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Edge-disjoint spanning trees and eigenvalues ☆

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ABSTRACT

Let $\tau(G)$ and $\lambda_2(G)$ be the maximum number of edge-disjoint spanning trees and the second largest eigenvalue of a graph G, respectively. Motivated by a question of Seymour on the relationship between eigenvalues of a graph G and $\tau(G)$, Cioabă and Wong conjectured that for any integers $k \geqslant 2$, $d \geqslant 2k$ and a d-regular graph G, if $\lambda_2(G) < d - \frac{2k-1}{d+1}$, then $\tau(G) \geqslant k$. They proved this conjecture for $k = 2, 3$. Gu, Lai, Li and Yao generalized this conjecture to simple graph and conjectured that for any integer $k \geqslant 2$ and a graph G with minimum degree δ and maximum degree Δ, if $\lambda_2(G) < 2\delta - \Delta - \frac{2k-1}{\delta+1}$ then $\tau(G) \geqslant k$. In this paper, we prove that $\lambda_2(G) \leqslant \delta - \frac{2k-2/k}{\delta+1}$ implies $\tau(G) \geqslant k$ and show the two conjectures hold for sufficiently large n.

1. Introduction

We only consider finite and simple graph in this paper. Undefined notation will follow Bondy and Murty [1]. Let G be a graph. We use $\kappa'(G)$ to represent the edge connectivity of G and $\tau(G)$ to represent the maximum number of edge-disjoint spanning trees of G. See Palmer's survey [11] for a literature review on $\tau(G)$.

Let G be a simple graph with vertex set $\{v_1, \ldots, v_n\}$. The adjacent matrix of G is the $n \times n$ matrix $A(G) := (a_{ij})$, where $a_{ij} = 1$ if v_i and v_j are adjacent and otherwise $a_{ij} = 0$. As G is simple and undirected, $A(G)$ is symmetric $(0, 1)$-matrix. The eigenvalues of G are the eigenvalues of $A(G)$. We use $\lambda_i(G)$ to denote the ith largest eigenvalue of G. So $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$. Motivated by

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Kirchhoff's matrix tree theorem [8] and by a problem of Seymour (see Ref. [19] of [5]), Cioabă and Wong [5] considered the following problem.

Problem 1.1. (See [5].) Let G be a connected graph. Determine the relationship between $\tau(G)$ and eigenvalues of G.

Cioabă and Wong proposed the following conjecture.

Conjecture 1.2. *(Cioabă and Wong [5]) Let k and d be two integers with $d \geqslant 2k \geqslant 4$. If G is a d-regular graph with $\lambda_2(G) < d - \frac{2k-1}{d+1}$, then $\tau(G) \geqslant k$.*

A fundamental theorem of Nash-Williams and Tutte characterizes graphs with at least k edge-disjoint spanning trees. Let $(V_1, \ldots, V_t)$ be a sequence of disjoint vertex subsets of $V(G)$ and $e(V_1, \ldots, V_t)$ means the number of edges whose ends lie in different V_i's.

Theorem 1.3. *(Nash-Williams [10] and Tutte [13]) Let G be a connected graph and let $k > 0$ be an integer. Then $\tau(G) \geqslant k$ if and only if for any partition $(V_1, \ldots, V_t)$ of $V(G)$, $e(V_1, \ldots, V_t) \geqslant k(t-1)$.*

Using this theorem, Cioabă and Wong [5] proved Conjecture 1.2 for $k = 2, 3$ and also constructed some examples to show the bound is essentially best possible. For general k, using the following result of Cioabă [4], Cioabă and Wong [5] obtained Theorem 1.5.

Theorem 1.4. *(Cioabă [4]) Let k and d be two integers with $d \geqslant k \geqslant 2$. If G is a d-regular graph with $\lambda_2(G) < d - \frac{(k-1)n}{(d+1)(n-d-1)}$ then $\kappa'(G) \geqslant k$.*

Theorem 1.5. *(Cioabă and Wang [5]) Let k and d be two integers with $d \geqslant 2k \geqslant 4$. If G is a d-regular graph with $\lambda_2(G) < d - \frac{2(2k-1)}{d+1}$, then $\tau(G) \geqslant k$.*

Later, Gu, Lai, Li and Yao [9] generalize this investigation into general simple graph and propose the following conjecture.

Conjecture 1.6. *(Gu, Lai, Li and Yao [9]) Let k be an integer with $k \geqslant 2$ and G be a graph with minimum degree $\delta \geqslant 2k$ and maximum degree Δ. If $\lambda_2(G) < 2\delta - \Delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geqslant k$.*

In fact, Gu, Lai, Li and Yao [9] generalize Theorem 1.4 into general simple graph case and obtained the following result, and use this result to prove the their main theorem, stated as Theorem 1.8.

Theorem 1.7. *(Gu, Lai, Li and Yao [9]) Let $k \geqslant 2$ be an integer and G be a graph with minimum degree $\delta \geqslant 2k$ and maximum degree Δ. If $\lambda_2(G) < 2\delta - \Delta - \frac{2(k-1)}{\delta+1}$, then $\kappa'(G) \geqslant k$.*

Theorem 1.8. *(Gu, Lai, Li and Yao [9]) Let $k \geqslant 2$ be an integer, G be a graph with minimum degree δ and maximum degree Δ.*

(i) *If $\delta \geqslant 4$ and $\lambda_2(G) < 2\delta - \Delta - \frac{3}{\delta+1}$, then $\tau(G) \geqslant 2$.*
(ii) *If $\delta \geqslant 6$ and $\lambda_2(G) < 2\delta - \Delta - \frac{5}{\delta+1}$, then $\tau(G) \geqslant 3$.*
(iii) *For $k \geqslant 4$, if $\delta \geqslant 2k$ and $\lambda_2(G) < 2\delta - \Delta - \frac{3k-1}{\delta+1}$, then $\tau(G) \geqslant k$.*

Utilizing Theorem 1.3, Gusfield [6] proved a relationship between edge-connectivity of G and $\tau(G)$ for a graph G. (A generalization of this result can be found in [3].)

Theorem 1.9. *If $\kappa'(G) \geqslant 2k$, then $\tau(G) \geqslant k$.*

The purpose of this paper is to make further investigation of Conjecture 1.2 and Conjecture 1.6. The following results are obtained.

Theorem 1.10. *Let G be a graph with minimum degree $\delta \geqslant k \geqslant 2$ and of order n. If $\lambda_2(G) \leqslant \delta - \frac{(k-1)n}{(\delta+1)(n-\delta-1)}$, then $\kappa'(G) \geqslant k$.*

Theorem 1.11. *Let G be a graph with minimum degree $\delta \geqslant 2k \geqslant 4$ and of order n. If $\lambda_2(G) \leqslant \delta - \frac{2k-2/k}{\delta+1}$ or $\lambda_2(G) \leqslant \delta - \frac{2k-1}{\delta+1}$ and $n \geqslant (2k-1)(\delta+1)$, then $\tau(G) \geqslant k$.*

Theorem 1.10 is a generalization to simple graph of Theorem 1.4 which is better than Theorem 1.7. When $k = 2$, Theorem 1.11 is slightly stronger than the two conjectures. For general k, Theorem 1.11 suggests the two conjectures hold for sufficiently large n.

The main tool of this paper is eigenvalue interlacing. In the next section, some preliminaries about eigenvalue interlacing and quotient matrices, which will be used in this paper, are displayed. In Section 3 and Section 4, Theorem 1.10 and Theorem 1.11 will be proved, respectively.

2. Preliminaries

Given two non-increasing real sequences $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m$ with $n > m$, the second sequence is said to *interlace* the first one if $\lambda_i \geqslant \mu_i \geqslant \lambda_{n-m+i}$ for $i = 1, \dots, m$. If there exists i such that $\mu_j = \lambda_j$ for $1 \leqslant j \leqslant i$ and $\mu_j = \lambda_{n-m+j}$ for $i+1 \leqslant j \leqslant m$, then the interlacing is called *tight*. For convenience, we say the eigenvalues of a matrix B interlace the eigenvalues of a matrix A, it means the non-increasing eigenvalue sequence of B interlaces that of A.

Let $(V_1, \dots, V_t)$ be a partition of $V(G)$. For $1 \leqslant i, j \leqslant t$, let b_{ij} denote the average number of neighbors in V_j of the vertices in V_i. The *quotient matrix* of this partition is the $t \times t$ matrix B whose (i, j)th entry equals b_{ij}. The partition is called *equitable* if for each i, j, every vertex in V_i has the same number of neighbors in V_j. Haemers [7] showed the eigenvalues of the quotient matrix are in fact interlacing the eigenvalues of G.

Theorem 2.1. *(Haemers [7]) Let G be a graph. Then the eigenvalues of any quotient matrix B of G interlace the eigenvalues of G. Furthermore, if the interlacing is tight then the partition is equitable.*

The next theorem is known as the Cauchy Interlace Theorem. A proof of this theorem can be found on page 186 of [12].

Theorem 2.2. *If H is an induced subgraph of G, then the eigenvalues of H interlace the eigenvalues of G.*

For a graph G, we use $\bar{d}(G)$ to represent the average degree of G.

Theorem 2.3. *(Proposition 3.1.2 in [2]) Let G be a graph with largest eigenvalue λ_1 and average degree $\bar{d}$. Then $\lambda_1 \geqslant \bar{d}$. Furthermore, if the equality holds then G is regular.*

Note that spectrum of a disconnected graph is the union of the spectrum of its components. From the above two theorem, the following corollary can be obtained easily.

Corollary 2.4. *(Cioabă and Wong [4]) Let S and T be disjoint vertex subsets of G and $e(S, T) = 0$. Then $\lambda_2(G) \geqslant \lambda_2(G[S \cup T]) \geqslant \min\{\lambda_1(G[S]), \lambda_1(G[T])\} \geqslant \min\{\bar{d}(G[S]), \bar{d}(G[T])\}$.*

3. Eigenvalues and edge connectivity

In this section, we prove Theorem 1.10 which is useful to deduce Theorem 1.11.

Proof of Theorem 1.10. Suppose that G is not k-edge connected. Then there is a partition (A,B) of G such that $e(A,B)=r\leqslant k-1$. Let $|A|=n_1$, $|B|=n_2$. Then $n_1+n_2=n$. First, we show that

$$\text{at least one vertex in } A \text{ has no neighbor in } B. \tag{1}$$

Suppose, by contrary, that for all $v\in A$, $N(v)\cap B\neq\emptyset$. Then pick $u\in A$ and then $r=e(A,B)=|N(u)\cap B|+|e(A-\{u\},B)|\geqslant|N(u)\cap B|+|A-\{u\}|\geqslant|N(u)\cap B|+|N(u)\cap A|=d(u)\geqslant\delta\geqslant k$, a contradiction, and (1) holds. By (1), pick $v_0\in A$ so that $N(v_0)\cap B=\emptyset$. Then $n_1=|A|\geqslant|N(v_0)\cup\{v_0\}|\geqslant\delta+1$. Similarly, $n_2\geqslant\delta+1$. Together with $n_1+n_2=n$, we have $\delta+1\leqslant n_1\leqslant n-\delta-1$. Thus, $n_1n_2=n_1(n-n_1)\geqslant(\delta+1)(n-\delta-1)$. Let $d_1=\frac{1}{n_1}\sum_{v\in A}d(v)$ and $d_2=\frac{1}{n_2}\sum_{v\in B}d(v)$. Then $d_1,d_2\geqslant\delta$, and the quotient matrix of the partition (A,B) is

$$A_2=\begin{bmatrix} d_1-\frac{r}{n_1} & \frac{r}{n_1}\\ \frac{r}{n_2} & d_2-\frac{r}{n_2}\end{bmatrix}.$$

As the characteristic polynomial of A_2 is $\lambda^2-(d_1-\frac{r}{n_1}+d_2-\frac{r}{n_2})\lambda+(d_1-\frac{r}{n_1})(d_2-\frac{r}{n_2})-\frac{r^2}{n_1n_2}$, by quadratic formula,

$$\begin{aligned}
\lambda_2(A_2)&=\frac{1}{2}\Bigg(d_1-\frac{r}{n_1}+d_2-\frac{r}{n_2}\\
&\qquad-\sqrt{\left(d_1-\frac{r}{n_1}+d_2-\frac{r}{n_2}\right)^2-4\left(d_1-\frac{r}{n_1}\right)\left(d_2-\frac{r}{n_2}\right)+4\frac{r^2}{n_1n_2}}\Bigg)\\
&=\frac{1}{2}\left(d_1-\frac{r}{n_1}+d_2-\frac{r}{n_2}-\sqrt{\left(d_1-\frac{r}{n_1}-d_2+\frac{r}{n_2}\right)^2+4\frac{r^2}{n_1n_2}}\right)\\
&=\frac{1}{2}\left(d_1+d_2-\frac{r}{n_1}-\frac{r}{n_2}-\sqrt{(d_1-d_2)^2+\left(\frac{r}{n_1}+\frac{r}{n_2}\right)^2+2(d_1-d_2)\left(\frac{r}{n_2}-\frac{r}{n_1}\right)}\right)\\
&\geqslant\frac{1}{2}\left(d_1+d_2-\frac{r}{n_1}-\frac{r}{n_2}-\sqrt{(d_1-d_2)^2+\left(\frac{r}{n_1}+\frac{r}{n_2}\right)^2+2|d_1-d_2|\left(\frac{r}{n_2}+\frac{r}{n_1}\right)}\right)\\
&=\frac{1}{2}\left(d_1+d_2-\frac{r}{n_1}-\frac{r}{n_2}-\left(|d_1-d_2|+\frac{r}{n_1}+\frac{r}{n_2}\right)\right)\\
&=\min\{d_1,d_2\}-\frac{rn}{n_1n_2}\\
&\geqslant\delta-\frac{(k-1)n}{(\delta+1)(n-\delta-1)}.
\end{aligned}$$

On the other hand, by Theorem 2.1, $\lambda_2(A_2)\leqslant\lambda_2(G)\leqslant\delta-\frac{(k-1)n}{(\delta+1)(n-\delta-1)}$. Thus we must have $\lambda_2(A_2)=\lambda_2(G)=\delta-\frac{(k-1)n}{(\delta+1)(n-\delta-1)}$, and so both $r=k-1$ and $d_1=d_2=\delta$. Hence G must be δ-regular. Then, also by quadratic formula,

$$\begin{aligned}
\lambda_1(A_2)&=\frac{1}{2}\Bigg(\delta-\frac{r}{n_1}+\delta-\frac{r}{n_2}\\
&\qquad+\sqrt{\left(\delta-\frac{r}{n_1}+\delta-\frac{r}{n_2}\right)^2-4\left(d_1-\frac{r}{n_1}\right)\left(d_2-\frac{r}{n_2}\right)+4\frac{r^2}{n_1n_2}}\Bigg)\\
&=\frac{1}{2}\left(2\delta-\frac{r}{n_1}-\frac{r}{n_2}+\sqrt{\left(\delta-\frac{r}{n_1}-\delta+\frac{r}{n_2}\right)^2+4\frac{r^2}{n_1n_2}}\right)\\
&=\frac{1}{2}\left(2\delta-\frac{r}{n_1}-\frac{r}{n_2}+\sqrt{\left(\frac{r}{n_1}-\frac{r}{n_2}\right)^2+4\frac{r^2}{n_1n_2}}\right)
\end{aligned}$$

$$= \frac{1}{2}\left(2\delta - \frac{r}{n_1} - \frac{r}{n_2} + \left(\frac{r}{n_1} + \frac{r}{n_2}\right)\right)$$
$$= \delta = \lambda_1(G).$$

Hence, the interlacing is tight. By Theorem 2.1, the partition is equitable. So, every vertex in A has the same number of neighbors in B. This, together with (1), forces $r = e(A, B) = 0$. However $r = k - 1 > 0$, a contradiction, which completes the proof. $\square$

4. Eigenvalues and edge-disjoint trees

In this section, we give the proof of Theorem 1.11. In order to simplify the proof, we restate the theorem here as follows.

Theorem 4.1. *Let G be a graph with minimum degree $\delta \geqslant 2k \geqslant 4$ and of order n and $\lambda_2(G) \leqslant \delta - \frac{2k-\mu}{\delta+1}$. If $\mu = 2/k$ or $\mu = 1$ and $n \geqslant (2k-1)(\delta+1)$, then $\tau(G) \geqslant k$.*

Proof. Suppose, by contrary and by Theorem 1.3, there is a partition $(V_1, \dots, V_t)$ of $V(G)$ such that $e(V_1, \dots, V_t) \leqslant k(t-1) - 1$. For each $i = 1, 2, \dots,$ let $x_i = |\{j \mid d(V_j) = i\}|$. Then $2e(V_1, \dots, V_t) = \sum_{i=1}^{\infty} ix_i$ and $\sum_{i=1}^{\infty} x_i = t$. It follows that $\sum_{i=1}^{\infty} 2kx_i = 2kt$ and $\sum_{i=1}^{\infty} ix_i \leqslant 2k(t-1) - 2$. The difference of them implies $\sum_{i=1}^{\infty}(2k-i)x_i \geqslant 2k+2$. Note that $(2k-i)x_i \leqslant 0$ for $i \geqslant 2k$. So

$$\sum_{i=1}^{2k-1}(2k-i)x_i \geqslant 2k+2. \tag{2}$$

For each V_i $(1 \leqslant i \leqslant t)$, if $d(V_i) \leqslant 2k-1 < \delta$ then $|V_i| \geqslant \delta + 1$. For, otherwise, every vertex in V_i has at most $\delta - 1$ neighbors in V_i and thus has at least one neighbor outside V_i. Let $v \in V_i$. Then $d(V_i) \geqslant |N(v) - V_i| + |V_i - \{v\}| \geqslant |N(v) - V_i| + |N(v) \cap V_i| = d(v) \geqslant \delta \geqslant 2k$, a contradiction. So, among $V_1, \dots, V_t$ there are at least $\sum_{i=1}^{2k-1} x_i$ sets whose orders are at least $\delta + 1$. Hence $n = |V(G)| \geqslant (\delta+1)\sum_{i=1}^{2k-1} x_i$. Let $M := \sum_{i=1}^{2k-1} x_i$. Then

$$n \geqslant M(\delta+1). \tag{3}$$

Denote $\gamma \triangleq \lfloor 2k+1-\mu - \frac{\delta+1}{n}(2k-\mu)\rfloor$. Then it is easy to see that $\delta - \frac{2k-\mu}{\delta+1} \leqslant \delta - \frac{(\gamma-1)n}{(\delta+1)(n-\delta-1)}$. Thus, by the assumption of the theorem, $\lambda_2(G) \leqslant \delta - \frac{(\gamma-1)n}{(\delta+1)(n-\delta-1)}$. By assigning the value of γ to k in Theorem 1.10, $\kappa'(G) \geqslant \gamma$. If $\mu = 1$ and $n \geqslant (\delta+1)(2k-1)$, then

$$\kappa'(G) \geqslant 2k-1. \tag{4}$$

On the other hand, if $\mu = \frac{2}{k}$ we shall show (4) holds too.

In fact, by (3), $\kappa'(G) \geqslant \gamma \geqslant 2k+1-\lceil \mu + \frac{2k-\mu}{M}\rceil$. Denote

$$r = 2k+1-\left\lceil \frac{2}{k} + \frac{2k-2/k}{M}\right\rceil. \tag{5}$$

Then $\kappa'(G) \geqslant r$, which implies $x_i = 0$ for $i = 1, \dots, r-1$. Since G is a counterexample, we have $\tau(G) \leqslant k-1$. By Theorem 1.9, we must have $\kappa'(G) \leqslant 2k-1$. It follows from $\kappa'(G) \geqslant r$ that

$$r \leqslant 2k-1, \tag{6}$$

and

$$\begin{aligned} 2k+2 &\leqslant \sum_{i=1}^{2k-1}(2k-i)x_i = \sum_{i=r}^{2k-1}(2k-i)x_i \\ &\leqslant (2k-r)\sum_{i=r}^{2k-1} x_i \\ &= (2k-r)M. \end{aligned}$$

Substituting (5) into the above inequality, we have

$$\begin{aligned} 2k+2 &\leqslant \left(\left\lceil \frac{2}{k} + \frac{2k-2/k}{M} \right\rceil - 1\right) M \\ &< \left(\frac{2}{k} + \frac{2k-2/k}{M}\right) M = 2k - \frac{2}{k} + \frac{2}{k} M. \end{aligned}$$

It follows that $M > k+1$. Substitute it into (5),

$$r \geqslant 2k+1 - \left\lceil \frac{2}{k} + \frac{2k-2/k}{k+1} \right\rceil = 2k-1,$$

which implies (4) holds.

Combining (4) with (2), one can see that for each $i = 1, 2, \ldots, 2k-2$, $x_i = 0$ and $M = x_{2k-1} \geqslant 2k+2$. Without loss of generality, we may assume that V_i satisfies $d(V_i) = 2k-1$ for $i = 1, \ldots, 2k+2$. Considering V_1, as $d(V_1) = 2k-1$, there exists $i \in \{2, \ldots, 2k+2\}$ such that $e(V_1, V_i) = 0$, $i = 2$ say. Then by Corollary 2.4, $\lambda_2(G) \geqslant \min\{\lambda_1(G[V_1]), \lambda_1(G[V_2])\} \geqslant \min\{\bar{d}(G[V_1]), \bar{d}(G[V_2])\} = \min\{\frac{1}{|V_i|}\sum_{u \in V_i} d(u) - d(V_i)\colon i = 1, 2\} \geqslant \delta - \frac{2k-1}{\delta+1}$. By the assumption of the theorem, the equality must hold, which implies there exists $i \in \{1, 2\}$ such that $\lambda_1(G[V_i]) = \bar{d}(G[V_i]) = \frac{1}{|V_i|}\sum_{u \in V_i} d(u) - d(V_i)$. By Theorem 2.3, $G[V_i]$ is regular, which forces $2k-1 = d(V_i) = 0$, a contradiction. The proof is complete. $\square$

When $k \geqslant 3$, for regular graphs, there still exists a small gap between the conjecture and Theorem 4.1. However, the gap is small enough to make us give the following conjecture.

Conjecture 4.2. *Let G be a graph with minimum degree $\delta \geqslant 2k \geqslant 4$. If $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geqslant k$.*

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, New York, 2008.
[2] A.E. Brouwer, W.H. Haemers, Spectra of Graphs, Universitext, Springer, 2012, 250 pp. monograph, available online at http://homepages.cwi.nl/aeb/math/ipm.pdf.
[3] P.A. Catlin, H.-J. Lai, Y. Shao, Edge-connectivity and edge-disjoint spanning trees, Discrete Math. 309 (2009) 1033–1040.
[4] S.M. Cioabă, Eigenvalues and edge-connectivity of regular graphs, Linear Algebra Appl. 432 (2010) 458–470.
[5] S.M. Cioabă, W. Wong, Edge-disjoint spanning trees and eigenvalues of regular graphs, Linear Algebra Appl. 437 (2012) 630–647.
[6] D. Gusfield, Connectivity and edge-disjoint spanning trees, Inform. Process. Lett. 16 (1983) 87–89.
[7] W.H. Haemers, Interlacing eigenvalues and graphs, Linear Algebra Appl. 226 (228) (1995) 593–616.
[8] G. Kirchhoff, Über die Auflöung der Gleichungen, auf welche man bei der untersuchung der linearen verteilung galvanischer Ströme geführt wird, Ann. Phys. Chem. 72 (1847) 497–508.
[9] X. Gu, H. Lai, P. Li, S. Yao, Edge-disjoint spanning trees, edge connectivity and eigenvalues in graphs, submitted for publication.
[10] C.St.J.A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. London Math. Soc. 36 (1961) 445–450.
[11] E.M. Palmer, On the spanning tree packing number of a graph: a survey, Discrete Math. 230 (2001) 13–21.
[12] B.N. Parlett, The Symmetric Eigenvalue Problems, Prentice Hall, Englewood Cliffs, NJ, 1980.
[13] W.T. Tutte, On the problem of decomposing a graph into n factors, J. London Math. Soc. 36 (1961) 221–230.