 5}(j - 4)d_j - 4$.*

Proof As $|V(G)| = \sum_{j \ge 1} d_j$ and $2|E(G)| = \sum_{j \ge 1} j d_j$, by Theorem 2.3, we have $2F(G) = 3d_1 + 2d_2 + d_3 - \sum_{j \ge 5}(j - 4)d_j - 4$. □

Theorem 2.5 ([9, Theorem 3.1]) *Let G be a 3-edge-connected graph with $F(G) = 3$. If G is nonsupereulerian and reduced, then each of the following holds:*

(i) *G has no edge joining two vertices of even degree;*

(ii) *G has girth at least 5;*

(iii) *G has no subgraph H with $\kappa'(H) \ge 2$ and $F(H) = 2$.*

For a graph G, an edge-cut $X \subset E(G)$ is called *essential edge-cut*, if each component of $G - X$ has at least one edge.

Lemma 2.6 *Let G be a 3-edge-connected nonsupereulerian reduced graph with $F(G) = 3$. Then every edge-cut of size 3 is not an essential edge-cut (i.e., the number of edge-cut of size 3 is equal to $d_3(G)$).*

Proof Let $X \subseteq E(G)$ be an edge-cut of size 3, and H_1 and H_2 the two components of $G - X$. By (iv) of Theorem 2.1, H_1 and H_2 are both reduced. Then by Theorem 2.3,

$$\begin{aligned} F(G) &= 2|V(G)| - |E(G)| - 2 \\ &= 2(|V(H_1)| + |V(H_2)|) - (|E(H_1)| + |E(H_2)| + |X|) - 2 \\ &= 2|V(H_1)| - |E(H_1)| - 2 + 2|V(H_2)| - |E(H_2)| - 3 \\ &= F(H_1) + F(H_2) - 1, \end{aligned}$$

and so $F(G)+1 = F(H_1)+F(H_2)$. Since $F(G) = 3$, $\min\{F(H_1), F(H_2)\} \le 2$ (say $F(H_1) \le 2$). By Theorem 2.2, $H_1 \in \{K_1, K_2, K_{2,t}(t \ge 1)\}$. If $H_1 = K_1$, then X is not an essential edge-cut. If $H_1 = K_2$ or $H_1 = K_{2,1}$, then vertex of degree 2 will appear, contrary to $\kappa'(G) \ge 3$. Hence $H_1 = K_{2,t}$ $(t \ge 2)$. Since $K_{2,t}$ $(t \ge 2)$ contains C_4, this is contrary to (ii) of Theorem 2.5. This completes the proof of the lemma. □

3 Proof of Theorem 1.3

Let G' be the reduction of G. By Theorem 2.1 (iii), it suffices to show that G' either is supereulerian or is the Petersen graph. We shall show that G is contractible to the Petersen graph with the following assumption:

$$G' \text{ is not supereulerian.} \tag{3.1}$$

Since G has at most 11 edge cut of size 3, G' has at most 11 edge cut of size 3. Thus $d_3(G') \le 11$. Since $\kappa'(G') \ge \kappa'(G) \ge 3$, $d_1(G') = d_2(G') = 0$. By Corollary 2.4, we have

$$2F(G') = 3d_1(G') + 2d_2(G') + d_3(G') - \sum_{j\ge5}(j-4)d_j(G') - 4 = d_3(G') - \sum_{j\ge5}(j-4)d_j(G') - 4. \tag{3.2}$$

By (3.2) and by $d_3(G') \le 11$, $F(G') \le 3$. If $F(G') \le 2$, then by Theorem 2.2, $G' \in \{K_1, K_2, K_{2,t}(t \ge 1)\}$. By (3.1), $G' \ne K_1$, and so $G' \in \{K_2, K_{2,t}(t \ge 1)\}$, contrary to the fact that $\kappa'(G') \ge 3$. Hence $F(G') = 3$.

In the rest of the proof, we will write d_j for $d_j(G')$, $j \ge 1$. By (3.2) and by $F(G') = 3$,

$$10 = d_3 - \sum_{j\ge5}(j-4)d_j. \tag{3.3}$$

Thus $11 \ge d_3 \ge 10$. If $d_3 = 10$, by Lemma 2.6, G' has exactly 10 edge-cuts of size 3. Hence by Theorem 1.1, G' is the Petersen graph. If $d_3 = 11$, then by (3.3), $d_5 = 1$, $d_j = 0$, $j \ge 6$. Thus $V(G') = D_3(G') \cup D_4(G') \cup D_5(G')$. Then by Theorem 2.5, $G' \in \mathcal{F}$. Thus by Theorem 1.4, G' is supereulerian, contrary to (3.1). This completes the proof of Theorem 1.3. □

4 Proof of Theorem 1.4

Let $G \in \mathcal{F}$ be a graph. Throughout this section, we always use $w \in V(G)$ to denote the unique vertex of degree 5. Let H be the subgraph induced by the vertices of distance at least 2 from w in G and $G_0 = G - E(H)$. Define $S = N(w) \cap D_4(G)$, $T = N(w) \cap D_3(G)$, $S_1 = \bigcup_{u\in S} N(u) - w$, $T_1 = (\bigcup_{v\in T} N(v)) \cap D_3(G)$ and $T_2 = (\bigcup_{v\in T} N(v)) \cap D_4(G)$. Let $W = V(H) - (S_1 \cup T_1 \cup T_2)$, and let

$$a = |D_3(G) \cap W| \quad \text{and} \quad b = |D_4(G) \cap W|.$$

Lemma 4.1 *With the notations above, each of the following holds.*

(i) $N(w) = S \cup T$.

(ii) $V(G_0) = V(G)$ *and* $E(G_0) = \bigcup_{u \in S \cup T} E(u)$.

(iii) $\forall u, v \in S \cup T$ *with* $u \neq v$, $N(u) \cap N(v) - w = \emptyset$.

(iv) G_0 *is acyclic.*

(v) $(S_1 \cup T_1 \cup T_2) \subseteq V(H)$ *and* $S_1 \subseteq D_3(G)$.

(vi) $|S_1| = 3|S|$ *and* $|T_1| + |T_2| = 2|T|$.

(vii) $d_3(G) = |S_1| + |T| + |T_1| + a$ *and* $d_4(G) = |S| + |T_2| + b$.

(viii) $|E(H[V(H) \cap D_3(G)])| = \frac{1}{2}((3a + 2(|S_1| + |T_1|)) - (4b + 3|T_2|))$, *and* $4b + 3|T_2| \leq 3a + 2(|S_1| + |T_1|)$.

Proof (i) follows from (F1) and (F2). The definition of H implies (ii). (iii) and (iv) follow from (F3) and (v) follows from (F4). Since $S \subseteq D_4(G)$ and $T \subseteq D_3(G)$, for every $u \in S$, $|N(u) \cap V(H)| = 3$ and for every $v \in T$, $|N(v) \cap V(H)| = 2$. These imply (vi).

By the definitions of S_1, T_1 and T_2 and by (F3), S_1, T_1 and T_2 are mutually disjoint. Then direct computation yields (vii). By the definition of H, $|V(H)| = a + b + |S_1| + |T_1| + |T_2|$. Let $H_1 = H[V(H) \cap D_3(G)]$. Then counting $\sum_{v \in V(H_1)} d_G(v)$ in two different ways, we obtain

$$3a + 3(|S_1| + |T_1|) = \sum_{v \in V(H_1)} d_G(v) = 2|E(H_1)| + |S_1| + |T_1| + 4b + 3|T_2|,$$

and so (viii) follows. □

By (F1), $11 = d_3(G) = 3|S| + |T| + |T_1| + a \geq 3|S| + |T| = 3|S| + 5 - |S|$, and so

$$|S| \leq 3, \text{ where } |S| = 3 \text{ only if } |T_1| + a = 0. \tag{4.1}$$

Throughout this section, let

$$\begin{aligned} &S = \{u_1, u_2, \ldots, u_{|S|}\}, \\ &N(u_i) \cap V(H) = \{w_{3i-2}, w_{3i-1}, w_{3i}\}, \text{ where } 1 \leq i \leq |S|, \\ &T = \{v_1, v_2, \ldots, v_{5-|S|}\}, \\ &N(v_i) \cap V(H) = \{w_{3|S|+2j-1}, w_{3|S|+2j}\}, \text{ where } 1 \leq j \leq 5 - |S| = |T|. \end{aligned} \tag{4.2}$$

As $|S| \leq 3$, $3|S| + 2(5 - |S|) \leq 13$. By (F3),

$$w_i \neq w_j \text{ if and only if } i \neq j \text{ for } 1 \leq i, j \leq 13. \tag{4.3}$$

Lemma 4.2 *G must be one of* 8 *possible graphs.*

Proof By (4.1), $|S| \leq 3$ and so we can analyze cases when $|S|$ takes different values.

Case 1 $|S| = 3$.

Then $|T| = 2$. By (4), $|T_1| + a = 0$. As $d_3(G) = 11$, $D_3(G) = T \cup S_1$ and $|T_2| = 2|T| - |T_1| = 4$. By Lemma 4.1 (viii), $0 \leq b \leq 1$. If $b = 1$, then $V(G) \cap W \cap D_4(G)$ has a vertex z. Since $|T_1| = 0$ and by (F4), $N(z) \subseteq S_1$. Since $d_G(z) = 4$, for some $i \in \{1, 2, 3\}$, $|N(z) \cap N(u_i)| \geq 2$, whence $G[(N(z) \cap N(u_i)) \cup \{z, u_1\}]$ induces a C_4, contrary to (F3). Therefore in Case 1, $b = 0$, and so there is only one possible graph, called G_2, as presented in Table 1 below.

Case 2 $|S| = 2$.

As $d_3(G) = 11$, $|T_1| = 11 - |T| - |S_1| - a = 2 - a$ and $|T_2| = 6 - (2 - a) = 4 + a$. Then by Lemma 4.1 (viii), $4b + 3(4 + a) \leq 3a + 2(6 + 2 - a)$, and so, $a + 2b \leq 2$. Therefore, there will be 4 different possible graphs in this case. Let G_1, G_3, G_4, G_5 denote such a graph when $a = 2$ and $b = 0$, or when $a = 0$ and $b = 1$, or when $a = 1$ and $b = 0$, or $a = 0$ and $b = 0$, respectively, as presented in Table 1 below.

Case 3 $|S| = 1$.

In this case, $|T_1| = 11 - |T| - |S_1| - a = 4 - a$ and $|T_2| = 8 - (4 - a) = 4 + a$. By Lemma 4.1 (viii), $4b + 3(4 + a) \leq 3a + 2(3 + 4 - a)$, and so $a + 2b \leq 1$. Let G_6, G_7 denote such a graph when $a = 1$ and $b = 0$, or when $a = 0$ and $b = 0$, respectively, as presented in Table 1 below.

Case 4 $|S| = 0$.

Then $S = S_1 = \emptyset$. Again by $d_3(G) = 11$, $|T_1| = 11 - |T| - |S_1| - a = 6 - a$ and $|T_2| = 10 - (6 - a) = 4 + a$. Then by Lemma 4.1 (viii), $4b + 3(4 + a) \leq 3a + 2(0 + 6 - a)$, and so $a = 0$ and $b = 0$. Thus there is one such graph, denoted by G_8, as presented in Table 1 below.

Summing up, we list the 8 possibilities of G in the following Table 1, with $n = |V(G)|$.

G	n	S	S_1	T	$\|T_1\|$	$T_1 \cup T_2$	a	b
G_1	20	$\{u_1, u_2\}$	$\{w_1, w_2, \ldots, w_6\}$	$\{v_1, v_2, v_3\}$	0	$\{w_7, w_8, \ldots, w_{12}\}$	2	0
G_2	19	$\{u_1, u_2, u_3\}$	$\{w_1, w_2, \ldots, w_9\}$	$\{v_1, v_2\}$	0	$\{w_{10}, w_{11}, w_{12}, w_{13}\}$	0	0
G_3	19	$\{u_1, u_2\}$	$\{w_1, w_2, \ldots, w_6\}$	$\{v_1, v_2, v_3\}$	2	$\{w_7, w_8, \ldots, w_{12}\}$	0	1
G_4	19	$\{u_1, u_2\}$	$\{w_1, w_2, \ldots, w_6\}$	$\{v_1, v_2, v_3\}$	1	$\{w_7, w_8, \ldots, w_{12}\}$	1	0
G_5	18	$\{u_1, u_2\}$	$\{w_1, w_2, \ldots, w_6\}$	$\{v_1, v_2, v_3\}$	2	$\{w_7, w_8, \ldots, w_{12}\}$	0	0
G_6	18	$\{u_1\}$	$\{w_1, w_2, w_3\}$	$\{v_1, v_2, v_3, v_4\}$	3	$\{w_4, w_5, \ldots, w_{11}\}$	1	0
G_7	17	$\{u_1\}$	$\{w_1, w_2, w_3\}$	$\{v_1, v_2, v_3, v_4\}$	4	$\{w_4, w_5, \ldots, w_{11}\}$	0	0
G_8	16	$\emptyset$	$\emptyset$	$\{v_1, v_2, v_3, v_4, v_5\}$	6	$\{w_1, w_2, \ldots, w_{10}\}$	0	0

Table 1 The graphs G_i $(1 \leq i \leq 8)$

This proves the lemma. □

Throughout the rest of this section, the graphs G_i $(1 \leq i \leq 8)$, will be these graphs defined in Table 1.

Lemma 4.3 *If* $G \in \{G_1, G_3, G_6, G_8\}$, *then* $|E(H[V(H) \cap D_3(G)])| = 0$ *and* $4b + 3|T_2| = 3a + 2(|S_1| + |T_1|)$.

Proof By Lemma 4.1 (viii), it suffices to show that $4b + 3|T_2| = 3a + 2(|S_1| + |T_1|)$.

If $G = G_1$, then $a = 2$, $b = 0$, $|T_1| = 0$ and $|S_1| = 6$. By Lemma 4.1 (vi), $|T_2| = 2|T| = 6$. Thus $4b + 3|T_2| = 18 = 3a + 2(|S_1| + |T_1|)$. If $G = G_3$, then $a = 0$, $b = 1$, $|T_1| = 2$, $|T_2| = 4$ and $|S_1| = 6$. Thus $4b + 3|T_2| = 16 = 3a + 2(|S_1| + |T_1|)$. If $G = G_6$, then $a = 1$, $b = 0$, $|T_1| = 3$, $|T_2| = 5$ and $|S_1| = 3$. Thus $4b + 3|T_2| = 15 = 3a + 2(|S_1| + |T_1|)$. If $G = G_8$, then $a = b = 0$, $|T_1| = 6$, $|T_2| = 4$ and $|S_1| = 0$. Thus $4b + 3|T_2| = 12 = 3a + 2(|S_1| + |T_1|)$. □

Lemma 4.4 $G \neq G_3$.

Proof Suppose $G = G_3$. Then as shown in Table 1, G_0 is isomorphic to G'_3 in Figure 1 (see Section 6). Thus $S = \{u_1, u_2\}$, $S_1 = \{w_1, w_2, w_3, w_4, w_5, w_6\}$, $T = \{v_1, v_2, v_3\}$, $a = 0$, $b = 1$,

$|T_1| = 2$ and $|T_2| = 4$. Denote the vertex of degree 4 in $V(G_3) \cap W$ by x. If the two vertices in T_1 have one common neighbor in T (say $v_1 \in N(w_7) \cap N(w_8)$, and so $T_1 = \{w_7, w_8\}$), then by (F4), $N(x) \subseteq S_1 \cup T_1$. Since $|N(x)| = 4$, either $T_1 \subseteq N(x)$, whence $G[\{v_1, x\} \cup T_1]$ contains a 4-cycle, contrary to (F3); or for some $i = 1, 2$, $|N(x) \cap N(u_i)| \geq 2$, whence $G[(N(x) \cap N(u_i)) \cup \{x, u_i\}]$ has a 4-cycle, contrary to (F3). Hence by symmetry, we may assume that $T_1 = \{w_7, w_9\}$. By (F3) and (F4), $w_7, w_9 \in N(x)$, $w_8 \in N(w_9)$ and $w_{10} \in N(w_7)$, and so $N_H(w_{11}) \subseteq S_1 = N(u_1) \cup N(u_2)$. Since $w_{11} \in D_3(H)$, then for some $i \in \{1, 2\}$, $|N_H(w_{11}) \cap N(u_i)| \geq 2$, and so $G\{w_{11}, u_i\} \cup (N_H(w_{11}) \cap N(u_i))$ contains a 4-cycle, contrary to (F3). □

Lemma 4.5 *$G \neq G_6$.*

Proof Suppose $G = G_6$. Then as shown in Table 1, G_0 is isomorphic to G'_6 in Figure 1. Thus we have $S = \{u_1\}$, $S_1 = \{w_1, w_2, w_3\}$, $T = \{v_1, v_2, v_3, v_4\}$, $a = 1$ and $b = 0$. Then $|T_1| = 3$ and $|T_2| = 5$. Denote the vertex of degree 3 in $V(G_6) \cap W$ by x. By Lemma 4.3, $|E(H[V(H) \cap D_3(G)])| = |E(G_6[S_1 \cup T_1])| = 0$, and so $N_H(w_1) \cup N_H(w_2) \cup N_H(w_3) \subseteq T_2$.

By (F3), $N_H(w_i) \cap N_H(w_j) = \emptyset$ for all $i \neq j, 1 \leq i \leq 3, 1 \leq j \leq 3$, and so $|N_H(w_1) \cup N_H(w_2) \cup N_H(w_3)| = 6$, contrary to the fact that $|T_2| = 5$. □

Lemma 4.6 *$G \neq G_7$.*

Proof Suppose $G = G_7$. Then as shown in Table 1, G_0 is isomorphic to G'_7 in Figure 1. Thus we have $S = \{u_1\}$, $S_1 = \{w_1, w_2, w_3\}$, $T = \{v_1, v_2, v_3, v_4\}$ and $a = b = 0$. Then $|T_1| = 4$ and $|T_2| = 4$. By (F4) and Lemma 4.1 (viii), $|E(G_7[S_1] \cup T_1)| = |E(H[V(H) \cap D_3(G)])| = \frac{1}{2}((3a + 2(|S_1| + |T_1|)) - (4b + 3|T_2|)) = 1$. By (F3), for any $i \neq j$ with $i, j \in \{1, 2, 3\}$, $N_H(w_i) \cap N_H(w_j) = \emptyset$. As in this case, $\{w_1, w_2, w_3\} \subseteq D_2(H)$, and so $|N_H(w_1) \cup N_H(w_2) \cup N_H(w_3)| = 6$. Since $|E(G_7[S_1] \cup T_1)| = 1$, we have $|(N_H(w_1) \cup N_H(w_2) \cup N_H(w_3)) \cap T_1| \leq 1$, and so $|(N_H(w_1) \cup N_H(w_2) \cup N_H(w_3)) \cap T_2| \geq 5$ by $N_H(w_1) \cup N_H(w_2) \cup N_H(w_3) \subseteq T_1 \cup T_2$, contrary to the fact that $|T_2| = 4$. □

Lemma 4.7 *If $G = G_1$, then G is supereulerian.*

Proof Suppose $G = G_1$. We use the notation in Table 1 for G_1. As $a = 2$, let $D_3(G) \cap W = \{x, y\}$. By Lemma 4.3, $E(G[S_1 \cup \{x, y\}]) = \emptyset$. Hence $N(x) \cup N(y) \subseteq T_2 = \{w_7, w_8, w_9, w_{10}, w_{11}, w_{12}\}$.

If $N(x) \cap N(y) \neq \emptyset$, then there is a vertex in T_2 (say w_7) which is adjacent to neither x nor y. Hence $N_H(w_7) \subseteq S_1$. Since vertex w_7 has degree 3 in H, C_4 must be induced. Therefore $N(x) \cap N(y) = \emptyset$.

Without loss of generality, by (F3) we may assume that $x \in N(w_7) \cap N(w_9) \cap N(w_{11})$, and $y \in N(w_8) \cap N(w_{10}) \cap N(w_{12})$. Thus $|N(w_7) \cap S_1| = |N(w_8) \cap S_1| = 2$. By (F3), without loss of generality, we may assume that $w_7 \in N(w_1) \cap N(w_4)$ and $w_8 \in N(w_2) \cap N(w_5)$. Hence $|N(w_3) \cap \{w_9, w_{10}, w_{11}, w_{12}\}| = |N(w_6) \cap \{w_9, w_{10}, w_{11}, w_{12}\}| = 2$. By symmetry and by (F3), we may also assume that $w_3 \in N(w_9) \cap N(w_{11})$ and $w_6 \in N(w_{10}) \cap N(w_{12})$.

By the assumptions above, we got a graph $G'_1 = G[E(G_0) \cup \{xw_7, xw_9, xw_{11}, yw_8, yw_{10}, yw_{12}, w_1w_7, w_4w_7, w_2w_8, w_5w_8, w_3w_9, w_3w_{11}, w_6w_{10}, w_6w_{12}\}]$ (see Figure 1). Then G'_1 is a spanning subgraph of G. Since $G'_1 - \{wv_1, wv_2, wv_3, w_3w_{11}, w_6w_{12}, xw_9, yw_{10}\}$ is a spanning eulerian subgraph of G'_1, G is supereulerian. □

Lemma 4.8 *If $G = G_2$, then G is supereulerian.*

Proof Suppose $G = G_2$. We use the notation in Table 1 for G_2. Then $T_1 = \emptyset$, and so by Lemma 4.1 (vi), $T_2 = \{w_{10}, w_{11}, w_{12}, w_{13}\}$. As $a = b = 0$, $3a + 2(|S_1| + |T_1|) - 4b + 3|T_2| = 18 - 12 = 6$, and so by Lemma 4.1 (viii) and by (F4), $|E(G[S_1])| = 3$. Let $H_1 = H - E(G[S_1])$.

By (F3), $g(G) \geq 5$, and so $N_{H_1}(w_{10}) \cap N_{H_1}(w_{11}) = \emptyset$ and $N_{H_1}(w_{12}) \cap N_{H_1}(w_{13}) = \emptyset$. Let $P = N_{H_1}(w_{10}) \cup N_{H_1}(w_{11})$ and $Q = N_{H_1}(w_{12}) \cup N_{H_1}(w_{13})$. Then by (F4),

$$P \cup Q \subseteq S_1. \tag{4.4}$$

As $\{w_{10}, w_{11}, w_{12}.w_{13}\} \subseteq D_3(H_1)$, $|N_{H_1}(w_{10})| = |N_{H_1}(w_{11})| = |N_{H_1}(w_{12})| = |N_{H_1}(w_{13})| = 3$. Thus $|P| = |Q| = 6$. If $|P \cap Q| \geq 5$, then $N_{H_1}(w_{10}) \subseteq (P \cap Q)$ or $N_{H_1}(w_{11}) \subseteq (P \cap Q)$. We suppose $N_{H_1}(w_{10}) \subseteq (P \cap Q)$. By $|N_{H_1}(w_{10})| = 3$, w_{10} has two neighbors in some member of $\{N_{H_1}(w_{12}), N_{H_1}(w_{13})\}$, say in $N_{H_1}(w_{12})$. Thus the two neighbors and $\{w_{10}, w_{12}\}$ together induce a 4-cycle in G, contrary to (F3). If $|P \cap Q| \leq 2$, then $|P \cup Q| \geq 10 > 9 = |S_1|$, contrary to (4.4). Hence $3 \leq |P \cap Q| \leq 4$.

Case 1 $|P \cap Q| = 4$.

Since $|P \cap Q| = 4$ and $|S| = 3$, for some $u_i \in S$, $|(P \cap Q) \cap N(u_i)| \geq 2$. Hence we may assume that $w_1, w_2 \in (P \cap Q) \cap N(u_1)$. By (F3), $N_{H_1}(w_1) \cap N_{H_1}(w_2) = \emptyset$. As $\{w_1, w_2\} \subseteq (P \cap Q) \cap D_2(H)$, we have $|N_{H_1}(w_1) \cap \{w_{10}, w_{11}\}| = |N_{H_1}(w_1) \cap \{w_{12}, w_{13}\}| = 1$ and $|N_{H_1}(w_2) \cap \{w_{10}, w_{11}\}| = |N_{H_1}(w_2) \cap \{w_{12}, w_{13}\}| = 1$. Hence by $N_{H_1}(w_1) \cap N_{H_1}(w_2) = \emptyset$, $\{w_{10}, w_{11}, w_{12}, w_{13}\} \subseteq N_{H_1}(w_1) \cup N_{H_1}(w_2)$. Without loss of generality, assume that $\{w_1w_{10}, w_1w_{12}, w_2w_{11}, w_2w_{13}\} \subseteq E(G_2)$. By symmetry and (F3), we may further assume $\{w_{10}w_4, w_{10}w_7, w_{11}w_5, w_{11}w_8\} \subseteq E(G_2)$. As $|P \cup Q| = |P| + |Q| - |P \cap Q| = 8 < 9 = |S_1|$ and by (4.4), $|S_1 - P \cup Q| = 1$. If $w_3 \in P \cup Q$, then by $\{w_{10}, w_{11}, w_{12}, w_{13}\} \subseteq N_{H_1}(w_1) \cup N_{H_1}(w_2)$, for some $i \in \{10, 11, 12, 13\}$, $|N(w_i) \cap N_{H_1}(u_1)| \geq 2$, say $|N(w_{10}) \cap N_{H_1}(u_1)| \geq 2$. Then $G[\{u_1, w_{10}\} \cup (N(w_{10}) \cap N_{H_1}(u_1))]$ contains a 4-cycle, contrary to (F3). Therefore, $w_3 \notin P \cup Q$.

It follows that either $w_6 \in N(w_{12})$ and $w_9 \in N(w_{13})$ or $w_6 \in N(w_{13})$ and $w_9 \in N(w_{12})$. By symmetry, we assume $w_6 \in N(w_{12})$ and $w_9 \in N(w_{13})$. Thus w_3 must be adjacent to one of vertices w_4, w_5 and w_6. The proofs for each of these subcases will be similar, and so we shall only prove the case when $w_3w_4 \in E(G)$ and omit the others.

Let $G_2' = G_0 + \{w_1w_{10}, w_1w_{12}, w_2w_{11}, w_2w_{13}, w_{10}w_4, w_{10}w_7, w_{11}w_5, w_{11}w_8, w_6w_{12}, w_9w_{13}, w_3w_4\}$ (see Figure 1). Then G_2' is a spanning subgraph of G. As $G_2' - \{wv_2, v_1w_{10}, w_1w_{12}, w_2w_{13}\}$ is a spanning eulerian subgraph of G_2', G is supereulrian.

Case 2 $|P \cap Q| = 3$.

By (4.4) and $|P \cup Q| = |P| + |Q| - |P \cap Q| = 9 = |S_1|$, $P \cup Q = S_1$, and so $\Delta(G_2[S_1]) = 1$. Let $P \cup Q = \{z_1, z_2, z_3\}$. Hence $\{N_{H_1}(z_1), N_{H_1}(z_2), N_{H_1}(z_3)\} \subset \{\{w_{10}, w_{12}\}, \{w_{10}, w_{13}\}, \{w_{11}, w_{12}\}, \{w_{11}, w_{13}\}\}$. By symmetry, we may assume $N_{H_1}(z_1) = \{w_{10}, w_{12}\}$, $N_{H_1}(z_2) = \{w_{10}, w_{13}\}$ and $N_{H_1}(z_3) = \{w_{11}, w_{12}\}$. Let $G_2'' = G_0 + E(H_1)$. Then G_2'' is a spanning subgraph of G. (An example with $z_1 = w_1, z_2 = w_4, z_3 = w_7$ is shown in Figure 1.) By $|E(G_2[S_1])| = 3$ and $\Delta(G_2[S_1]) = 1$, $O(G_2'') = \{w, v_1, v_2, z_1, z_2, z_3\}$. It follows that $G_2'' - \{wv_1, z_1w_{10}, z_2w_{10}, z_3w_{12}, v_2w_{12}\}$ is a spanning eulerian subgraph of G_2'', and so G is supereulrian. □

Lemma 4.9 *If $G = G_4$, then G is supereulerian.*

Proof Suppose $G = G_4$. We use the notation in Table 1 for G_4. As $a = 1$, let $D_3(G) \cap W = \{x\}$. Since $|T_1| = 1$, by Lemma 4.1 (vi), $|T_2| = 2|T| - |T_1| = 5$. Without loss of generality, let $T_1 = \{w_7\}$ and so $T_2 = \{w_8, w_9, w_{10}, w_{11}, w_{12}\}$. By Lemma 4.1 (viii), $|E(G_4[S_1 \cup \{w_7, x\}])| = 3a + 2(|S_1| + |T_1|) - 4b + 3|T_2| = 1$. Let $E(G_4[S_1 \cup \{w_7, x\}]) = \{e\}$.

Case 1 x is not incident with e.

Since $E(G_4[S_1 \cup \{w_7, x\}]) = \{e\}$, x is an isolated vertex in $G_4[S_1 \cup \{w_7, x\}]$ and so $N(x) \subseteq T_2$. If $N(x) \subseteq T_2 - \{w_8\}$, then by $|N(x)| = 3$, for some $i \in \{2, 3\}$, $|N(v_i) \cap N(x)| \geq 2$, and so $G[\{x, v_i\} \cup (N(v_i) \cap N(x))]$ has a 4-cycle, contrary to (F3). Hence $x \in N(w_8)$. Without loss of generality, we may assume $x \in N(w_9) \cap N(w_{11})$.

Thus by (F4), $N_H(w_{10}) \subseteq S_1 \cup \{w_7\}$. If $N_H(w_{10}) \subseteq S_1$, then as $|N_H(w_{10})| = 3$, for some $i \in \{1, 2\}$, $|N(u_i) \cap N_H(w_{10})| \geq 2$, and so $G[\{u_i, w_{10}\} \cup (N(u_i) \cap N_H(w_{10}))]$ has a 4-cycle, contrary to (F3). Hence $w_{10} \in N(w_7)$. Similarly, $w_{12} \in N(w_7)$. Since $|N_H(w_8)| = 3$, $w_8 \notin N(w_7)$ and $x \in N_H(w_8)$, we have $|N_H(w_8) \cap S_1| = 2$. Then by (F3), w_8 must be adjacent to one vertex in $\{w_1, w_2, w_3\}$ and to one vertex in $\{w_4, w_5, w_6\}$. Thus we may assume $w_8 \in N(w_1) \cap N(w_4)$. Since e cannot be incident with two vertices in $\{w_1, w_2, w_3\}$, with $w_8 \in N(w_1)$, one of $\{w_2, w_3\}$ must be adjacent to two vertices in $\{w_9, w_{10}, w_{11}, w_{12}\}$. Similarly, one of $\{w_5, w_6\}$ must be adjacent to two vertices in $\{w_9, w_{10}, w_{11}, w_{12}\}$. Without loss of generality, let $|N(w_2) \cap \{w_9, w_{10}, w_{11}, w_{12}\}| = |N(w_5) \cap \{w_9, w_{10}, w_{11}, w_{12}\}| = 2$. By (F3), $\{N_H(w_2), N_H(w_5)\} = \{\{w_9, w_{12}\}, \{w_{10}, w_{11}\}\}$ and $N_H(w_2) \neq N_H(w_5)$. By symmetry, we assume $\{w_2w_9, w_2w_{12}, w_5w_{10}, w_5w_{11}\} \subseteq E(G_4)$. Note that $N_H(w_3) \cap \{w_7, w_8\} = \emptyset$ and $N_H(w_6) \cap \{w_7, w_8\} = \emptyset$.

Under these assumptions, we shall show $e = w_3w_6$. If $N_H(w_3) \subseteq \{w_9, w_{10}, w_{11}, w_{12}\}$, then $N_H(w_3) \in \{\{w_9, w_{10}\}, \{w_9, w_{11}\}, \{w_9, w_{12}\}, \{w_{10}, w_{11}\}, \{w_{10}, w_{12}\}, \{w_{11}, w_{12}\}\}$ by $|N_H(w_3)| = 2$. In any case, G would have a 4-cycle (see Table 2), contrary to (F3).

$N_H(w_3)$ is in	G has a 4-cycle in
w_9, w_{10}	$G[\{w_3, w_9, w_{10}, v_2\}]$
w_9, w_{11}	$G[\{w_3, w_9, w_{11}, x\}]$
w_9, w_{12}	$G[\{w_3, w_9, w_{12}, w_2\}]$
w_{10}, w_{11}	$G[\{w_3, w_{10}, w_{11}, w_5\}]$
w_{10}, w_{12}	$G[\{w_3, w_{10}, w_{12}, w_7\}]$
w_{11}, w_{12}	$G[\{w_3, w_{11}, w_{12}, v_3\}]$

Table 2 Possible 4-cycles in G

Hence, by $N_H(w_3) \cap \{w_7, w_8\} = \emptyset$, $|N_H(w_3) \cap \{w_4, w_5, w_6\}| \geq 1$. If $|N_H(w_3) \cap \{w_4, w_5, w_6\}| \geq 2$, then $G[N(w_3) \cap N(u_2) \cup \{w_3\}]$ contains a 4-cycle, contrary to (F3). Hence $|N_H(w_3) \cap \{w_4, w_5, w_6\}| = 1$. By symmetry, $|N_H(w_6) \cap \{w_1, w_2, w_3\}| = 1$. As $\{e\} = E(G_1[S_1 \cup \{w_7, x\}])$, we have $e = w_3w_6$.

Let $G_4' = G_0 + \{xw_8, xw_9, xw_{11}, w_7w_{10}, w_7w_{12}, w_1w_8, w_4w_8, w_2w_9, w_2w_{12}, w_5w_{10}, w_5w_{11}, w_3w_6\}$. Thus we obtained a spanning subgraph G_4' of G_4 (see Figure 1). Since $G_4' - \{wv_1, wv_2, wv_3, w_7w_{10}, w_5w_{11}, w_2w_{12}, xw_9\}$ is a spanning eulerian subgraph of G_4', G_4 is supereulerian.

Case 2 x is incident with e.

If $e = xw_7$, then as $|E(G_1[S_1 \cup \{w_7, x\}])| = 1$, $N_H(w_1) \cup N_H(w_2) \cup N_H(w_3) \subseteq \{w_8, w_9, w_{10}, w_{11}, w_{12}\}$ and by $g(G_4) \geq 5$, $N_H(w_i) \cap N_H(w_j) = \emptyset$ $(i \neq j, i = 1,2,3, j = 1,2,3)$. Hence $|N_H(w_1) \cup N_H(w_2) \cup N_H(w_3)| = 6$, contrary to $N_H(w_1) \cup N_H(w_2) \cup N_H(w_3) \subseteq \{w_8, w_9, w_{10}, w_{11}, w_{12}\}$. Therefore $x \notin N(w_7)$ and so $|N(x) \cap S_1| = 1$. Thus by (F3), for every $v \in \{w_8, w_9, w_{10}, w_{11}, w_{12}\}$, $N_H(v) \cap \{x, w_7\} \neq \emptyset$. Therefore,

$$\{w_8, w_9, w_{10}, w_{11}, w_{12}\} \subseteq N_H(x) \cup N_H(w_7),$$

and so

$$|N_H(x) \cup N_H(w_7) \cap \{w_8, w_9, w_{10}, w_{11}, w_{12}\}| \geq 5.$$

But as $d_H(x) = 3$, $d_H(w_7) = 2$ and $|N_H(x) \cap S_1| = 1$,

$$|(N_H(w_7) \cup N_H(x)) \cap \{w_8, w_9, w_{10}, w_{11}, w_{12}\}| \leq 4,$$

contrary to $|N_H(x) \cup N_H(w_7) \cap \{w_8, w_9, w_{10}, w_{11}, w_{12}\}| \geq 5$. □

Lemma 4.10 *If $G = G_5$, then G is supereulerian.*

Proof Suppose $G = G_5$. We use the notation in Table 1 for G_5, and so $S = \{u_1, u_2\}$, $S_1 = \{w_1, w_2, w_3, w_4, w_5, w_6\}$, $T = \{v_1, v_2, v_3\}$ and $a = b = 0$. Since $|T_1| = 2$, by Lemma 4.1 (vi), $|T_2| = 2|T| - |T_1| = 4$. By Lemma 4.1 (viii),

$$|E(G_5[S_1 \cup T_1])| = 3a + 2(|S_1| + |T_1|) - 4b + 3|T_2| = 2.$$

Denote

$$E(G_5[S_1 \cup T_1]) = \{e_1, e_2\}.$$

As $|T_1| = 2$, we may assume that $T_1 = \{w_7, w'\}$ for some $w' \in \{w_8, w_9, \dots, w_{12}\}$.

Case 1 $w' \in N(v_1)$. Then $w' = w_8$.

Without loss of generality, we may assume that w_1, w_4 and $w_7 \in N(w_9)$, and that w_2, w_5 and $w_8 \in N(w_{10})$. Then each of w_{11} and w_{12} must be adjacent to one in $\{w_7, w_8\}$. By symmetry, assume $w_7w_{11}, w_8w_{12} \in E(G_5)$. As $w_9, w_{11} \in N(w_7)$ and as $w_{10}, w_{12} \in N(w_8)$, both of e_1 and e_2 can only be adjacent to vertices in S_1. By (F3), $g(G_5) \geq 5$, and so e_1 is not adjacent to e_2. Since $N_H(w_9) = \{w_1, w_4, w_7\}$ and $N_H(w_{10}) = \{w_2, w_5, w_8\}$, each of w_3 and w_6 is adjacent to at least one in $\{w_{11}, w_{12}\}$. Thus we may assume that $w_3w_{11}, w_6w_{12} \in E(G_5)$ (the proofs for the other cases $w_3w_{12}, w_6w_{11} \in E(G_5)$ or $w_3w_{11}, w_6w_{11} \in E(G_5)$ or $w_3w_{12}, w_6w_{12} \in E(G_5)$ are similar).

Let $G_5' = G_0 + \{w_1w_9,\ w_4w_9, w_7w_9,\ w_2w_{10},\ w_5w_{10}, w_8w_{10},\ w_7w_{11}, w_8w_{12},\ w_3w_{11}, w_6w_{12}\}$. Then G_5' is a spanning subgraph of G_5 (see Figure 1). Since $G_5' - \{wv_1, wv_2, wv_3, w_7w_{11}, w_8w_{12}\}$ is a spanning eulerian subgraph of G_5', G_5 is supeulerian.

Case 2 $w' \notin N(v_1)$. Thus we may assume that $w' = w_9$.

Then by (F3), w_8w_9, $w_{10}w_7 \in E(G_5)$. By symmetry, each of w_{11} and w_{12} must be adjacent to one in $\{w_7, w_9\}$, to one in $\{w_1, w_2, w_3\}$ and one in $\{w_4, w_5, w_6\}$. Without loss of generality, we assume vertex w_1, w_4 and $w_7 \in N(w_{11})$, and w_2, w_5 and $w_9 \in N(w_{12})$. Let $G_5'' = G_0 + \{w_8w_9, w_7w_{10},\ w_1w_{11}, w_4w_{11},\ w_7w_{11}, w_2w_{12},\ w_5w_{12}, w_9w_{12}\}$. Thus G_5'' is a spanning subgraph G_5 (see Figure 1).

As $N_H(w_7) = \{w_{10}, w_{11}\}$ and $N_H(w_9) = \{w_8, w_{12}\}$, $E(G_5[S_1] \cup T_1) = E(G_5[S_1])$. By (F3), $\Delta(G_5[S_1]) = 1$. Since $N_H(w_{11}) = \{w_1, w_4, w_7\}$ and $N_H(w_{12}) = \{w_2, w_5, w_9\}$, each of w_3

and w_6 is adjacent to w_8 or w_{10}. If $\{w_3w_{10}, w_6w_8\} \subset E(G_5)$ (or similarly, $\{w_3w_8, w_6w_{10}\} \subset E(G_5)$), then $G_5'' + \{w_3w_{10}, w_6w_8\} - \{wv_1, wv_2, wv_3, w_8w_9, w_7w_{10}\}$ is an eulerian subgraph of $G_5'' + \{w_3w_{10}, w_6w_8\}$ which spans G_5, and so G_5 is supereulerian.

If $\{w_3w_8, w_6w_8\} \subset E(G_5)$ (or similarly, $\{w_3w_{10}, w_6w_{10}\} \subset E(G_5)$), then $G_5''+\{w_3w_8, w_6w_8\} - \{wv_3, v_1w_7, v_2w_9\}$ is a spanning eulerian subgraph of $G_5'' + \{w_3w_8, w_6w_8\}$ that spans G_5, and so G_5 must be supereulerian. □

Lemma 4.11 *If $G = G_8$, then G is supereulerian.*

Proof Suppose $G = G_8$. We use the notation in Table 1 for G_8, and so $S = \emptyset$, $T = \{v_1, v_2, v_3, v_4, v_5\}$. By Lemma 4.3,

$$|E(H[V(H) \cap D_3(G)]) = \emptyset,$$

and so

$$H \text{ is a bipartite graph with a vertex bipartition } (T_1, T_2). \tag{4.5}$$

By (F3), for any i with $1 \le i \le 5$,

$$N_H(w_{2i-1}) \cap N_H(w_{2i}) = \emptyset. \tag{4.6}$$

Without loss of generality, assume that $w_8, w_{10} \in T_2$. Define

$$T' = \{v \in T : N_H(v) \subseteq T_2\}.$$

If $|T'| \ge 2$, as $|T_1| = 6$ and $|T_2| = 4$, we may assume $\{w_7, w_8, w_9, w_{10}\} = T_2$. By (F3) and (4.5), $N_H(w_7) \cup N_H(w_8) = \{w_1, w_2, w_3, w_4, w_5, w_6\} = N_H(w_9) \cup N_H(w_{10})$, $|N_H(w_7)| = |N_H(w_8)| = |N_H(w_9)| = |N_H(w_{10})| = 3$ and $N_H(w_7) \cap N_H(w_8) = N_H(w_9) \cap N_H(w_{10}) = \emptyset$. It follows that either $|N_H(w_7) \cap N_H(w_9)| \ge 2$ or $|N_H(w_7) \cap N_H(w_{10})| \ge 2$, forcing G_8 to has a 4-cycle, contrary to (F3). Hence $|T'| \le 1$.

Case 1 $|T'| = 1$.

We may assume that $T' = \{v_5\}$, and so by symmetry, assume that $T_2 = \{w_6, w_8, w_9, w_{10}\}$. By (4.6) and (F3), we have that $N_H(w_9) \cup N_H(w_{10}) = \{w_1, w_2, w_3, w_4, w_5, w_7\}$. By symmetry, let $\{w_1w_9, w_3w_9, w_5w_9\} \subset E(G_8)$, it follows $\{w_2w_{10}, w_4w_{10}, w_7w_{10}\} \subset E(G_8)$. By (F3), $w_6w_7, w_5w_8 \in E(G_8)$. Let $G_8' = G_0 + \{w_1w_9,\ w_3w_9,\ w_5w_9,\ w_2w_{10}, w_4w_{10},\ w_7w_{10}, w_6w_7,\ w_5w_8\}$. Thus G_8' is a spanning subgraph of G_8 (see Figure 1). Since $G_8' - \{wv_1, wv_2, wv_3, w_5w_9, w_9v_5, w_7v_4\}$ is eulerian, G_8 is supereulerian.

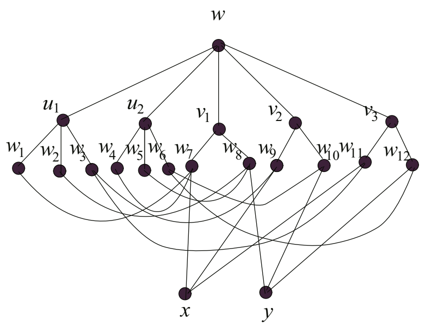
Case 2 $|T'| = 0$.

Then we may assume that $T_2 = \{w_4, w_6, w_8, w_{10}\}$. By (4.6) and by symmetry, we may assume that $\{w_1w_4, w_1w_6, w_2w_8, w_2w_{10}\} \subset E(G_8)$. As $w_8, w_{10} \in N(w_2)$, by (F3), $w_3 \notin N(w_8) \cap N(w_{10})$, and so $w_3w_6 \in E(G_8)$. Similarly, $w_4w_5, w_7w_{10}, w_8w_9 \in E(G_8)$. Let $G_8'' = G_0 + \{w_1w_4,\ w_1w_6, w_2w_8,\ w_2w_{10}, w_3w_6,\ w_4w_5, w_7w_{10},\ w_8w_9\}$. Thus G_8'' is a spanning subgraph of G_8 (see Figure 1). As $wv_3w_5w_4v_2w_3w_6w_1v_1w_2w_{10}w_7v_4w_8w_9v_5w$ is a Hamilton cycle of G_8'', G_8 is supereulerian. □

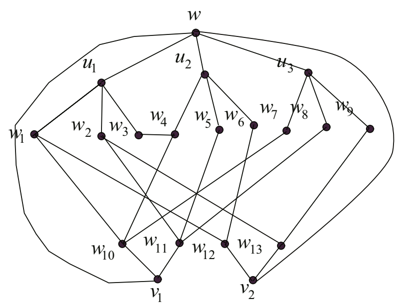
5 Remarks

Remark 5.1 Both Theorem 1.1 (Theorem 3.12 of [9]) and Theorem 1.3 in this paper raise the following a question: if G is a 3-edge-connected graph and if the number of 3-edge-cuts of

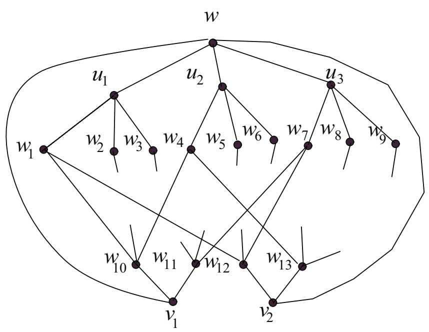
G is k, what is the largest value of k such that every 3-edge-connected graph G with at most k edge-cuts of size 3 is supereulerian if and only if G cannot be contracted to the Petersen graph? Theorem 1.1 says that $k \geq 10$ and in this paper we prove $k \geq 11$. However, since either of the two Blanusa snarks (see [2] or [14]) is 3-edge-connected and nonsupereulerian, has exactly 18 edge-cuts of size 3, and cannot be contracted to the Petersen graph, we have $k \leq 17$. We conclude this section by conjecturing that $k = 17$.



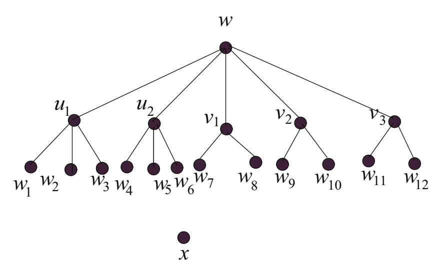
G_1'



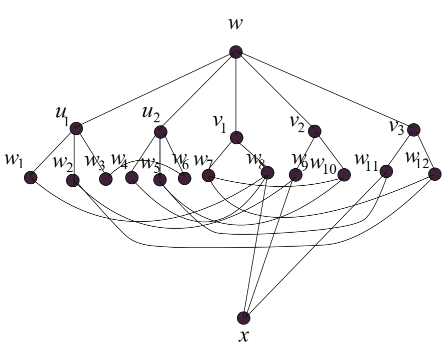
G_2'



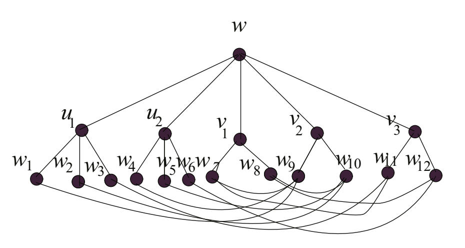
G_2''



G_3'



G_4'



G_5'

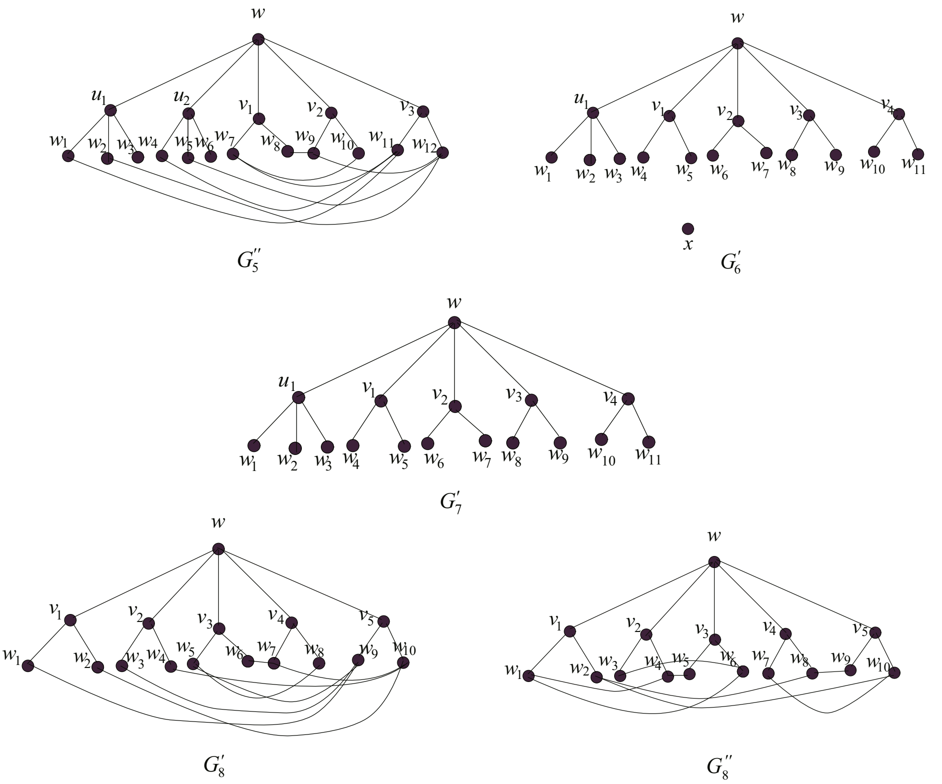


Figure 1 $G_1', G_2', \ldots, G_8', G_8''$

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