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Spanning 3-connected index of graphs

Wei Xiong · Zhao Zhang · Hong-Jian Lai

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Abstract For an integer $s > 0$ and for $u, v \in V(G)$ with $u \neq v$, an $(s; u, v)$-path-system of G is a subgraph H of G consisting of s internally disjoint (u, v)-paths, and such an H is called a spanning $(s; u, v)$-path system if $V(H) = V(G)$. The spanning connectivity $\kappa^*(G)$ of graph G is the largest integer s such that for any integer k with $1 \leq k \leq s$ and for any $u, v \in V(G)$ with $u \neq v$, G has a spanning $(k; u, v)$-path-system. Let G be a simple connected graph that is not a path, a cycle or a $K_{1,3}$. The spanning k-connected index of G, written $s_k(G)$, is the smallest nonnegative integer m such that $L^m(G)$ is spanning k-connected. Let $l(G) = \max\{m :\ G$ has a divalent path of length m that is not both of length 2 and in a $K_3\}$, where a divalent path in G is a path whose interval vertices have degree two in G. In this paper, we prove that $s_3(G) \leq l(G) + 6$. The key proof to this result is that every connected 3-triangular graph is 2-collapsible.

Keywords Spanning k-connected index · 3-triangular graph · Line graph · 2-collapsible

1 Introduction

We refer to Bondy and Murty (2008) for terminologies and notations not defined here and consider finite connected graphs only. For a graph G and a vertex $v \in V(G)$, denote

W. Xiong · Z. Zhang (✉) · H.-J. Lai
College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, People's Republic of China
e-mail: hxhzz@163.com

H.-J. Lai
Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

$N_G(v) = \{u \in V(G) : u$ is adjacent to v in $G\}$ and $E_G(v) = \{e \in E(G) : e$ is incident with v in $G\}$. Following Bondy and Murty (2008), we use $c(G)$, $\delta(G)$, $\kappa(G)$, and $\kappa'(G)$ to represent the number of components, the minimum degree, the connectivity, and the edge connectivity of graph G, respectively. For subsets $X, Y \subseteq V(G)$, define $[X, Y]_G = \{xy \in E(G) : x \in X, y \in Y\}$. When H, K are subgraphs of G, we use $[H, K]_G$ for $[V(H), V(K)]_G$. The subscript G is often omitted when G is understood from the context.

The *line graph* of graph G, written as $L(G)$ or $L^1(G)$, has $E(G)$ as its vertex set, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have a common vertex. For an integer $m \geq 1$, we define the *m-th iterated line graph* $L^m(G) = L(L^{m-1}(G))$, where $L^0(G) = G$. Intuitively, when $m \to \infty$, $L^m(G)$ becomes more and more "locally dense". It has been shown that for a connected graph G such that G is not a path, cycle or a $K_{1,3}$, the connectivity and edge-connectivity of $L^m(G)$ grows very fast as the number m increases. See Knor and Niepel (2003), Shao (2010), Shao (2012), Zhang et al. (2012), among others. Therefore, for a graphical property P, it is of interest to study what is the smallest integer m such that $L^m(G)$ has property P. Such smallest integer m is called the *P-index of G*. The concept of hamiltonian index was first introduced by Chartrand and Wall (1973), who showed that the hamiltonian index exists as a finite number. Clark and Wormald (1983) generalized this idea and introduced hamiltonian-like indices.

A path with initial vertex u and terminal vertex v will be referred to as an (u, v)-path. For an integer $s > 0$ and for $u, v \in V(G)$ with $u \neq v$, an *$(s; u, v)$-path-system* of G is a subgraph H of G consisting of s internally disjoint (u, v)-paths, and such an H is called a *spanning $(s; u, v)$-path system* if $V(H) = V(G)$. A graph G is *spanning s-connected* if for any $u, v \in V(G)$ with $u \neq v$ and for any k with $1 \leq k \leq s$, G has a spanning $(k; u, v)$-path-system. The *spanning connectivity* $\kappa^*(G)$ of graph G is the largest integer s such that G is spanning s-connected. Thus $\kappa^*(G) \geq 1$ if and only if G is hamiltonian-connected.

The concept of $(s; u, v)$-path system has been used in the design and the implementation of parallel routing and efficient information transmission in larger-scale networking systems (see Akers and Krishnamurthy 1989, Hsu 1994, Lin et al. 2007 and references therein). By the definition, spanning s-connectivity is a combined generalization of s-connectivity and hamiltonicity, which is closely related with faulty-tolerance of networks. The study on spanning connectivity has received a lot of attention recently (see Albert et al. 2001, Lin et al. 2007, Tsai et al. 2004 and references therein).

In this paper, we study *spanning k-connected index* $s_k(G)$, which is the smallest nonnegative integer m such that $L^m(G)$ is spanning k-connected. A key to the main result of this paper is a result on 2-collapsible graphs.

Catlin (1988), introduced collapsible graphs as a tool to study supereulerian graphs. Motivated by Catlin's work, the concept of collapsible graphs has been generalized to s-collapsible graphs in Chen et al. (2012b) and Li (2012). Let $O(G)$ denote the set of vertices in G with odd degree.

Definition 1.1 A graph G is *s-collapsible* if for any subset $X \subseteq V(G)$ with $|X| \equiv 0 \pmod 2$, G has a spanning subgraph L_X such that

(i) both $O(L_X) = X$ and $\kappa'(L_X) \geq s - 1$, and
(ii) $G - E(L_X)$ is connected.
Such L_X is called an (s, X)-*subgraph* of G.

The concept of collapsible graphs defined in Catlin (1988) coincides with 1-collapsible graphs in Definition 1.1.

An (u, v)-path P of G with $u \neq v$ is called a non-closed path. A non-closed path P of G is called a *divalent path* in G if all the internal vertices of P have degree 2 in G. Following Lai (1988), we define $l(G) = \max\{m : G$ has a divalent path of length m which is not a 2-path in a $K_3\}$. By definition, we have $l(K_3) = 1$. A graph G is *k-triangular* if each edge of G lies in at least k triangles, and 1-triangular is abbreviated as *triangular*. We use J_3^k to denote the collection of k-triangular graphs.

In this paper, we shall prove the following theorems:

Theorem 1.2 *Every connected 3-triangular graph G is 2-collapsible.*

Theorem 1.3 *Let G be a simple connected graph that is not a path, a cycle or* $K_{1,3}$. *Then* $s_3(G) \leq l(G) + 6$.

Theorem 1.2 is best possible in the sense that there exists an infinite family graphs in J_3^2 which is not 2-collapsible, as shown in the example below.

Example 1.4 Let $G \cdot H$ denote the graph obtained from $G \cup H$ by identifying a vertex u of G with a vertex v of H. The graph $K_4 \in J_3^2 - J_3^3$ and K_4 is not 2-collapsible. It can be seen that $K_4 \cdot K_4 \cdot \ldots \cdot K_4$ is an infinite family of J_3^2 which is not 2-collapsible.

2 Some notations and preliminaries

The line graph of $K_{1,3}$ is a 3-cycle. The iterated line graph of a cycle is always isomorphic to the same cycle, and the lth iterated line graph of a path of length l is K_1. These are trivial cases. Thus we always assume that G is a connected graph that is not a path, a cycle, or a $K_{1,3}$.

Let $G[X]$ denote the subgraph of G induced by an edge subset $X \subseteq E(G)$. When no confusion arises, we shall adopt the convention that an edge subset $X \subseteq E(G)$ is also used to denote subgraph $G[X]$.

For two edge sets X and Y, the *symmetric difference* of X and Y is $X \oplus Y = (X \cup Y) - (X \cap Y)$. For graphs G and H, we define $G \oplus H$ to be the spanning subgraph of $G \cup H$ with edge set $E(G) \oplus E(H)$, called the symmetric difference of G and H. For a cycle $C = u_1u_2 \ldots u_k$, sometimes we may use $G \oplus u_1u_2 \ldots u_k$ to denote $G \oplus C$.

For a graph G, and for $X \subseteq E(G)$, *the contraction* G/X is obtained from G by identifying the two ends of each edge in X and then by deleting the resulting loops. If H is a subgraph of G, then we write G/H for $G/E(H)$. We use $D_i(G)$ to denote the set of all vertices of degree i in G. The concept of core is defined as follows, which was first introduced in the dissertation of Shao (2005). Let G be a graph such that $\kappa(L(G)) \geq 3$ and such that $L(G)$ is not complete. For each $v \in D_2(G)$, let $E_G(v) = \{e_1^v, e_2^v\}$ and define

$$X_1(G) = \bigcup_{v \in D_1(G)} E_G(v), \quad \text{and} \quad X_2(G) = \{e_2^v : v \in D_2(G)\}. \tag{1}$$

Since $\kappa(L(G)) \geq 3$, the minimum degree of $L(G)$ is at least 3, and thus $D_2(G)$ is an independent set of G, and for any $v \in D_2(G)$, $|X_2(G) \cap E_G(v)| = 1$. The *core* of graph G is defined to be

$$G_0 = G/(X_1(G) \cup X_2(G)) = (G - D_1(G))/X_2(G). \tag{2}$$

Let $\mathcal{C}_s$ denote the collection of all s-collapsible graphs. By definition, for $s \geq 1$, any $(s+1, X)$-subgraph of G is also an (s, X)-subgraph of G. This implies that

$$\mathcal{C}_{s+1} \subseteq \mathcal{C}_s, \text{ for any positive integer } s. \tag{3}$$

The following previous results are needed in our proofs.

Lemma 2.1 (Proposition 2.2 of Chen et al. 2012b) *Let G be a graph, and let $s \geq 1$ be an integer. Then the following are equivalent.*

(i) $G \in \mathcal{C}_s$.
(ii) *For any $X \subseteq V(G)$ with $X \equiv 0 \pmod 2$, G has a spanning connected subgraph L_X such that $O(L_X) = X$ and $\kappa'(G - E(L_X)) \geq s - 1$.*

Lemma 2.2 (Corollary 2.4 of Chen et al. 2012b) *Let $s \geq 1$ be an integer. Then $\mathcal{C}_s$ satisfies the following.*

(i) $K_1 \in \mathcal{C}_s$
(ii) *If $G \in \mathcal{C}_s$ and if $e \in E(G)$, then $G/e \in \mathcal{C}_s$.*
(iii) *If H is a subgraph of G and if $H, G/H \in \mathcal{C}_s$, then $G \in \mathcal{C}_s$.*

Lemma 2.3 (Corollary 2.6 (ii) of Zhang et al. 2012) *Let $s \geq 3$ be an integer. Then $L^{l+s}(G)$ is 2^{s-2}-triangular.*

Lemma 2.4 (Corollary 1 of Catlin 1988) *Let G be a graph. If G contains a spanning tree T such that each edge of T is in a collapsible subgraph of G, then G is collapsible.*

Theorem 2.5 (Theorem 4.3 (iii) of Chen et al. 2012a) *Let G be a graph with core G_0, and let $k \geq 3$ be an integer. If for any $e, e' \in E(G_0)$ with $e \neq e'$, $G_0 - \{e, e'\}$ is a $(k-1)$-collapsible graph, then $\kappa^*(L(G)) \geq k$.*

3 Proof of main results

Proof of Theorem 1.2 Suppose G is a counterexample.

By Lemma 2.4 and the fact that K_3 is collapsible, G is 1-collapsible. By Lemma 2.1, for any vertex subset $X \subseteq V(G)$ with $|X| \equiv 0 \pmod 2$, G has

$$\text{a spanning connected subgraph } L_X \text{ with } O(L_X) = X. \tag{4}$$

A subgraph L_X satisfying (4) is said to *have Property A_X*. By Lemma 2.1 and since G is not 2-collapsible, there exists a vertex subset $X \subseteq V(G)$ with $|X| \equiv 0 \pmod 2$ such that for *any* subgraph L_X with property A_X, $G - E(L_X)$ is not connected. Choose

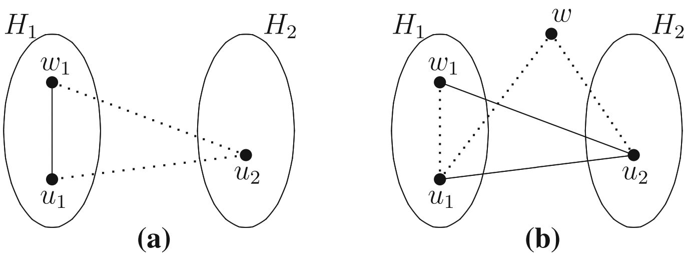

Fig. 1 Illustration for the proof of Claim 1. In **a** *dotted lines* belong to L_{X_0}, while in **b** *dotted lines* are in L_Y. The same notation is adopted in following figures

X_0 to be such a vertex subset and choose L_{X_0} to be a subgraph with property A_{X_0} such that

$$r = c(G - E(L_{X_0})) \text{ is as small as possible.} \tag{5}$$

Then $r \geq 2$.

The following observations follow by the definition of symmetric difference and the definition of $G - L_{X_0}$.

Observation 3.1 (i) *For a cycle C, $O(L_{X_0} \oplus C) = O(L_{X_0})$.*
(ii) *If e is an edge joining vertices in different components of $G - E(L_{X_0})$, then $e \in E(L_{X_0})$.*

Let $H_1, H_2, \ldots, H_r$ be the connected components of $G - E(L_{X_0})$. For an edge u_1u_2, denote $W_{u_1u_2} = \{w \in V(G) \mid G[\{u_1u_2w\}] \text{ is a 3-cycle}\}$. Since G is 3-triangular, we have $|W_{u_1u_2}| \geq 3$. For an edge $u_1u_2 \in [H_i, H_j]$ with $i \neq j$, define

$$f_{w,u_1u_2} = \begin{cases} u_1w, & \text{if } \mathrm{w} \in \mathrm{V(H_i)}, \\ u_2w, & \text{if } \mathrm{w} \in \mathrm{V(H_j)}, \end{cases}$$

$$e_{w,u_1u_2} = \begin{cases} u_1w, & \text{if } \mathrm{w} \in \mathrm{V(H_j)}, \\ u_2w, & \text{if } \mathrm{w} \in \mathrm{V(H_i)}. \end{cases}$$

In other words, for a triangle containing an edge $u_1u_2 \in [H_i, H_j]$ (see Fig. 1a for an illustration), f_{w,u_1u_2} is the edge lying in a component, while u_1u_2 and e_{w,u_1u_2} are the two edges crossing the two components. For example, in Fig. 1a, $f_{w_1,u_1u_2} = u_1w_1$ and $e_{w_1,u_1u_2} = u_2w_1$. Further denote

$$W'_{u_1u_2} = W_{u_1u_2} \cap (V(H_i) \cup V(H_j)) \quad \text{and} \quad F_{u_1u_2} = \{f_{w,u_1u_2} \mid w \in W'_{u_1u_2}\}.$$

It should be pointed out that the following Claims 1–4 hold for *any* edge crossing different components of $G - E(L_{X_0})$. For simplicity of notation, we assume that the edge is $u_1u_2 \in [H_1, H_2]$.

We use Fig. 1 to illustrate the proof of Claim 1.

Claim 1 *If there exists a vertex $w_1 \in W'_{u_1u_2}$ such that $f_{w_1,u_1u_2} \notin E(L_{X_0})$, then for any $w \in W_{u_1u_2} \setminus \{w_1\}$, $\{u_1w, u_2w\} - E(L_{X_0}) \neq \emptyset$.*

Proof Let C be the 3-cycle $u_1u_2w_1u_1$ and let $L_Y = L_{X_0} \oplus C$. By definition, $L_Y = (L_{X_0} - \{u_1u_2, u_2w_1\}) \cup \{u_1w_1\}$. If L_Y is not connected, then since u_1w_1 is an edge of L_Y and since L_{X_0} is connected, u_1, u_2 must be in different components of L_Y. If there exists a vertex $w \in W_{u_1u_2}$ such that $\{u_1w, u_2w\} \subseteq E(L_{X_0})$, then u_1wu_2 is a 2-path in L_Y connecting vertex u_1 and u_2, contrary to the fact that u_1 and u_2 are in different component of L_Y. Hence L_Y is connected, and so by Observation 3.1 (i), L_Y has property A_{X_0}. By the definition $L_Y = L_{X_0} \oplus C$, $(G - E(L_Y))\,[V(H_1) \cup V(H_2)]$ is connected, and so $H_1 \cup H_2, H_3, H_4, \ldots, H_r$ are the components of $G - E(L_Y)$. It follows that $c(G - E(L_Y)) = c(G - E(L_{X_0})) - 1$, contrary to (5). □

Claim 2 *If $W'_{u_1u_2} \neq \emptyset$, then $F_{u_1u_2} \cap E(L_{X_0}) = \emptyset$.*

Proof Notice that for any $w \in W'_{u_1u_2}$, $\{u_1w, u_2w\} = \{f_{w,u_1u_2}, e_{w,u_1u_2}\}$. Since $e_{w,u_1u_2} \in [H_1, H_2]$, by Observation 3.1 (ii), $e_{w,u_1u_2} \in E(L_{X_0})$. Hence, if $F_{u_1u_2} \not\subseteq E(L_{X_0})$, then by Claim 1, for any $w \in W'_{u_1u_2}$, $f_{w,u_1u_2} \notin E(L_{X_0})$, and thus $F_{u_1u_2} \cap E(L_{X_0}) = \emptyset$.

Suppose, by contradiction, that $F_{u_1u_2} \cap E(L_{X_0}) \neq \emptyset$. Then $F_{u_1u_2} \subseteq E(L_{X_0})$. Pick $w_1 \in W'_{u_1u_2}$. Suppose, without loss of generality, that $w_1 \in V(H_1)$ (see Fig. 2). Let $L_Y = L_{X_0} \oplus u_1u_2w_1u_1 = L_{X_0} - \{u_1u_2, u_2w_1, w_1u_1\}$. Let $w_2 \in W_{u_1u_2} - \{w_1\}$. If $w_2 \in \bigcup_{j=3}^{r} V(H_j)$, then $\{u_1w_2, u_2w_2\} \subseteq E(L_{X_0})$. If $w_2 \in V(H_1) \cup V(H_2)$, then since $F_{u_1u_2} \subseteq E(L_{X_0})$, we have $\{u_1w_2, u_2w_2\} \subseteq E(L_{X_0})$. In any case, $u_1w_2, u_2w_2 \in E(L_Y)$ and so u_1, u_2 are in the same component of L_Y. With a similar argument by replacing u_1u_2 by w_1u_2, we conclude that for some $w_3 \in W'_{w_1u_2} - \{u_1\}$, $w_1w_3, w_3u_2 \in E(L_Y)$. It follows that L_Y is connected (see Fig. 2(b)), and so by Observation 3.1(i), L_Y has property A_{X_0}. Furthermore, since $G - E(L_Y) = (G - E(L_{X_0})) \cup \{u_1u_2, u_1w_1, u_2w_1\}$, we have $c(G - E(L_Y)) = c(G - E(L_{X_0})) - 1$, contrary to (5). □

Claim 3 $W'_{u_1u_2} = W_{u_1u_2}$.

Proof Suppose $W'_{u_1u_2} \neq \emptyset$. Let w_1 be a vertex in $W'_{u_1u_2}$. Then by Claim 2, $f_{w_1,u_1u_2} \notin E(L_{X_0})$. Since for any vertex $w \in W_{u_1u_2} \cap \left(\bigcup_{j=3}^{r} V(H_j)\right)$, $\{u_1w, u_2w\} \subseteq E(L_{X_0})$, it follows from Claim 1 that $W_{u_1u_2} = W'_{u_1u_2}$. This proves

$$\text{either } W'_{u_1u_2} = W_{u_1u_2} \text{ or } W'_{u_1u_2} = \emptyset. \tag{6}$$

By (6), to prove Claim 3, we assume that $W'_{u_1u_2} = \emptyset$ to derive a contradiction. Pick $w_1 \in W_{u_1u_2}$. Since $W'_{u_1u_2} = \emptyset$, we may assume that $w_1 \in V(H_3)$ (see Fig.3). Let $L_Y = L_{X_0} \oplus u_1u_2w_1u_1$. Then $L_Y = L_{X_0} - \{u_1u_2, u_1w_1, u_2w_1\}$. Since $u_2 \in W_{u_1w_1} - (V(H_1) \cup V(H_3))$, it follows by (6) that $W_{u_1w_1} = W'_{u_1w_1}$. Let $w_2 \in W_{u_1w_1} - \{u_2\}$. Then $w_2 \notin V(H_1) \cup V(H_3)$, and so $\{u_1w_2, w_1w_2\} \subseteq E(L_{X_0})$. It follows that $u_1w_2, w_2w_1 \in E(L_Y)$. Similarly, for some $w_3 \in W_{w_1u_2}$, $w_1w_3, u_2w_3 \in E(L_Y)$. Hence $\{w_1, u_1, u_2\}$ is connected by the path $u_1w_2w_1w_3u_2$. Thus L_Y is connected and so by Observation 3.1 (i), L_Y has property A_{X_0}. By the definition $L_Y = L_{X_0} \oplus C$,

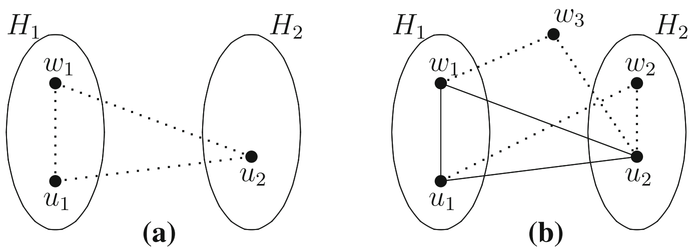

Fig. 2 Illustration for the proof of Claim 2

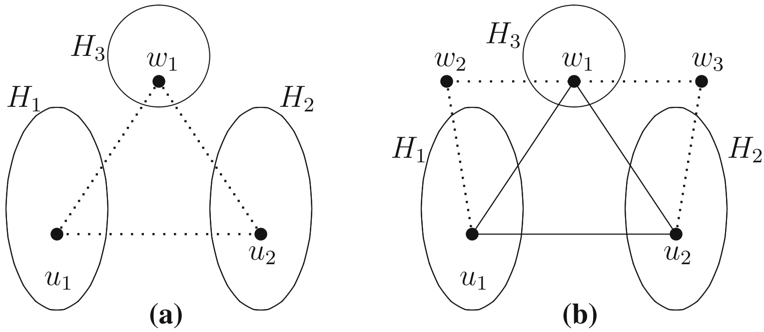

Fig. 3 Illustration for the proof of Claim 3

$(G - E(L_Y))\,[V(H_1 \cup H_2 \cup H_3)]$ is connected, and so $H_1 \cup H_2 \cup H_3, H_4, \ldots, H_r$ are the components of $G - E(L_Y)$. It follows that $c(G - E(L_Y)) = c(G - E(L_{X_0})) - 2$, contrary to (5). □

Claim 4 *For any vertex $w \in W_{u_1u_2}$, f_{w,u_1u_2} is a cut edge for the component of $G - E(L_{X_0})$ which contains w.*

Proof By Claim 3, $W_{u_1u_2} \subseteq V(H_1) \cup V(H_2)$. Let w be an arbitrary vertex of $W_{u_1u_2}$. For simplicity of statement, assume $w \in V(H_2)$. Then $f_{w,u_1u_2} = wu_2$. We are to show wu_2 is a cut edge of H_2.

First, suppose that there exists a vertex $w_1 \in W_{u_1u_2} \cap V(H_1)$. By Claim 2, both $u_1w_1, u_2w \notin E(L_{X_0})$. Let C be the 4-cycle $u_1w_1u_2wu_1$ and let $L_Y = L_{X_0} \oplus C$. By the definition of $L_{X_0} \oplus C$, L_Y is connected (see Fig. 4b), and so by Observation 3.1 L_Y has property A_{X_0}. In order not to violate the choice of L_{X_0}, we must have $c(G - E(L_Y)) \geq c(G - E(L_{X_0}))$, which is possible only if w_1u_1 is a cut edge of H_1 and wu_2 is a cut edge of H_2 (see Fig. 4b).

Next, suppose all vertices of $W_{u_1u_2}$ lie in H_2. Since $G \in J_3^3$, $W_{u_1u_2} \setminus \{w\}$ has two distinct vertices w_1, w_2. By Claim 2, $u_2w_i \notin E(L_{X_0})$ for $i = 1, 2$. Let $L_Y = L_{X_0} \oplus u_1w_1u_2w_2u_1$ (see Fig. 4c, d). Similarly to the above, in order not to violate the choice of L_{X_0}, we have $c(G - E(L_Y)) \geq c(G - E(L_{X_0}))$, which is possible only when the removal of $\{u_2w_1, u_2w_2\}$ from H_2 separates u_2 from $\{w_1, w_2\}$. Let H_{u_2} be the component of $H_2 - \{u_2w_1, u_2w_2\}$ containing u_2, and $H_{w_1,w_2} = H_2 - H_{u_2}$ (see Fig. 4e). Then $w \in V(H_{u_2})$. Let $L_Z = L_{X_0} \oplus u_1w_1u_2wu_1$ (Fig. 4f). Again, in order

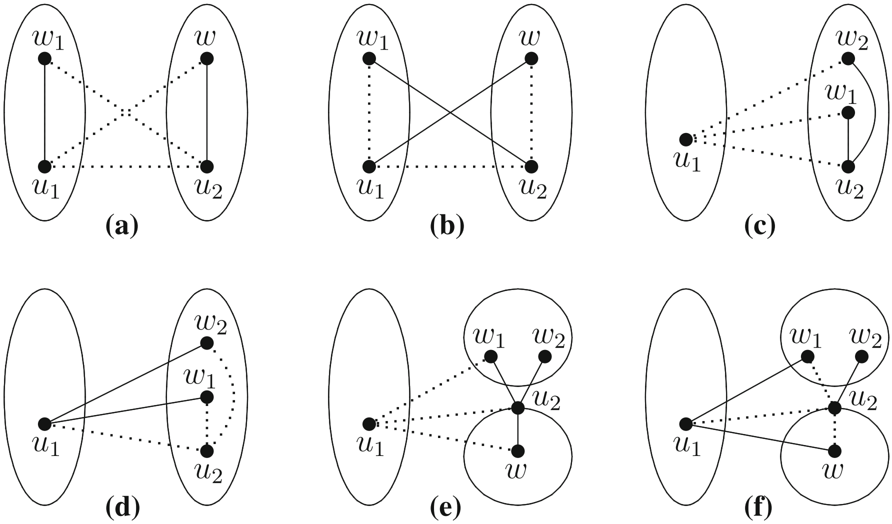

Fig. 4 Illustration for the proof of Claim 4

not to violate the choice of L_{X_0}, we have $c(G - E(L_Z)) \geq c(G - E(L_{X_0}))$. Then, vertex u_2 must be separated from vertex w in $G - E(L_Z)$, which is possible only when edge $u_2 w$ is a cut edge of H_{u_2}, and thus a cut edge of H_2 (Fig. 4f). □

For $i = 1, 2$, denote $F_{H_i} = \bigcup_{u_1 \in V(H_1), u_2 \in V(H_2)} \left(F_{u_1 u_2} \cap E(H_i)\right)$. By Claim 3, at least one of F_{H_1} and F_{H_2} is nonempty, say F_{H_1}. By Claim 4, every edge $f \in F_{H_1}$ is a cut edge of H_1. Choose such a cut edge f that one component of $H_1 - f$ contains no edge of F_{H_1}. Suppose $f = f_{w_1, u_1 u_2}$, and that the two components of $H_1 - f$ are H_{w_1} and H_{u_1}, such that

$$E(H_{w_1}) \cap F_{H_1} = \emptyset. \tag{7}$$

Claim 5 *There exists an infinite sequence of vertices $\{u_3, u_4, \ldots\}$ and a sequence of corresponding subgraphs $\{H_{u_3}, H_{u_4}, \ldots\}$ satisfying the following conditions:*

(i) $u_{i+2} \in W(w_1 u_{i+1}) \setminus \{u_1, u_i\}$ $(i = 1, 2, \ldots)$,
(ii) $u_{i+2} \in V(H_{u_{i+1}})$, *$u_{i+1} u_{i+2}$ is a cut edge of $H_{u_{i+1}}$, and $H_{u_{i+2}}$ is the component of $H_{u_{i+1}} - u_{i+1} u_{i+2}$ containing vertex u_{i+2} (where $H_{u_2} = H_2$).*

Proof We prove this claim by induction, and illustrate the proof by Fig. 5.

For the basic step, let u_3 be a vertex in $W_{w_1 u_2} \setminus \{u_1\}$. By Claim 2 , $f_{u_3, w_1 u_2} \in E(G) - E(L_{X_0})$. Since $w_1 u_1$ is the only edge of $G - E(L_{X_0})$ between H_{w_1} and H_{u_1}, $u_3 \notin V(H_{u_1})$. If $u_3 \in V(H_{w_1})$, then by Claim 4, $w_1 u_3$ is a cut edge of H_1. Then $w_1 u_3$ is an edge in $E(H_{w_1}) \cap F_{H_1}$, contradicting (7). Hence, $u_3 \notin V(H_{w_1})$. By Claim 3 , we have $u_3 \in V(H_2)$. By Claim 4, $u_2 u_3$ is a cut edge of H_2. Let H_{u_3} be the components of $H_2 - u_2 u_3$ containing u_3. Then u_3 and H_{u_3} satisfy the conditions of the claim for $i = 1$.

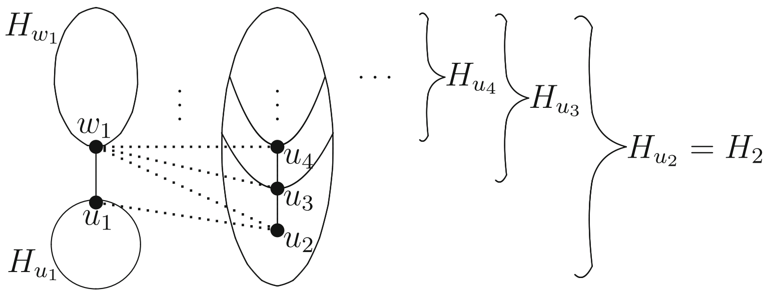

Fig. 5 Illustration for Claim 5

Suppose by induction that for some positive integer k, one has found a vertex sequence $\{u_3, \ldots, u_{k+2}\}$ and the corresponding subgraphs $\{H_{u_3}, \ldots, H_{u_{k+2}}\}$ satisfying (i) and (ii). Let u_{k+3} be a vertex in $W_{w_1 u_{k+2}} \setminus \{u_1, u_{k+1}\}$ (such vertex exists by $|W_{w_1 u_{k+2}}| \geq 3$). By Claim 2, $f_{u_{k+3}, w_1 u_{k+2}} \in E(G) - E(L_{X_0})$. Since $w_1 u_1$ is the only edge of $G - E(L_{X_0})$ between H_{w_1} and H_{u_1}, $u_{k+3} \notin V(H_{u_1})$. If $u_{k+3} \in V(H_{w_1})$, then by Claim 4, $w_1 u_{k+3}$ is a cut edge of H_1. Then $w_1 u_{k+3}$ is an edge in $E(H_{w_1}) \cap F_{H_1}$, contradicting (7). Hence, $u_{k+3} \notin V(H_{w_1})$. By Claim 3, we have $u_{k+3} \in V(H_2)$. By Claim 4, $u_2 u_{k+3}$ is a cut edge of H_2. Since $u_{k+1} u_{k+2} \in E(G - E(L_{X_0}))$ is a cut edge of H_2, $u_{k+3} \in H_{u_{k+2}}$. Then $u_{k+2} u_{k+3}$ is a cut edge of $H_{u_{k+2}}$. Let $H_{u_{k+3}}$ be the components of $H_{u_{k+2}} - u_{k+2} u_{k+3}$ containing u_{k+3}. Then, sequences $\{u_3, \ldots, u_{k+2}, u_{k+3}\}$ and $\{H_{u_3}, \ldots, H_{u_{k+2}}, H_{u_{k+3}}\}$ satisfy the conditions of the claim. This completes the induction step. □

By Claim 5, $H_{u_{i+2}}$ is a *proper* subgraph of $H_{u_{i+1}}$ for $i \geq 1$. Combining this with $\{H_{u_3}, H_{u_4}, \ldots\}$ being an *infinite* sequence of subgraphs of H_2, we obtain a contradiction to the assumption that G is finite. Theorem 1.2 is proved.

Proof of Theorem 1.3 By Lemma 2.3, $L^{l+5}(G)$ is 8-triangular. Hence, for any distinct edges $e, e' \in E(L^{l+5}(G))$, $L^{l+5}(G) - \{e, e'\}$ is 6-triangular, and thus $L^{l+5}(G)$ is 2-collapsible by Theorem 1.2. Then by Theorem 2.5, $\kappa^*(L^{l+6}(G)) \geq 3$. Theorem 1.3 is proved.

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References

Akers SB, Krishnamurthy B (1989) A group-theoretic model for symmetric interconnection networks. IEEE Trans Comput 38:555–566

Albert M, Aldred REL, Holton D (2001) On 3*-connected graphs. Australas J Comb 24:193–208

Bondy JA, Murty USR (2008) Graph theory. Springer, New York

Catlin PA (1988) A reduction method to find spanning Eulerian subgraphs. J Graph Theory 12:199–211

Chartrand G, Wall CE (1973) On the hamiltonian index of a graph. Stud Sci Math Hung 8:38–43

Chen Y, Chen Z-H, Lai H-J, Li P, Wei E (2012a) On spanning disjoint paths in line graphs, Graphs Combi, accepted

Chen Y, Lai H-J, Li H, Li P (2012b) Supereulerian graphs with width s and s-collapsible graphs, submitted

Clark LH, Wormald NC (1983) Hamiltonian-like indices of the graphs. Ars Comb 15:131–148

Hsu DF (1994) On container width and length in graphs, groups, and netwoeks. IEICE Trans Fund E77–A:668–680

Knor M, Niepel L (2003) Connectivity of iterated line graphs. Discret Appl Math 125:255–266

Lai H-J (1988) On the hamiltonian index. Discret Math 69:43–53

Li P (2012) Bases and cycles of matroids and graphs. West Virginia University, PhD. Dissertation (2012)

Lin CK, Huang HM, Hsu LH (2007) On the spanning connectivity of graphs. Descret Math 307:285–289

Lin CK, Tan JJM, Hsu DF, Hsu LH (2007) On the spanning connectivity and spanning laceability of hypercube-like networks. Theor Comput Sci 381:218–229

Shao YH (2005) Claw-free graphs and line graphs. West Virginia University, Ph. D. Dissertation (2005)

Shao YH (2010) Connectivity of iterated line graphs. Discret Appl Math 158:2081–2087

Shao YH (2012) Essential edge connectivity of line graphs, submitted

Tsai CH, Tan JJM, Hsu LH (2004) The super connected property of recursive circulant graphs. Inf Process Lett 91:293–298

Zhang LL, Shao YH, Chen GH, Xu XP, Zhou J (2012) s-Vertex pancyclic index. Graphs Comb 28:393–406