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Spanning trails in essentially 4-edge-connected graphs

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ABSTRACT

A connected graph G is essentially 4-edge-connected if for any edge cut X of G with $|X| < 4$, either $G - X$ is connected or at most one component of $G - X$ has edges. In this paper, we introduce a reduction method and investigate the existence of spanning trails in essentially 4-edge-connected graphs. As an application, we prove that if G is 4-edge-connected, then for any edge subset $X_0 \subseteq E(G)$ with $|X_0| \leq 3$ and any distinct edges $e, e' \in E(G)$, G has a spanning (e, e')-trail containing all edges in X_0, which solves a conjecture posed in [W. Luo, Z.-H. Chen, W.-G. Chen, Spanning trails containing given edges, Discrete Math. 306 (2006) 87–98].

1. Introduction

Graphs considered in this paper are finite and loopless. Undefined terms will follow [2]. A *trail* is a finite sequence $T = u_0e_1u_1e_2u_2 \cdots e_ru_r$, whose terms are alternately vertices and edges, with $e_i = u_{i-1}u_i$ $(1 \leq i \leq r)$, where the edges are distinct. A trail T is a *closed trail* if $u_0 = u_r$ and is called a (u, v)-trail if $u_0 = u$ and $u_r = v$, and is called a (e, e')-trail if $e = e_1$ and $e' = e_r$. A closed trail is also called an Eulerian subgraph. A trail T is called a *spanning trail* if $V(T) = V(G)$. A graph is called *supereulerian* if it has a spanning closed trail.

A graph G is *nontrivial* if $E(G) \neq \emptyset$. An edge cut X of a graph G is *essential* if both components of $G - X$ are nontrivial; and G is *essentially k-edge-connected* if G is connected and does not have an essential edge cut of size less than k. It follows from the definition, we have the following proposition:

Proposition 1.1. *Let G be an essentially k-edge-connected graph with the minimum degree $\delta(G)$ and the edge-connectivity $\kappa'(G)$. Then $\kappa'(G) = \min\{\delta(G), k\}$.*

For a graph G, the *line graph* of G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent in $L(G)$. It follows from the definitions that a line graph $L(G)$ is k-connected if and only if G is essentially k-edge-connected. For line graphs, Thomassen has a well known conjecture [12]: "every 4-connected line graph is Hamiltonian". By a theorem of Harary and Nash-Williams [6], to prove Thomassen's conjecture, one can prove the equivalent version: every essentially 4-edge-connected graph has a closed trail that contains at least one vertex of every edge in G.

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On the other hand, motivated by the Chinese postman problem, Boesch et al. [1] introduced the supereulerian problem, that is to determine if a graph G has a spanning closed trail. Pulleyblank [11] showed that this is an NP-complete problem. Catlin [3] and Jaeger [7] proved the following:

Theorem 1.2 (*Catlin [3] and Jaeger [7]*). *A 4-edge-connected graph has a spanning closed trail.*

As shown in [10], Theorem 1.2 can be improved in the sense that a 4-edge-connected graph can have spanning closed trail containing some fixed edges. In [10], Luo et al. defined a graph G to be *r-edge-Eulerian-connected* if for any edge subset $X \subseteq E(G)$ with $|X| \le r$ and any distinct edges $e, e' \in E(G)$, G has a spanning (e, e')-trail containing all edges in X. Define $\xi(r)$ to be the smallest integer k such that every k-edge-connected graph is r-edge-Eulerian-connected. They proved the following:

Theorem 1.3 (*Luo, Chen and Chen [10]*). *Let $r \ge 0$ be an integer. Then*

$$\xi(r) = \begin{cases} 4, & 0 \le r \le 2, \\ r+1, & r \ge 4. \end{cases}$$

For $r = 3$, Luo et al. [10] indicated that $4 \le \xi(3) \le 5$, and conjectured $\xi(3) = 4$.

In this paper, we introduce a reduction method on essentially 4-edge-connected graphs and investigate spanning trails in essentially 4-edge-connected graphs. As an application, we prove the following:

Theorem 1.4. *If G is a 4-edge-connected graph, then for any $X_0 \subseteq E(G)$ with $|X_0| \le 3$ and any distinct edges $e, e' \in E(G)$, G has a spanning (e, e')-trail T such that $X_0 \subseteq E(T)$. Thus, G is 3-edge-Eulerian-connected and so $\xi(3) = 4$.*

Theorem 1.4 confirmed the conjecture above, and so all the values of $\xi(r)$ are determined for all integer $r \ge 0$.

In the rest of the paper, we provide the theory of Catlin's reduction method which is an important tool to solve problems related to spanning trails, and introduce a new reduction method on essentially 4-edge-connected graphs in Section 2. The results of spanning trails in essentially 4-edge-connected graphs are given in Section 3. We will discuss 3-edge-Eulerian-connected graphs and give the proof of the conjecture $\xi(3) = 4$ in Section 4.

2. Reductions of essentially 4-edge-connected graphs

In this section, we shall develop a reduction method for essentially 4-edge-connected graphs and prove some associate results on spanning trails that will be needed in the proof of Theorem 1.4.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For vertex disjoint subsets $V_1, V_2 \subseteq V(G)$, let $[V_1, V_2]_G$ denotes the set of all edges in G with one end in V_1 and the other in V_2. For vertex disjoint subgraphs H, L of G, we write $[H, L] = [V(H), V(L)]_G$, and define $\partial_G(H) = [V(H), V(G) - V(H)]_G$, called the *boundary of H in G*. When $H = K_1$ is a single vertex v, we denote $\partial_G(v)$ as $\partial_G(H)$ and $|\partial_G(v)| = d_G(v)$.

For a graph G and $X \subseteq E(G)$, the *contraction* G/X is obtained from G by identifying the two ends of each edge in X and then by deleting the resulting loops. If H is a subgraph of G, then we write G/H for $G/E(H)$. When H is connected, we use v_H to denote the vertex in G/H onto which H is contracted. Note that $E(G/H) = E(G) - E(H)$ and $V(G/H) = (V(G) - V(H)) \cup \{v_H\}$. For an edge xy in $E(G)$, we let $\theta(xy)$ be the vertex in G/xy onto which the edge xy is contracted.

A graph G is *collapsible* [3] if for any subset $S \subseteq V(G)$ with $|S| \equiv 0 \pmod 2$, G has a spanning connected subgraph L_S such that the set of odd degree vertices in L_S is precisely S. As shown in [3], if G is a simple graph and H is a maximal collapsible subgraph of G, then G/H is also a simple graph. Furthermore, Catlin [3] showed that any graph G has a unique collection of vertex disjoint maximally collapsible subgraphs $H_1, H_2, \ldots, H_c$, and $G/(H_1 \cup H_2 \cup \cdots \cup H_c)$ obtained by contracting each H_i into a single vertex v_{H_i}, is called the *reduction* of G. As always, K_1 is considered both supereulerian and collapsible, and has infinity edge-connectivity. It was shown in [3] if G' is the reduction of G, then G' is simple and K_3-free and $\kappa'(G') \ge \kappa'(G)$. A graph G is *reduced* if its reduction is G itself. The theory on collapsible graphs is useful for both simple graphs and multigraphs. Let $F(G)$ be the minimum number of additional edges that must be added to G to result in a graph G^* with at least two edge-disjoint spanning trees. The following are some useful theorems which will be needed.

Theorem 2.1. *Let G be a graph and let H be a collapsible subgraph of G. Let v_H be the vertex in G/H onto which H is contracted.*

(i) [3] *Suppose that $u \ne v_H$ and G/H has a (u, v)-trail T' containing v_H. If $v \ne v_H$, then G has a (u, v)-trail T with $E(T') \subseteq E(T)$ and $V(T) = (V(T') - \{v_H\}) \cup V(H)$. If $v = v_H$, then for any $v' \in V(H)$, G has a (u, v')-trail T with $E(T') \subseteq E(T)$ and $V(T) = (V(T') - \{v_H\}) \cup V(H)$.*

(ii) *(Theorem 1.3 of [4]) If $\kappa'(G) \ge 2$ and $F(G) \le 2$, then the reduction of G is in $\{K_1, K_{2,t}$ for some integer $t \ge 2\}$.*

(iii) [3] *If G is reduced, then $F(G) = 2|V(G)| - |E(G)| - 2$.*

(iv) *(Theorem 2.3(iii) of [9]) If G is collapsible, then for any $u, v \in V(G)$, G has a spanning (u, v)-trail.*

(v) [3] *G is supereulerian if and only if G/H is supereulerian. In particular, G is supereulerian if and only if the reduction of G is supereulerian.*

Next, we introduce a new reduction method for preserving essentially 4-edge-connected property of graphs, which develops the ideas deployed in the proof of Theorem 3.1 in [8].

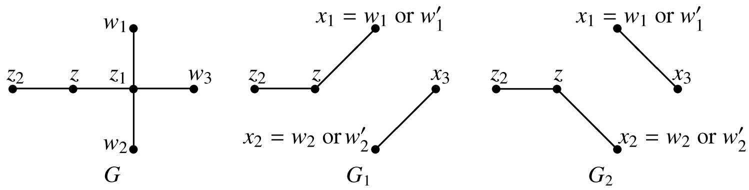

Fig. 1. The graphs G_1 and G_2 from G in Theorem 2.2.

For a graph G and for each integer $i > 0$, define

$$D_i(G) = \{v \in V(G) : d_G(v) = i\}.$$

Let $z \in D_2(G)$ with $N_G(z) = \{z_1, z_2\}$ such that $z_1 \in D_4(G)$ and $N_G(z_1) = \{z, w_1, w_2, w_3\}$. For $i \in \{1, 2, 3\}$, if $w_i \in D_2(G)$, then let $N_G(w_i) = \{z_1, w'_i\}$. For $j \in \{1, 2\}$, let $G_j^- = (G - \{z_1\}) + \{zw_j, w_{3-j}w_3\}$, and $W(G_j^-) = \{e = xy \in E(G_j^-) : x, y \in D_2(G_j^-)\}$. Define

$$G_j = G_j^-/W(G_j^-). \quad (1)$$

For an essentially 4-edge-connected graph G, if $w_i \in D_2(G)$, then $N_G(w_i) = \{z_1, w'_i\} \cap D_2(G) = \emptyset$. Thus, if an edge $e \in W(G_j^-)$, then $e \in \{zw_j, w_{3-j}w_3\}$ (see Fig. 1).

Theorem 2.2. *Let G be an essentially 4-edge-connected graph with $\delta(G) \geq 2$ and $D_3(G) = \emptyset$. Let $z \in D_2(G)$ with $N_G(z) = \{z_1, z_2\}$ such that $z_1 \in D_4(G)$ and $N_G(z_1) = \{z, w_1, w_2, w_3\}$. For $i \in \{1, 2, 3\}$, if $w_i \in D_2(G)$, then let $N_G(w_i) = \{z_1, w'_i\}$. Let G_1 and G_2 be the graphs defined by (1) above. Then either G_1 or G_2 is also essentially 4-edge-connected and $\delta(G_j) \geq 2$ and $D_3(G_j) = \emptyset$ ($j = 1, 2$).*

Proof. Since G is essentially 4-edge-connected with $\delta(G) \geq 2$, by Proposition 1.1, G is 2-edge-connected. Then by the definition of G_j ($j = 1, 2$), G_j is connected with $\delta(G_j) \geq 2$ and $D_3(G_j) = \emptyset$. It suffices to show that either G_1 or G_2 is essentially 4-edge-connected. For $j \in \{1, 2\}$, by (1), when $w_{3-j}w_3 \in W(G_j^-)$, we shall use w_{3-j} to denote the vertex $\theta(w_{3-j}w_3)$ in G_j; and when $w_j \in D_2(G)$, use z to denote the vertex $\theta(zw_j)$ in G_j. Let x_1, x_2 and x_3 denote the vertices in G_1 and G_2 such that

$$x_1 = \begin{cases} w_1 & \text{if } w_1 \notin D_2(G) \\ w'_1 & \text{if } w_1 \in D_2(G), \end{cases} \qquad x_2 = \begin{cases} w_2 & \text{if } w_2 \notin D_2(G) \\ w'_2 & \text{if } w_2 \in D_2(G), \end{cases} \quad (2)$$

and

$$x_3 = \begin{cases} w_3 & \text{if } w_{3-j} \notin D_2(G) \text{ in } G_j, j \in \{1, 2\} \\ w_2 & \text{if } w_2 \in D_2(G) \text{ in } G_1 \\ w_1 & \text{if } w_1 \in D_2(G) \text{ in } G_2. \end{cases} \quad (3)$$

The notation x_3 in (3) is for the convenience in our discussion below for G_1 and G_2, respectively. In G_1, if $w_2 \in D_2(G)$, then (3) defines $x_3 = w_2$ in G_1; if $w_2 \notin D_2(G)$, then (3) defines $x_3 = w_3$ (see Fig. 2 for G_1). Similarly, one can find what x_3 is in G_2 from (3).

Since G is essentially 4-edge-connected, by $D_3(G) = \emptyset$ and by (2),

$$d_G(x_i) \geq 4, \quad \text{if } 1 \leq i \leq 2. \quad (4)$$

By way of contradiction, suppose both G_1 and G_2 are not essentially 4-edge-connected. Then G_1 and G_2 have minimum essential edge cuts X and Y, respectively, such that $2 \leq |X| \leq 3$ and $2 \leq |Y| \leq 3$.

Claim 1. *For any essential edge cuts X in G_1 and Y in G_2 with $2 \leq |X| \leq 3$ and $2 \leq |Y| \leq 3$, $X \cap \{zx_1, x_2x_3\} = \emptyset$, and $Y \cap \{zx_2, x_1x_3\} = \emptyset$.*

We will prove the case for X only. The proof for Y is similar and hence omitted. By way of contradiction, suppose X contains either zx_1 or x_2x_3, (we may, without lose of generality, assume that z and x_2 are in the same component of $G_1 - X$), then define

$$X' = \begin{cases} (X - zx_1) \cup \{z_1w_1\} & \text{if } zx_1 \in X \text{ and } x_2x_3 \notin X \\ (X - x_2x_3) \cup \{z_1w_3\} & \text{if } x_2x_3 \in X \text{ and } zx_1 \notin X \\ (X - \{zx_1, x_2x_3\}) \cup \{z_1w_1, z_1w_3\} & \text{if } x_2x_3 \in X \text{ and } zx_1 \in X. \end{cases}$$

Thus, X' is an essential edge cut of G with $|X'| = |X|$, contrary to the assumption that G is essentially 4-edge-connected. Claim 1 is proved.

Since $X \cap \{zx_1, x_2x_3\} = \emptyset$, zx_1 and x_2x_3 must be in distinct components of $G_1 - X$. Let A_1 and A_2 be the two components of $G_1 - X$ with $zx_1 \in E(A_1)$ and $x_2x_3 \in E(A_2)$.

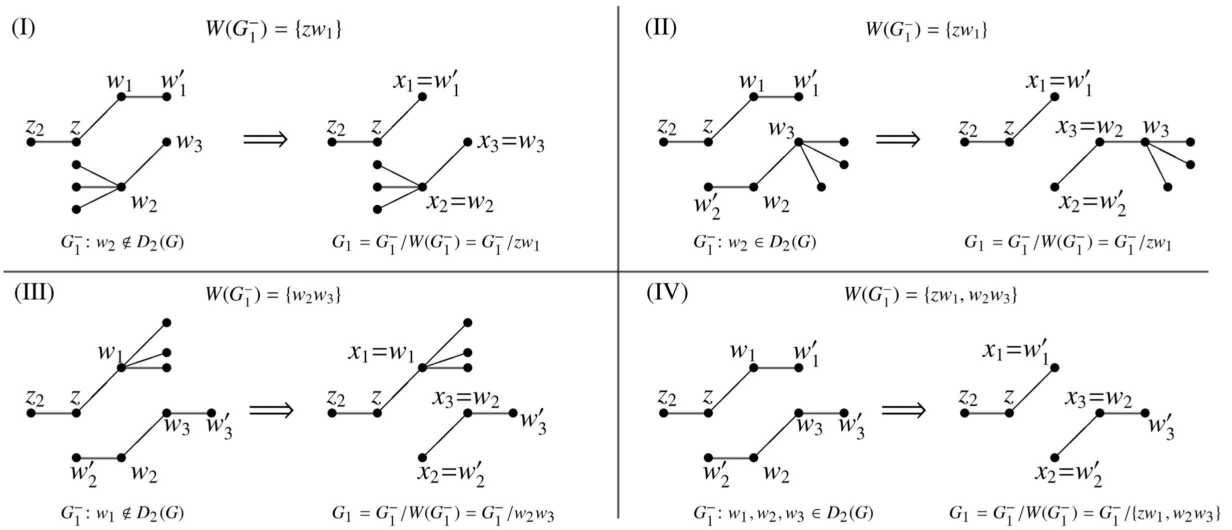

Fig. 2. All the cases of G_1 with labels x_1, x_2, and x_3 from G_1^- with $W(G_1^-) \neq \emptyset$.

Similarly, since $\{zx_2, x_1x_3\} \cap Y = \emptyset$, zx_2, x_1x_3 are in distinct components of $G_2 - Y$. Let B_1 and B_2 be the two components of $G_2 - Y$ such that $zx_2 \in E(B_1)$ and $x_1x_3 \in E(B_2)$. Hence

$$|\partial_{G_1}(A_1)| = |\partial_{G_1}(A_2)| = |X| \leq 3, \quad \text{and} \quad |\partial_{G_2}(B_1)| = |\partial_{G_2}(B_2)| = |Y| \leq 3. \tag{5}$$

By the definition of G_1 and G_2, $A_1 \cap B_1, A_1 \cap B_2, A_2 \cap B_1$ and $A_2 \cap B_2$ are subgraphs of G. Furthermore, we may assume that $z \in V(A_1 \cap B_1)$, $x_1 \in V(A_1 \cap B_2)$, and $x_2 \in V(A_2 \cap B_1)$.

Claim 2. $|\partial_G(A_1 \cap B_2)| \geq 4$ *and* $|\partial_G(A_2 \cap B_1)| \geq 4$.

By symmetry, we prove $|\partial_G(A_1 \cap B_2)| \geq 4$ only. By contradiction, suppose $|\partial_G(A_1 \cap B_2)| \leq 3$. Since G is 2-edge-connected and essentially 4-edge-connected with $D_3(G) = \emptyset$, we must have $|\partial_G(A_1 \cap B_2)| = 2$ and so $|V(A_1 \cap B_2)| = 1$. Hence $V(A_1 \cap B_2) = \{x_1\}$, contrary to (4). This proves Claim 2.

In the following, we define $\alpha_1 = |[A_1 \cap B_2, A_2 \cap B_2]|$, $\alpha_2 = |[A_1 \cap B_2, A_2 \cap B_1]|$, $\alpha_3 = |[A_1 \cap B_1, A_2 \cap B_1]|$, $\beta_1 = |[A_1 \cap B_1, A_1 \cap B_2]|$, $\beta_2 = |[A_1 \cap B_1, A_2 \cap B_2]|$, $\beta_3 = |[A_2 \cap B_1, A_2 \cap B_2]|$. Thus by (5),

$$\sum_{i=1}^{3} \alpha_i + \beta_2 = |X| \leq 3 \quad \text{and} \quad \sum_{i=1}^{3} \beta_i + \alpha_2 = |Y| \leq 3$$

and so

$$\alpha_1 + \alpha_2 + \alpha_3 \leq 3 - \beta_2 \quad \text{and} \quad \beta_1 + \beta_3 + \alpha_2 \leq 3 - \beta_2. \tag{6}$$

Note that

$$\partial_G(A_1 \cap B_2) \subseteq [A_1 \cap B_2, A_1 \cap B_1] \cup [A_1 \cap B_2, A_2 \cap B_1] \cup [A_1 \cap B_2, A_2 \cap B_2],$$
$$\partial_G(A_2 \cap B_1) \subseteq [A_2 \cap B_1, A_2 \cap B_2] \cup [A_2 \cap B_1, A_1 \cap B_1] \cup [A_2 \cap B_1, A_1 \cap B_2].$$

By Claim 2, we have

$$4 \leq |\partial_G(A_1 \cap B_2)| \leq \beta_1 + \alpha_2 + \alpha_1, \quad \text{and} \quad 4 \leq |\partial_G(A_2 \cap B_1)| \leq \beta_3 + \alpha_3 + \alpha_2. \tag{7}$$

By (7) and (6),

$$8 \leq \beta_1 + \beta_3 + \alpha_2 + \alpha_1 + \alpha_2 + \alpha_3 \leq 3 - \beta_2 + 3 - \beta_2 = 6 - 2\beta_2 \leq 6.$$

This contradiction establishes the theorem. □

3. Spanning trails in essentially 4-edge-connected graphs

For a reduced graph G with $\delta(G) \geq 2$, let $d_i = |D_i(G)|$. Then $|V(G)| = \sum_{i \geq 2} d_i$ and $2|E(G)| = \sum_{i \geq 2} id_i$, by Theorem 2.1(iii),

$$2F(G) = 4\sum_{i \geq 2} d_i - \sum_{i \geq 2} id_i - 4. \tag{8}$$

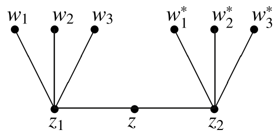

Fig. 3. $M(z)$ and $M^*(z)$ in G.

Hence, if $F(G) \geq 3$, then (8) implies

$$\sum_{i \geq 5}(i-4)d_i + 10 \leq 2d_2 + d_3. \tag{9}$$

We are now ready to prove the main result of this section, which will be needed to prove the conjecture $\xi(3) = 4$ in next section.

Theorem 3.1. *Let G be an essentially 4-edge-connected graph with $\delta(G) \geq 2$ and $|D_2(G) \cup D_3(G)| \leq 5$. Then each of the following holds.*

(i) *If $|D_2(G)| \leq 3$, then G is collapsible.*
(ii) *Either G is supereulerian or the reduction of G is $K_{2,5}$ such that all the vertices of degree 2 in the reduction are trivial.*
(iii) *If $|D_2(G)| \geq 2$, then for any pair of distinct vertices $u, v \in D_2(G)$, G has a spanning (u, v)-trail.*

Proof. Since G is an essentially 4-edge-connected graph with $\delta(G) \geq 2$, by Proposition 1.1, $\kappa'(G) \geq 2$. We argue by contradiction and assume that

$$G \text{ is a counterexample with } |V(G)| \text{ minimized.} \tag{10}$$

If G is collapsible, then Theorem 3.1(i) holds. Hence we may assume that G is not collapsible. Let G' be the reduction of G. Then $G' \neq K_1$ and $\kappa'(G') \geq 2$. If $F(G') \leq 2$, then by Theorem 2.1(ii) G' is a $K_{2,t}$ for some $t \geq 2$. Since G is essentially 4-edge-connected, we must have $t \in \{4, 5\}$ and any vertex in $D_2(G')$ must be a trivial contraction, and so we can view $D_2(G') \subseteq D_2(G)$. Thus, $|D_2(G)| \geq |D_2(G')| = t \geq 4$. If $t = 4$, then $K_{2,t} = K_{2,4}$ is Eulerian and so by Theorem 2.1(v) G is supereulerian. If G is not supereulerian, then the reduction of G must be $K_{2,5}$, and so Theorem 3.1(ii) must holds. Moreover, by inspection, if $u \in D_2(K_{2,t})$ and $v \in V(K_{2,t} - u)$, then $K_{2,t}$ always has a spanning (u, v)-trail, and so by Theorem 2.1(i), Theorem 3.1(iii) must hold. Hence we may assume that

$$\text{the reduction of } G \text{ is not a } K_{2,t} \text{ for any integer } t \geq 2. \tag{11}$$

Thus by Theorem 2.1(ii), $F(G') \geq 3$. By (10), we may assume that G is reduced. Thus, $G = G'$. By (9), $d_2 + d_3 \leq 5$. It follows from (9) that we must have $d_2 = 5$, $d_3 = 0$ and

$$V(G) = D_2(G) \cup D_4(G). \tag{12}$$

Hence, G must be Eulerian, and we are done for the proof of Theorem 3.1(i) and (ii). It remains to prove Theorem 3.1(iii).

We introduce the following notations in our argument. For each vertex $z \in D_2(G)$, let $N_G(z) = \{z_1, z_2\}$. As G is essentially 4-edge-connected, $z_1, z_2 \in D_4(G)$. Let $N_G(z_1) = \{w_1, w_2, w_3, z\}$ and $N_G(z_2) = \{w_1^*, w_2^*, w_3^*, z\}$. Define (see Fig. 3)

$$M(z) = \{w_1, w_2, w_3\} \quad \text{and} \quad M^*(z) = \{w_1^*, w_2^*, w_3^*\}.$$

By (10), there exist a pair of distinct vertices $u, v \in D_2(G)$ such that

$$G \text{ has not spanning } (u, v)\text{-trails.} \tag{13}$$

We proceed our proof by verifying the following claims and let $D_2(G) = \{a, b, c, u, v\}$.

Claim 1. *For any $z \in \{a, b, c\} = D_2(G) - \{u, v\}$,*

(a) *$|M(z) \cap D_2(G)| \geq 2$ and $|M^*(z) \cap D_2(G)| \geq 2$;*
(b) *$|M(z) \cap \{u, v\}| \geq 1$ and $|M^*(z) \cap \{u, v\}| \geq 1$.*

Proof of Claim 1(a). By symmetry, it suffices to show that $|M(z) \cap D_2(G)| \geq 2$. By contradiction, suppose $|M(z) \cap D_2(G)| \leq 1$. Then we may assume $M(z) \cap D_2(G) \subseteq \{w_3\}$.

Using the reduction method and the same notations in Theorem 2.2, we obtain two graphs G_1 and G_2 from G with $\delta(G_i) \geq 2$ and $D_3(G_i) = \emptyset\,(i = 1, 2)$. By Theorem 2.2, we may assume that G_1 is essentially 4-edge-connected. Since $M(z) \cap D_2(G) \subseteq \{w_3\}$, $w_1, w_2 \notin D_2(G)$, and by (1), we have $G_1 = (G - \{z_1\}) + \{zw_1, w_2w_3\}$, $x_1 = w_1$, $x_2 = w_2$ and $x_3 = w_3$. Thus we may view $D_2(G_1) = D_2(G)$. By (10), G_1 has a spanning (u, v)-trail H_1'. Since z has degree 2 in G_1 and $z \notin \{u, v\}$, $zx_1 \in E(H_1')$. Define

$$H_1 = \begin{cases} G[E(H_1' - zx_1) \cup \{zz_1, z_1w_1\}] & \text{if } x_2x_3 \notin E(H_1') \\ G[E(H_1' - \{zx_1, x_2x_3\}) \cup \{zz_1, z_1w_1, w_2z_1, z_1w_3\}] & \text{if } x_2x_3 \in E(H_1'). \end{cases}$$

Then H_1 is a spanning (u, v)-trial of G, contrary to (13). This proves Claim 1(a).

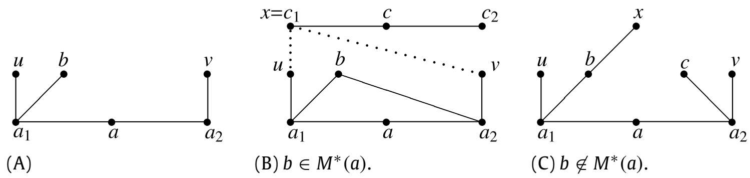

Fig. 4. Graphs in the proof of Claim 3 of Theorem 3.1.

Proof of Claim 1(b). By way of contradiction, suppose Claim 1(b) is not true. Let z be a vertex in $\{a, b, c\}$ such that $M(z)\cap\{u, v\} = \emptyset$. We may assume that $z = a$. By Claim 1(a), $|M(z)\cap D_2(G)| \geq 2$. Since $z = a \notin M(z)$ and $M(z)\cap\{u, v\} = \emptyset$, $M(z) \cap D_2(G) = D_2(G) - \{a, u, v\} = \{b, c\}$. We may assume that $w_1 = b$, and $w_2 = c$, and so $d_G(w_1) = d_G(w_2) = 2$ and $d_G(w_3) = 4$. Let $N_G(w_i) = \{z, w_i'\}$ $(i = 1, 2)$. Again using the reduction method on G as in Theorem 2.2, we obtained two graphs G_1 and G_2 with $\delta(G_i) \geq 2$ and $D_3(G_i) = \emptyset$ $(i = 1, 2)$. By Theorem 2.2, we may assume that G_1 is essentially 4-edge-connected. Then since $d_G(z) = d_G(w_1) = 2$, and $d_G(w_3) = 4$, $G_1^- = (G-\{z_1\}) + \{zw_1, w_2w_3\}$ with $W(G_1^-) = \{zw_1\} = \{zb\}$, and so $G_1 = G_1^-/zw_1$ with $z = \theta(zw_1)$ and $zw_1' \in E(G_1)$, and with $x_1 = w_1'$, $x_2 = w_2'$ and $x_3 = w_2 = c$ (see Fig. 2(II) for G_1). Thus, by (10), G_1 has a spanning (u, v)-trail H_0.

Since $\{z, x_3\} = \{a, c\} \subseteq D_2(G_1) - \{u, v\}$, $zx_1 = zw_1'$ and $x_2x_3 = w_2'w_2$ are both in $E(H_0)$. Since $d_{G_1}(w_2) = d_{G_1}(c) = d_G(c) = 2$ and $c \notin \{u, v\}$, w_2w_3 is also in $E(H_0)$. Define

$$H_1 = (H_0 - \{zx_1, w_2w_3\}) + \{zz_1, z_1w_1, w_1w_1', z_1w_2, z_1w_3\}.$$

Then H_1 is a spanning (u, v)-trail in G, a contradiction. Thus, Claim 1(b) is proved.

Claim 2. *For any $z \in D_2(G)$, $|D_2(G) \cap M(z) \cap M^*(z)| \leq 1$.*

By the definition of $M(z)$ and $M^*(z)$, $|D_2(G) \cap M(z) \cap M^*(z)| \leq 3$, where equality holds if and only if $G = K_{2,4}$. Since $|D_2(G)| = d_2 = 5$, $G \neq K_{2,4}$, and so $|D_2(G) \cap M(z) \cap M^*(z)| \leq 2$. If $|D_2(G) \cap M(z) \cap M^*(z)| = 2$, then we may assume that $w_1 = w_1^*$ and $w_2 = w_2^*$ in $D_2(G)$. Then $\{z_1w_3, z_2w_3^*\}$ is an essential edge cut of G, contrary to that G is essentially 4-edge-connected. This proves Claim 2.

Claim 3. *For all $y \in \{u, v\}$, $M(y) \cap M^*(y) \cap \{a, b, c\} = \emptyset$.*

Without loss of generality, we may assume $y = u$. By way of contradiction, suppose there is a vertex z in $\{a, b, c\}$ such that $z \in M(u)\cap M^*(u)$. Let $N_G(u) = \{u_1, u_2\}$. Then zu_1 and zu_2 are the two edges incident with z. Let $G_0 = G/zu_2$ with $u_2 = \theta(zu_2)$. Then $u_1u_2 \in E(G_0)$. Note G_0 has the same essentially edge-connectivity as G and $\delta(G_0) \geq 2$ with $|V(G_0)| < |V(G)|$. Therefore, by (10), G_0 has a spanning (u, v)-trail H_0.

If $u_1u_2 \in E(H_0)$, then $H = H_0 - u_1u_2 + \{u_1z, zu_2\}$ is a spanning (u, v)-trail in G, contrary to (13). If $u_1u_2 \notin E(H_0)$, then since H_0 is a spanning (u, v)-trail in G_0, one and only one of uu_1 or uu_2 (say uu_1) is in H_0, then $H = H_0 - uu_1 + \{uu_2, u_2z, zu_1\}$ is a spanning (u, v)-trail in G, a contradiction again. Claim 3 is proved.

For $\{a, b, c\} = D_2(G) - \{u, v\}$, let $N_G(a) = \{a_1, a_2\}$, $N_G(b) = \{b_1, b_2\}$, and $N_G(c) = \{c_1, c_2\}$. Then since G is essentially 4-edge-connected and by (12), $d(a_i) = d(b_i) = d(c_i) = 4$ where $i = 1, 2$. Let $S = N_G(a) \cup N_G(b) \cup N_G(c)$. If $|S| = 2$, then $S = N_G(a) = N_G(b) = N_G(c)$, contrary to Claim 2. Thus, $|S| \geq 3$. In the following, we assume $N_G(a) = \{a_1, a_2\} \subseteq S$ and let $x \in S - \{a_1, a_2\}$. Thus,

$$S = \{a_1, a_2, x, \ldots\}.$$

By Claim 1(a) and (b), $|M(a) \cap D_2(G)| \geq 2$, $|M^*(a) \cap D_2(G)| \geq 2$, $|M(a) \cap \{u, v\}| \geq 1$ and $|M^*(a) \cap \{u, v\}| \geq 1$. We may assume that $b \in M(a) = N_G(a_1) - \{a\}$, $u \in M(a) = N_G(a_1) - \{a\}$, and by Claim 3 $v \in M^*(a) = N_G(a_2) - \{a\}$ and so $u \notin M^*(a)$ and $v \notin M(a)$ (see the Fig. 4(A)).

Case 1. $b \in M^*(a)$ (see Fig. 4(B)).

Then $N_G(b) = \{a_1, a_2\} = N_G(a)$. Since $N_G(c) \subseteq S$, c must be adjacent to x, and so $x \in N_G(c)$. We may assume that $x = c_1$ and $M(c) = N_G(c_1) - \{c\}$. By Claim 1(a), $|M(c) \cap D_2(G)| \geq 2$, c_1 must be adjacent to another two degree 2 vertices in addition to c. Hence, since $N_G(a) = N_G(b)$, u and v must be the two vertices adjacent to c_1 and so $N_G(u) = \{a_1, c_1\}$ and $N_G(v) = \{a_2, c_1\}$. Therefore, the another vertex c_2 in $N_G(c)$ is not in $\{a_1, a_2\}$. Otherwise, $c \in M(u) \cap M^*(u)$ or $c \in M(v) \cap M^*(v)$, contrary to Claim 3. Note $M^*(c) = N_G(c_2) - \{c\}$. Thus,

$$D_2(G) \cap M^*(c) = \{a, b, u, v, c\} \cap M^*(c) = \emptyset,$$

contrary to Claim 1(a) that $|M^*(c) \cap D_2(G)| \geq 2$.

Case 2. $b \notin M^*(a)$ (see Fig. 4(C)).

Then by Claim 1(a), $M^*(a) = N_G(a_2) - \{a\}$ must have at least two degree 2 vertices, and so $c \in M^*(a) = N_G(a_2) - \{a\}$. Since $b \notin M^*(a)$, $N_G(b) \cap S \neq \emptyset$, and so we may assume $x \in N_G(b) - \{a_1\}$ (see Fig. 4(C)). Then since both u and b are adjacent to a_1, by Claim 3 u is not adjacent to x. By Claim 1(a), $M^*(b) = N_G(x) - \{b\}$ must have at least two degree 2 vertices other than b and u. Thus, v and c must be in $M^*(b) = N_G(x) - \{b\}$. Therefore, $N_G(v) = \{a_2, x\} = N_G(c)$, contrary to Claim 3.

We have a contradiction for each case above, and so the statement (13) is false. The theorem is proved. □

In Theorem 3.12 of [5], Catlin and Lai proved that if a 3-edge-connected graph G has at most 9 edge cuts of size 3, then G is supereulerian. For an essentially 4-edge-connected graph G with $\delta(G) \geq 3$, we have the following:

Theorem 3.2. *If G is an essentially 4-edge-connected graph with $\delta(G) \geq 3$ and $|D_3(G)| < 10$, then G is collapsible and has a spanning (u, v)-trail for any $u, v \in V(G)$.*

Proof. Since G is essentially 4-edge-connected with $\delta(G) \geq 3$, by Proposition 1.1, $\kappa'(G) \geq 3$. Let G' be the reduction of G. By way of contradiction, suppose G is not collapsible. Then $G' \neq K_1$ and $\kappa'(G') \geq 3$. Let $d_i = |D_i(G')|$. Then since $\kappa'(G') \geq 3$, $d_1 = d_2 = 0$. Since G is essentially 4-edge-connected, G does not have an essential edge cut of size 3, and so $d_3 = |D_3(G')| \leq |D_3(G)| < 10$. If $F(G') \leq 2$, then by Theorem 2.1(ii), $G' \in \{K_1, K_{2,t}\}$ $(t \geq 2)$, contrary to $G' \neq K_1$ and $\kappa'(G') \geq 3$. Hence, $F(G') \geq 3$, then by (9) and $d_2 = 0$,

$$\sum_{i\geq 5}(i-4)d_i + 10 \leq 2d_2 + d_3;$$

$$10 \leq d_3 < 10,$$

a contradiction. Thus, G must be collapsible. By Theorem 2.1(iv), for any $u, v \in V(G)$, G has a spanning (u, v)-trail. The theorem is proved. □

Remark. The Petersen Graph shows that Theorem 3.2 is best possible in the sense that $|D_3(G)| < 10$ is necessary.

4. Graphs that are 3-edge-Eulerian-connected

In this section, we shall investigate what graphs are 3-edge-Eulerian-connected. First, we prove the following theorem, as stated in Theorem 1.4, which proves the conjecture posed in [10].

Theorem 4.1. *If G is a 4-edge-connected graph, then G is 3-edge-Eulerian-connected and so $\xi(3) = 4$.*

Proof. Let G be a graph with $\kappa'(G) \geq 4$, and let $X \subseteq E(G)$ be an edge set with $|X| = 3$. Pick any pair of edges $e', e'' \in E(G) - X$. Let L be the graph obtained from G by subdividing each edge $e \in X \cup \{e', e''\}$ exactly once. (That is, for each edge $e = a_eb_e \in X \cup \{e', e''\}$, we replace e by a path $a_ev_eb_e$ by inserting a new vertex v_e). Then $D_2(L)$ is the set of the five degree 2 vertices generated by the subdivision, and L is 2-edge-connected and essentially 4-edge-connected. By Theorem 3.1(iii), L has a spanning $(v_{e'}, v_{e''})$-trail. This implies that G has a spanning (e', e'')-trail containing X, and so by definition, G is 3-edge-Eulerian-connected. □

As we know many 3-edge-connected graphs such as the Petersen graph have no spanning closed trail, the edge-connectivity in Theorem 4.1 cannot be lowered to 3-edge-connected. However, a 3-edge-Eulerian-connected graph is not necessarily 4-edge-connected. For example, let G be a graph obtained from K_n $(n \geq 8)$ and a vertex v by joining v to v_1 and v_2 with two edges vv_1 and vv_2, where $v_1, v_2 \in V(K_n)$ and $v \notin V(K_n)$. Then G is a 3-edge-Eulerian connected graph with $d(v) = 2$. We have the following necessary conditions for 3-edge-Eulerian-connected graphs.

Proposition 4.2. *Let G be a 3-edge-Eulerian-connected graph with $|E(G)| \geq 6$. Then G must be essentially 4-edge-connected with $D_3(G) = \emptyset$.*

Proof. We shall first show that G does not have an edge cut of size 3. By contradiction, assume that G an edge cut of G with $|X| = 3$. Let H_1 and H_2 be the two components of $G - X$ with $|E(H_1)| \leq |E(H_2)|$. Since G is 3-edge-Eulerian-connected with $|E(G)| \geq 6$ and $|X| = 3$, we may assume that $|E(H_2)| \geq 2$. Let e_1 and e_2 be two distinct edges in $E(H_2)$. Then G has a spanning (e_1, e_2)-trail T with $X \subseteq E(T)$. Since both $e_1, e_2 \in E(H_2)$, $T' = T/(H_2 \cap T)$ is a spanning closed trail of G/H_2 that contains X. Since T' is a spanning closed trail and X is an edge cut, $|X| = |E(T') \cap X| \equiv 0 \pmod 2$, contrary to that $|X| = 3$. Hence G does not have an edge cut of size 3 and so $D_3(G) = \emptyset$.

To show G is essentially 4-edge-connected, it suffices to show that G does not have an essential edge cut X' with $|X'| = 2$. By way of contradiction, suppose that such an edge cut X' exists and $G - X'$ has two components H_1' and H_2'. Since X' is an essential edge cut, we can pick an edge $e_i' \in E(H_i')$, $(1 \leq i \leq 2)$. Since $|X'| = 2 < 3$ and G is 3-edge-Eulerian-connected, G has a spanning (e_1', e_2')-trail T' such that $X' \subseteq E(T')$. Let e'' be an edge not in G joining the two end vertices of T'. Then $T' + e''$ is a spanning closed trail of $G + e''$, which contains a 3-edge-cut $X' \cup \{e''\}$ of $G + e''$. This yields a contradiction as the intersection of any close trail and any edge cut must have an even number of edges. □

Let G be the graph shown in Fig. 5 with $s \geq 6$, where v is a vertex of degree 2, and $e' \in E(H_1)$ and $e'' \in E(H_2)$. Let $X = \{e_1, e_2, e_3\}$ be the set of the three edges shown in Fig. 5. As we can see that a trail started from e_1 in H_1 must ended in H_1 after tracing through the three edges in X and vertex v. Hence, there is no spanning (e', e'')-trail T in G such that $X \subseteq E(T)$ and $V(T) = V(G)$. Thus, an essentially 4-edge-connected graph G with $D_3(G) = \emptyset$ may not be 3-edge-Eulerian connected. It remains a problem to completely characterize the structures of 3-edge-Eulerian connected graphs.

Let $G_0 = G - \{v\} + v_1v_2$. Then G_0 is 4-edge-connected and $X_0 = \{e_1, e_2, e_3, v_1v_2\}$ is an edge-cut of G_0. And G_0 has no spanning (e', e'')-trails containing X_0. This shows that Theorem 4.1 is best possible in the sense that 4-edge-connected graph G cannot be 4-edge-Eulerian connected.

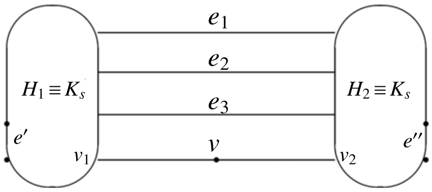

Fig. 5. G which is not 3-edge-Eulerian connected.

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