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On extremal k-supereulerian graphs

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ABSTRACT

A graph G is called *k-supereulerian* if it has a spanning even subgraph with at most k components. In this paper, we prove that any 2-edge-connected loopless graph of order n is $\lceil (n-2)/3 \rceil$-supereulerian, with only one exception. This result solves a conjecture in [Z. Niu, L. Xiong, Even factor of a graph with a bounded number of components, Australas. J. Combin. 48 (2010) 269–279]. As applications, we give a best possible size lower bound for a 2-edge-connected simple graph G with $n > 5k+2$ vertices to be k-supereulerian, a best possible minimum degree lower bound for a 2-edge-connected simple graph G such that its line graph $L(G)$ has a 2-factor with at most k components, for any given integer $k > 0$, and a sufficient condition for k-supereulerian graphs.

1. Introduction

Graphs in this paper are finite, undirected, and loopless. Undefined notation and terminology will follow [2]. Let G be a graph, and let $O(G)$ denote the set of all vertices in G with odd degrees. If $O(G) = \emptyset$, then G is called an *even* graph. An *Eulerian* graph is a connected graph G with $O(G) = \emptyset$. If a graph contains a spanning Eulerian subgraph, then it is called *supereulerian*. In particular, K_1 is supereulerian.

Boesch, Suffel, and Tindell [1] proposed the supereulerian graph problem: determine when a graph is supereulerian. They indicated that this might be a difficult problem. Pulleyblank [21] showed that such a decision problem, even when restricted to planar graphs, is NP-complete. Jaeger [14] and Catlin [5] independently showed that every 4-edge-connected graph is supereulerian.

Let G be a graph, and let $X \subseteq E(G)$. The *contraction* G/X is the graph obtained from G by contracting each edge of X and deleting the resulting loops. For $H \subset G$, we write G/H for $G/E(H)$. If H is a connected subgraph of G, and if v_H denotes the vertex in G/H to which H is contracted, then H is called the *preimage* of v_H. A vertex v in a contraction of G is *nontrivial* if v has a nontrivial preimage.

On extremal supereulerian graph problems, Cai [4] proved the following result.

Theorem 1 (*Cai, [4]*). *Let G be a 2-edge-connected simple graph of order n. If*

$$|E(G)| \geq \binom{n-4}{2} + 6, \tag{1}$$

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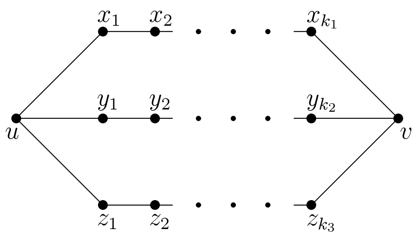

Fig. 1. $K_{2,3}(k_1, k_2, k_3)$.

then exactly one of the following holds.

(a) *G is supereulerian.*
(b) *Equality holds in (1), and G has a complete subgraph H of order $n-4$ such that $G/H = K_{2,3}$.*
(c) *G is either $K_{2,5}$ or the cube minus a vertex.*

For 3-edge-connected graphs, Catlin and Chen proved a similar result, which was conjectured by Cai [4].

Theorem 2 (*Catlin and Chen, [8]*). *Let G be a 3-edge-connected simple graph of order n. If $|E(G)| \geq \binom{n-9}{2} + 16$, then G is supereulerian.*

A graph G is called *k-supereulerian* if G has a spanning even subgraph with at most k components. Hence, a k-supereulerian graph is also $(k+1)$-supereulerian, but not vice versa. Let k_1, k_2, k_3 be three positive integers, u, v the vertices of $K_{2,3}$ with degree 3, and $K_{2,3}(k_1, k_2, k_3)$ the graph obtained from $K_{2,3}$ by replacing each $u-v$ path by a path of length k_i+1, as shown in Fig. 1. By definition, $K_{2,3}(1, 1, 1) = K_{2,3}$, and $K_{2,3}(k_1, k_2, k_3)$ is $(\min\{k_1, k_2, k_3\}+1)$-supereulerian, but not $(\min\{k_1, k_2, k_3\})$-supereulerian.

Motivated by the two results above, we investigate the extremal size of k-supereulerian graphs, and obtain the following result.

Theorem 3. *Let $k > 1$ be an integer, and G a 2-edge-connected simple graph of order $n > 5k+2$. If*

$$|E(G)| \geq \binom{n-3k-1}{2} + 3k + 3, \tag{2}$$

then exactly one of the following holds.

(a) *G is k-supereulerian.*
(b) *Equality holds in (2), and G has a complete subgraph H of order $n-3k-1$ such that $G/H = K_{2,3}(k, k, k)$, where $K_{2,3}(k, k, k)$ is depicted in Fig. 1 when $k_1 = k_2 = k_3 = k$.*

A graph H is *collapsible* if, for every subset $X \subseteq V(H)$ with $|X| \equiv 0 \pmod 2$, H has a spanning connected subgraph H_X with $O(H_X) = X$. In [5], Catlin showed that any graph G has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$ such that $\bigcup_{i=1}^{c} V(H_i) = V(G)$. The *reduction* of G, denoted by G', is the graph obtained from G by contracting each H_i $(1 \leq i \leq c)$ to a single vertex. A graph G is *reduced* if $G = G'$. The following result is key in the proof of Theorem 3.

Theorem 4. *Let G be a 2-edge-connected reduced graph of order n, and k a positive integer such that $n \leq 3k+2$. Then G is either k-supereulerian or isomorphic to the graph $K_{2,3}(k, k, k)$.*

Theorem 4 is indeed a conjecture in [19], which is equivalent to saying that every 2-edge-connected loopless graph G of order n is either $\lceil (n-2)/3 \rceil$-supereulerian or $n-2 \equiv 0 \pmod 3$, and $G \cong K_{2,3}(\frac{n-2}{3}, \frac{n-2}{3}, \frac{n-2}{3})$; see Theorem 20 and Proposition 21 for details. In [19], Niu and Xiong proved a similar result, stating that every 2-edge-connected reduced graph G of order $n \leq 3k+1 \leq 10$ is k-supereulerian, which was proved by analyzing the structure of G according to the different values of the circumference of G, and then by showing that G has a spanning even subgraph with at most k components. This proof technique fails when n is large, as the number of possible cases grows very quickly, and the structure of G becomes much more complicated. In this paper, we use a completely different approach, which utilizes the splitting lemma of Fleischner [12] and a result on perfect matchings in cubic graphs of Edmonds [11], to prove Theorem 4.

By a smallest graph in some collection of graphs we mean a graph with the least order, and with the least size amongst all graphs of that order in the collection. As an example, $K_{2,3}$ is the smallest 2-edge-connected non-supereulerian graph. As an extension, our result above implies that $K_{2,3}(k, k, k)$ is the smallest 2-edge-connected non-k-supereulerian graph.

In Section 2, we will assume the validity of Theorem 4 to prove Theorem 3, and present some other applications of Theorem 4, whose proof will be postponed to Section 3.

2. Applications of Theorem 4

2.1. Proof of Theorem 3

In this subsection, we use Theorem 4 to prove Theorem 3. First, we present some necessary results.

Theorem 5 (*Catlin, [5]*). *If G is reduced, then G is simple and triangle free, and with either $G \in \{K_1, K_2\}$ or $|E(G)| \le 2|V(G)|-4$.*

Catlin [5] proved that a connected graph G is supereulerian if and only if its reduction G' is supereulerian. Niu et al. extended this result to k-supereulerian graphs.

Theorem 6 (*Niu, Lai and Xiong, [18]*). *Let G be a connected graph, and G' the reduction of G. Then G is k-supereulerian if and only if G' is k-supereulerian.*

Let $F(G)$ denote the minimum number of edges that must be added to G in order to obtain a supergraph that has two edge-disjoint spanning trees. Catlin [6] showed that, if G is reduced, then

$$F(G) = 2|V(G)| - |E(G)| - 2. \tag{3}$$

Corollary 7 (*Niu, Lai and Xiong, [18]*). *Let G be a 2-edge-connected graph. If $F(G) \le k$, then G is k-supereulerian.*

Theorem 8 (*Catlin and Chen, [8]*). *Let G be a 2-edge-connected simple graph of order n, and let $p > 1$ be an integer. If*

$$|E(G)| \ge \binom{n-p+1}{2} + 2p - 4, \tag{4}$$

then one of the following holds.

(a) *The reduction of G has order less then p.*
(b) *Equality holds in (4), G has a complete subgraph H of order $n-p+1$, and the reduction of G is $G' = G/H$, a graph of order p and size $2p-4$.*
(c) *G is a reduced graph such that either $|E(G)| \in \{2n-4, 2n-5\}$ and $n \in \{p+1, p+2\}$, or $|E(G)| = 2n-4$ and $n = p+3$.*

Now, we prove Theorem 3.

Proof of Theorem 3. We need to discuss the following two cases by considering the size of G. Let G' be the reduction of G.

Case 1. $|E(G)| \ge \binom{n-3k-1}{2} + 6k$.

Let $p = 3k+2$. Then $n-p+1 = n-3k-1$ and $2p-4 = 6k$. Hence, (4) holds. In the following, we check the three cases of Theorem 8, and show that G is k-supereulerian in each case.

If (a) of Theorem 8 holds, then $|V(G')| < 3k+2$. Note that $|V(K_{2,3}(k,k,k))| = 3k+2$. By Theorem 4, G' is k-supereulerian. Then G is k-supereulerian by Theorem 6.

If (b) of Theorem 8 holds, then $|E(G)| = \binom{n-3k-1}{2} + 6k$. There exists a complete subgraph H of G with $|V(H)| = n-3k-1$, and $G' = G/H$. That is to say, $|V(G')| = 3k+2$, and $|E(G')| = 6k$. Note that $|E(K_{2,3}(k,k,k))| = 3k+3 < 6k$. By Theorem 4, G' is k-supereulerian. Then G is k-supereulerian by Theorem 6.

If (c) of Theorem 8 holds, then $G = G'$, $|E(G)| \in \{2n-4, 2n-5\}$, and $n \in \{p+1, p+2, p+3\}$. Hence, by (3), $F(G) \in \{2, 3\}$. If $F(G) \le k$, then, by Corollary 7, G is k-supereulerian. So we need to consider the remaining case when $k = 2$ and $F(G) = 3$. Hence, $p = 8$, and then $n \in \{9, 10, 11\}$, contrary to $n > 5k+2 = 12$.

Case 2. $\binom{n-3k-1}{2} + 3k + 3 \le |E(G)| \le \binom{n-3k-1}{2} + 6k - 1$.

As K_1 is supereulerian, we may assume that G' is 2-edge-connected and that $|V(G')| \ge 2$.

By (3), $F(G') = 2|V(G')| - |E(G')| - 2$. If $F(G') \le k$, then, by Corollary 7, G' is k-supereulerian, and then G is k-supereulerian by Theorem 6. Hence, it suffices to consider $F(G') \ge k+1$ in the following.

Let $e = |E(G)|$, $n' = |V(G')|$, and $e' = |E(G')|$. Then $\binom{n-3k-1}{2} + 3k + 3 \le e \le \binom{n-3k-1}{2} + 6k - 1$. For any graph H, we use $e(H)$ to denote $|E(H)|$. Suppose that $H_1, H_2, \ldots, H_m$ are all the maximal collapsible subgraphs of G such that G' is obtained from G by contracting $H_1, H_2, \ldots, H_m$. Assume that $n_i = |V(H_i)|$ for each $i \in \{1, 2, \ldots, m\}$. Since contracting an induced subgraph H does not affected the validity of $e = e(H) + e(G/H)$, and since all maximal collapsible subgraphs are induced, we can contract $H_1, H_2, \ldots, H_m$ in succession, and then

$$\begin{aligned} e &= e' + e(H_1) + e(H_2) + \cdots + e(H_m) \\ &\le e' + \binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_m}{2} \end{aligned}$$

and

$$n = n' + (n_1 - 1) + (n_2 - 1) + \cdots + (n_m - 1),$$

i.e.,

$$n + m - n' = n_1 + n_2 + \cdots + n_m.$$

Since $F(G') \geq k + 1$, by (3), we have $2n' - e' - 2 \geq k + 1$, i.e., $e' \leq 2n' - k - 3$. So

$$\begin{aligned} e &\leq e' + \binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_m}{2} \\ &\leq 2n' - k - 3 + \binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_m}{2}. \end{aligned}$$

Now, we define a function

$$\begin{aligned} f(n_1, n_2, \ldots, n_m) &= 2n' - k - 3 + \binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_m}{2} \\ &= 2n' - k - 3 + \frac{1}{2}(n_1^2 - n_1) + \frac{1}{2}(n_2^2 - n_2) + \cdots + \frac{1}{2}(n_m^2 - n_m) \end{aligned}$$

subject to $n_1 + n_2 + \cdots + n_m = n + m - n'$. By convexity, $f(n_1, n_2, \ldots, n_m)$ reaches its maximum value when $m = 1$, i.e., $n_1 = n + 1 - n'$ and $n_2 = n_3 = \cdots = n_m = 0$. So $e \leq 2n' - k - 3 + \binom{n+1-n'}{2}$.

If G is reduced, then $e = e'$ and $n = n'$. Since $e' \leq 2n' - k - 3$ and $k > 1$, we have $e \leq 2n - 5$, contrary to (2) when $n > 5k + 2$. Hence, G has at least one nontrivial collapsible subgraph. Note that K_3 is the nontrivial collapsible simple graph with the smallest order. We have $n' \leq n - 2$. Define a new function

$$\begin{aligned} g(n') &= 2n' - k - 3 + \binom{n + 1 - n'}{2} \\ &= \frac{1}{2}n'^2 + \left(\frac{3}{2} - n\right)n' + \left(\frac{1}{2}n^2 + \frac{1}{2}n - k - 3\right). \end{aligned}$$

The symmetric axis of this parabolic function $g(n')$ is $n' = n - 3/2$. Then $g(n')$ is decreasing when $n' \leq n - 3/2$.

By the definitions of functions f and g, $g(n')$ is always an upper bound of e. If $n' = 3k + 3$, then

$$\begin{aligned} g(3k + 3) &= \frac{1}{2}n^2 - \frac{6k + 5}{2}n + \frac{9k^2 + 25k + 12}{2} \\ &= \frac{1}{2}n^2 - \frac{6k + 3}{2}n + \frac{9k^2 + 15k + 8}{2} - n + 5k + 2 \\ &= \binom{n - 3k - 1}{2} + 3k + 3 - (n - 5k - 2) \\ &< e, \end{aligned}$$

when $n > 5k + 2$, contrary to $e \leq g(n')$.

As $n' \leq n - 2$, $g(n')$ is decreasing. Hence, we have $n' \leq 3k + 2$. By Theorem 4, G' is either k-supereulerian or the graph $K_{2,3}(k, k, k)$. In the former case, G is k-supereulerian by Theorem 6, so (a) of Theorem 3 holds. In the latter case, $n' = 3k + 2$, $e' = 3k + 3$, and then $e \leq e' + \binom{n-n'+1}{2} = 3k + 3 + \binom{n-3k-1}{2}$. By (2), we have $e = 3k + 3 + \binom{n-3k-1}{2}$, which implies that G has a complete subgraph H of order $n - 3k - 1$ such that $G/H = K_{2,3}(k, k, k)$. Hence, (b) of Theorem 3 holds.

This completes the proof of Theorem 3. □

2.2. The number of components of an even factor

An *even factor* of G is a spanning subgraph of G in which every vertex has a positive even degree. A 2-*factor* of G is a spanning subgraph in which every vertex has degree 2. In this subsection, we use Theorem 4 to prove some sufficient conditions for even factors of a graph and 2-factors of its line graph.

Note that a graph is k-supereulerian if it has a spanning even subgraph with at most k components. If G has an even factor with at most k components, then G is k-supereulerian, whereas the converse is not true in general; see [18].

There exist many minimum degree conditions guaranteeing the existence of certain factors of a graph, such as Hamiltonian cycles and spanning Eulerian subgraphs; see, e.g., [5,7,10]. In [19], Niu and Xiong obtained several minimum degree conditions for a graph to have an even factor with a bounded number of components, one of which is the following.

Theorem 9 (*Niu and Xiong, [19]*). *Let G be a 2-edge-connected simple graph of order n, and $k \in \{1, 2, 3\}$ such that $\delta(G) \geq \lfloor \frac{n}{3k+1} \rfloor - 1$. If n is sufficiently large relative to k, then G has an even factor with at most k components.*

We extend this result to general cases, and give a bit weaker minimum degree condition, with only one exception.

Theorem 10. *Let G be a 2-edge-connected simple graph of order n, and k a positive integer such that $\delta(G) \geq \lfloor \frac{n}{3k+2} \rfloor - 1$. If n is sufficiently large relative to k, then exactly one of the following holds.*

(a) *G has an even factor with at most k components.*
(b) *G', the reduction of G, is $K_{2,3}(k, k, k)$, and G has an even factor with exactly $k + 1$ components.*

We first present a necessary result for our proof.

Theorem 11 (*Niu and Xiong, [19]*). *Let p be a positive integer, and G a connected simple graph of order n such that*

$$\delta(G) \geq \lfloor n/p \rfloor - 1. \tag{5}$$

If n is sufficiently large relative to p, then the reduction G' of G satisfies $|V(G')| \leq p$, and each vertex of G' is nontrivial.

Now, we prove Theorem 10.

Proof of Theorem 10. By Theorem 11, $|V(G')| \leq 3k+2$, and each vertex of G' is nontrivial. Then, by Theorem 4, G' is either k-supereulerian or the graph $K_{2,3}(k, k, k)$. In the former case, G' has a spanning even subgraph with at most k components $L_1, L_2, \ldots, L_l$, where $l \leq k$. For each L_i, let $L_i^* = G[\cup_{v \in V(L_i)} V(H_v)]$, where H_v is the preimage of $v \in V(L_i)$. Since each vertex of G' is nontrivial, then, by Theorem 6, each L_i^* is supereulerian and nontrivial. By the definitions of collapsible graphs and contraction, $\bigcup_{1 \leq i \leq l} V(L_i^*) = V(G)$ and $V(L_i^*) \cap V(L_j^*) = \emptyset$ for $i \neq j$. Hence, G has an even factor with l ($\leq k$) components, so (a) of Theorem 10 holds. In the latter case, G' is $(k+1)$-supereulerian. Then, by arguing similarly as the above case, G has an even factor with exactly $k+1$ components, so (b) holds. □

By Theorem 10, we obtain the following corollary immediately, which extends a theorem (Theorem 9 in [5]) of Catlin and improves a theorem (Theorem 8 in [18]) of Niu et al.

Corollary 12. *Let G be a 2-edge-connected simple graph of order n, and k a positive integer such that $\delta(G) \geq \lfloor \frac{n}{3k+2} \rfloor - 1$. If n is sufficiently large relative to k, then exactly one of the following holds.*

(a) *G is k-supereulerian.*
(b) *G', the reduction of G, is $K_{2,3}(k, k, k)$.*

Let $G = (V(G), E(G))$ be a graph. The *line graph* $L(G)$ of G is the graph with $V(L(G)) = E(G)$, and $x, y \in V(L(G))$ are adjacent as vertices if and only if they are adjacent as edges in G. Let G be a simple graph with $\delta(G) \geq 3$, and let S be a set of mutually edge-disjoint connected even nontrivial subgraphs and stars ($K_{1,s}$, where $s \geq 3$ is an integer). If each star has at least three edges, and every edge in $E(G) \setminus \cup_{L \in S} E(L)$ is incident to an even subgraph in S, then S is called a *system that dominates G*.

Theorem 13 (*Gould and Hynds, [13]*). *Let G be a simple graph. Then L(G) has a 2-factor with c components if and only if there is a system that dominates G with c elements.*

Theorem 13 shows a close relationship between a system that dominates G with c elements and a 2-factor of $L(G)$ with the same number of components. From Theorems 10 and 13, one can easily obtain the following result.

Corollary 14. *Let G be a 2-edge-connected simple graph of order n, L(G) the line graph of G, and k a positive integer such that $\delta(G) \geq \lfloor \frac{n}{3k+2} \rfloor - 1$. If n is sufficiently large relative to k, then exactly one of the following holds.*

(a) *L(G) has a 2-factor with at most k components.*
(b) *G', the reduction of G, is $K_{2,3}(k, k, k)$, and L(G) has a 2-factor with exactly $k + 1$ components.*

2.3. A sufficient condition for k-supereulerian graphs

A *bond* of G is a minimal nonempty edge cut. Let $l > 0$, $m \geq 0$ be integers, and let $C_2(l, m)$ denote the graph family such that a graph G of order n is in $C_2(l, m)$ if and only if G is 2-edge-connected and such that, for every bond $S \subset E(G)$ with $|S| \leq 3$, each component of $G - S$ has order at least $(n - m)/l$.

Catlin and Li [9] were the first to investigate characterizations of supereulerian graphs in $C_2(m, l)$. They proved that a graph $G \in C_2(5, 0)$ is supereulerian if and only if G is not contractible to $K_{2,3}$. Since then, a series of characterizations of supereulerian graphs in $C_2(m, l)$ has been done; see [3,15–17]. In [20], Niu and Xiong considered a similar problem on k-supereulerian graphs, and proved the following theorem.

Theorem 15 (*Niu and Xiong, [20]*). *Let $6 \le l \le 10$ be an integer, and $G \in C_2(l, 0)$ be a graph of order n. Then G is $(l-4)$-supereulerian.*

In this subsection, we extend this result to general cases.

Theorem 16. *Let $l \ge 6$ be an integer, and $G \in C_2(l, 0)$ be a graph of order n. Then G is $(l-4)$-supereulerian.*

Let $D_i(G) = \{v \in V(G) \mid d(v) = i\}$ and $d_i(G) = |D_i(G)|$.

Theorem 17 (*Catlin, [5]*). *If G is a nontrivial 2-edge-connected reduced graph, then $d_2(G) + d_3(G) \ge 4$. If $d_2(G) + d_3(G) = 4$, then G is Eulerian, and G has four vertices of degree 2.*

Lemma 18 (*Niu and Xiong, [20]*). *Let $G \in C_2(l, m)$ be a graph with $n = |V(G)| > (l+1)m$. Then either $G' = K_1$ or $d_2(G') + d_3(G') \le l$, where G' is the reduction of G.*

Lemma 19 (*Niu and Xiong, [20]*). *Let G be a 2-edge-connected reduced graph, and $d_i = d_i(G)$. Then*

$$2F(G) + 4 + \sum_{j\ge 5}(j-4)d_j = 2d_2 + d_3.$$

Now, we prove Theorem 16.

Proof of Theorem 16. By Theorem 15, we may assume that $l \ge 11$. Let G' be the reduction of G. By Theorem 6, it suffices to show that G' is $(l-4)$-supereulerian. Since K_1 is supereulerian, if $G' = K_1$, then we are done. So we may assume that G' is 2-edge-connected and nontrivial. Let $d_i = |D_i(G')|$.

By Theorem 17, if $d_2 + d_3 = 4$, then G' is Eulerian. By Lemma 18, $d_2 + d_3 \le l$. Therefore, we only consider the case when $5 \le d_2 + d_3 \le l$. We shall assume that

$$G' \text{ is not } (l-4)\text{-supereulerian}, \tag{6}$$

to find a contradiction.

Case 1. $5 \le d_2 + d_3 \le l - 1$.

If $F(G') \le l - 4$, by Corollary 7, G' is $(l-4)$-supereulerian, contrary to (6). So we may assume that $F(G') \ge l - 3$. From Lemma 19, and since $d_2 + d_3 \le l - 1$, we have

$$2(l-1) + \sum_{j\ge 5}(j-4)d_j \le 2F(G') + 4 + \sum_{j\ge 5}(j-4)d_j = 2d_2 + d_3 \le 2(d_2 + d_3) \le 2(l-1).$$

Hence, equalities must hold everywhere, implying that $d_2 = l - 1$, $d_3 = 0$, and $d_j = 0$ $(j \ge 5)$. Thus G' is Eulerian, contrary to (6).

Case 2. $d_2 + d_3 = l$.

Let $H_1, H_2, \ldots, H_l$ denote the subgraphs of G whose contraction images in G' are the vertices of degree at most 3 in G'. Since $G \in C_2(l, 0)$, for each i with $1 \le i \le l$, $|V(H_i)| \ge n/l$. It follows that

$$n = |V(G)| \ge \sum_{i=1}^{l} |V(H_i)| \ge \frac{ln}{l} = n,$$

and hence $|V(G')| = l$. Denote $l = 3k + j$, where $j \in \{0, 1, 2\}$. By Theorem 4, G' is either k-supereulerian or the graph $K_{2,3}(k, k, k)$, which is $(k+1)$-supereulerian. Since $l \ge 11$, we have $k < k + 1 \le l - 4$, and then G' is $(l-4)$-supereulerian, contrary to (6).

This completes the proof of Theorem 16. □

3. Proof of Theorem 4

In this section, for presentational convenience, we shall show the validity of Theorem 4 by proving the following equivalence form.

Theorem 20. *Let G be a 2-edge-connected graph of order $n \ge 3$. Then exactly one of the following holds.*

(a) *G is $\lceil \frac{n-2}{3} \rceil$-supereulerian.*
(b) *$n - 2 \equiv 0 \pmod 3$, and $G \cong K_{2,3}(\frac{n-2}{3}, \frac{n-2}{3}, \frac{n-2}{3})$.*

Proposition 21. *Theorem 4 is equivalent to Theorem 20.*

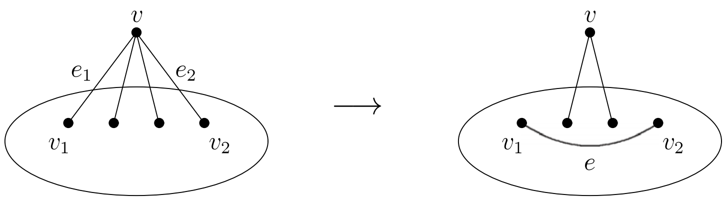

Fig. 2. Splitting off the edges e_1 and e_2 from v.

Proof. First, we show that Theorem 20 implies Theorem 4. Let G be a graph of order n satisfying the hypotheses of Theorem 4. If $n < 3$, then, since G is a 2-edge-connected reduced graph, we have $G \cong K_1$, which is supereulerian. Hence, we may assume that $n \geq 3$. By Theorem 20, G is either $\lceil \frac{n-2}{3} \rceil$-supereulerian or the graph $K_{2,3}(\frac{n-2}{3}, \frac{n-2}{3}, \frac{n-2}{3})$. Note that $n \leq 3k + 2$. In the former case, G is k-supereulerian since $\lceil \frac{n-2}{3} \rceil \leq k$ and by the definition of k-supereulerian graphs. In the latter case, we have $n - 2 \equiv 0 \pmod 3$. If $\frac{n-2}{3} < k$, then G is k-supereulerian; else $\frac{n-2}{3} = k$, i.e., $G \cong K_{2,3}(k, k, k)$. So Theorem 4 holds.

Conversely, let G be a graph satisfying the hypotheses of Theorem 20, let $n = 3k + j$, where k is a positive integer and $j \in \{0, 1, 2\}$, and let G' be the reduction of G. Then $n(G') \leq n = 3k + j \leq 3k + 2$. By Theorem 4, G' is either k-supereulerian or the graph $K_{2,3}(k, k, k)$. In the former case, G is $\lceil \frac{n-2}{3} \rceil$-supereulerian by the fact that $\lceil \frac{n-2}{3} \rceil = k$ and by Theorem 6. In the latter case, we have $n(G') = n = 3k + 2$, and then $\frac{n-2}{3} = k$. Theorem 20 holds. □

Before proving Theorem 20, we present several auxiliary results.

Let v be a vertex of a graph G, and let $e_1 = vv_1$ and $e_2 = vv_2$ be two edges of G incident to v. The operation of *splitting off* the edges e_1 and e_2 from v consists of deleting e_1 and e_2 and then adding a new edge e joining v_1 and v_2, depicted in Fig. 2. The following theorem, due to Fleischner, shows that under certain conditions this operation can be performed without creating cut edges.

Theorem 22 (*Fleischner, [12]*)**.** *Let G be a 2-connected graph, and v a vertex of G of degree at least four with at least two distinct neighbors. Then some two non-multiple edges incident to v can be split off so that the resulting graph is connected and has no cut edges.*

For $S \subseteq V(G)$ and $E \subseteq E(G)$, let $G - S$ and $G - E$ denote the subgraph obtained from G by deleting all the vertices in S and the subgraph obtained from G by deleting all the edges in E, respectively. For $H \subseteq G$, we denote $G - V(H)$ by $G - H$, for abbreviation. For $e = uv \notin E(G)$ with $u, v \in V(G)$, let $G + e$ denote the graph obtained by adding e to G. We present a lemma and a theorem of Edmonds, which are used in the proof of Theorem 20.

Lemma 23. *Let G be a 2-edge-connected graph, v a vertex of G, and e an edge of G.*

(a) *If G^* is a graph obtained from G by splitting off two edges incident to v, and $G^* \cong K_{2,3}(k, k, k)$, then G is k-supereulerian.*
(b) *If $G^* = G - e$ and $G^* \cong K_{2,3}(k, k, k)$, then G is k-supereulerian.*

Proof. (a) Note that $G^*(\cong K_{2,3}(k, k, k))$ is $(k + 1)$-supereulerian. It is easy to check that the number of supereulerian components of all the graphs obtained from G^* by deleting any edge u_1u_2 and adding two edges u_1u and u_2u, where $u \in V(G^*) \setminus \{u_1, u_2\}$ (this procedure can be looked upon as the reverse of splitting off two adjacent edges), will reduce by at least 1. Hence, G is k-supereulerian.

(b) Note that adding a new edge to G^* will reduce at least one supereulerian component. G is k-supereulerian. □

A graph is called *k-regular* if all vertices have degree k. A *perfect matching* in a graph is a spanning 1-regular subgraph.

Theorem 24 (*Edmonds, [11]*)**.** *For every 2-edge-connected 3-regular graph, there exists a constant p and $3p$ perfect matchings such that each edge is in p of them.*

For a path $P = x_0x_1 \dots x_{k-1}x_k$, the vertices $x_1, \dots, x_{k-1}$ are called the internal vertices of P. Let $\mathring{P} = x_1 \dots x_{k-1}$ be the subpath of P induced by its internal vertices. In the following, let $n_c(G)$ denote the number of components of G.

Now, we prove Theorem 20.

Proof of Theorem 20. We shall assume that Theorem 20 does not hold, to find a contradiction. Let G be a counterexample of Theorem 20 with $|E(G)|$ minimized.

First, we prove the following two claims.

Claim 1. *G is 2-connected.*

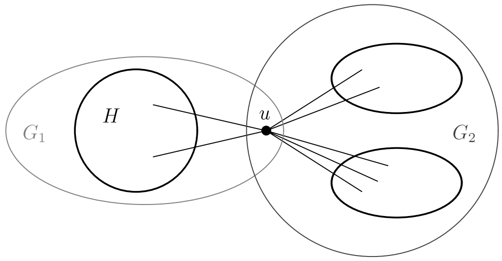

Fig. 3. The subgraphs G_1 and G_2 of G.

Proof of Claim 1. Suppose, to the contrary, that G has a cut vertex u. Let H be a component of $G-u$, $G_1 = G[V(H) \cup \{u\}]$, $n_1 = |V(G_1)|$and $G_2 = G - V(H)$, $n_2 = |V(G_2)|$, depicted in Fig. 3. Then $G_1 \cup G_2 = G$, $G_1 \cap G_2 = \{u\}$, $n = n_1 + n_2 - 1$, both G_1 and G_2 are 2-edge-connected.

For $i = 1, 2$, by the 2-edge-connectivity of G, we have $n_i \geq 3$. Since $|E(G_i)| < |E(G)|$ and by the minimality of G, either G_i is $\lceil \frac{n_i-2}{3} \rceil$-supereulerian or $n_i - 2 \equiv 0 \pmod 3$, and $G_i \cong K_{2,3}(\frac{n_i-2}{3}, \frac{n_i-2}{3}, \frac{n_i-2}{3})$. Now, we distinguish the following three cases.

Case 1. For $i = 1, 2$, G_i is $\lceil \frac{n_i-2}{3} \rceil$-supereulerian.

Denote $n_i = 3k_i + j_i$, where $j_i \in \{0, 1, 2\}$. Then $\lceil \frac{n_i-2}{3} \rceil = k_i$, and hence G_i is k_i-supereulerian. Note that $G_1 \cup G_2 = G$, $G_1 \cap G_2 = \{u\}$. G is $(k_1 + k_2 - 1)$-supereulerian. Since

$$k_1 + k_2 - 1 = \frac{3k_1 + 3k_2 - 1 - 2}{3} \leq \frac{3k_1 + j_1 + 3k_2 + j_2 - 1 - 2}{3} = \frac{n-2}{3} \leq \left\lceil \frac{n-2}{3} \right\rceil,$$

G is $\lceil \frac{n-2}{3} \rceil$-supereulerian, a contradiction.

Case 2. Exactly one of G_i $(i = 1, 2)$ $(G_1$, say) is $\lceil \frac{n_i-2}{3} \rceil$-supereulerian.

Denote $n_1 = 3k_1 + j$, where $j \in \{0, 1, 2\}$, and $n_2 = 3k_2 + 2$. Then $\lceil \frac{n_1-2}{3} \rceil = k_1$ and $\frac{n_2-2}{3} = k_2$, and hence G_1 is k_1-supereulerian, and G_2 is $(k_2 + 1)$-supereulerian. Thus, G is $(k_1 + k_2)$-supereulerian. Since

$$k_1 + k_2 = \left\lceil \frac{3k_1 + 3k_2 + 2 - 1 - 2}{3} \right\rceil \leq \left\lceil \frac{3k_1 + j + 3k_2 + 2 - 1 - 2}{3} \right\rceil = \left\lceil \frac{n-2}{3} \right\rceil,$$

G is $\lceil \frac{n-2}{3} \rceil$-supereulerian, a contradiction.

Case 3. For $i = 1, 2$, $n_i - 2 \equiv 0 \pmod 3$, and $G_i \cong K_{2,3}(\frac{n_i-2}{3}, \frac{n_i-2}{3}, \frac{n_i-2}{3})$.

Denote $n_i = 3k_i + 2$. Then $\frac{n_i-2}{3} = k_i$, and hence G_i is $(k_i + 1)$-supereulerian. Thus, G is $(k_1 + k_2 + 1)$-supereulerian. Since

$$k_1 + k_2 + 1 = \left\lceil \frac{3k_1 + 2 + 3k_2 + 2 - 1 - 2}{3} \right\rceil = \left\lceil \frac{n-2}{3} \right\rceil,$$

G is $\lceil \frac{n-2}{3} \rceil$-supereulerian, a contradiction.

This completes the proof of Claim 1. □

Claim 2. $\Delta(G) \leq 3$.

Proof of Claim 2. Suppose, to the contrary, that $\Delta(G) \geq 4$. Let v be a vertex of G with degree at least 4. By Claim 1, G is 2-connected. Hence, by Theorem 22, G contains two edges vv_1 and vv_2 incident to v that can be split off such that the resulting graph, denoted by G^* (i.e., $G^* = G - \{vv_1, vv_2\} + \{v_1v_2\}$), is connected and has no cut edges. Then $|V(G^*)| = |V(G)| = n$ and $|E(G^*)| = |E(G)| - 1 < |E(G)|$. By the minimality of G, we can obtain that G^* is either $\lceil \frac{n-2}{3} \rceil$-supereulerian or the graph $K_{2,3}(\frac{n-2}{3}, \frac{n-2}{3}, \frac{n-2}{3})$.

First, suppose that G^* is $\lceil \frac{n-2}{3} \rceil$-supereulerian, i.e., G^* has a spanning even subgraph L^* with $n_c(L^*) \leq \lceil \frac{n-2}{3} \rceil$. Then $v_1v_2 \in E(L^*)$; otherwise, L^* is also a spanning even subgraph of G, and then G is $\lceil \frac{n-2}{3} \rceil$-supereulerian, a contradiction. Let $L_1^* \subset L^*$ be the component containing v_1v_2, $L_2^* \subset L^*$ the component containing v, and let

$$L = \begin{cases} (L^* - L_1^* - L_2^*) \cup ((L_1^* - \{v_1v_2\}) \cup L_2^* \cup \{vv_1, vv_2\}), & \text{if } L_1^* \neq L_2^*; \\ (L^* - L_1^*) \cup ((L_1^* - \{v_1v_2\}) \cup \{vv_1, vv_2\}), & \text{if } L_1^* = L_2^*. \end{cases}$$

Then $n_c(L) \leq n_c(L^*)$. Hence, G has a spanning even subgraph L with at most $\lceil \frac{n-2}{3} \rceil$ components, i.e., G is $\lceil \frac{n-2}{3} \rceil$-supereulerian, a contradiction.

Next, suppose that $G^* \cong K_{2,3}(\frac{n-2}{3}, \frac{n-2}{3}, \frac{n-2}{3})$. Then, by (a) of Lemma 23, G is $\lceil \frac{n-2}{3} \rceil$-supereulerian, a contradiction.

This completes the proof of Claim 2. □

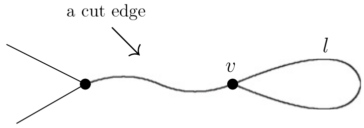

Fig. 4. G^3 has a loop.

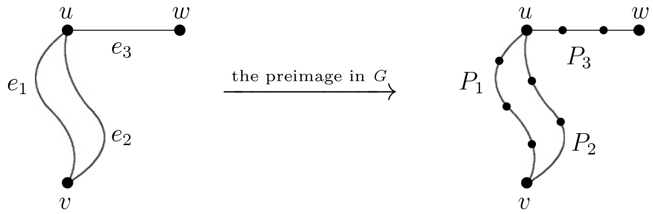

Fig. 5. Local structure of u and its neighbors in G^3 and the preimage in G.

Note that G is 2-edge-connected. By Claim 2, $2 \le \delta(G) \le \Delta(G) \le 3$. If $\Delta(G) = 2$, then G is a cycle, which is supereulerian, a contradiction. Hence, $\Delta(G) = 3$. For $i = 2, 3$, let $D_i(G)$ denote the set of all vertices of degree i in G, and $d_i(G) = |D_i(G)|$. In the following, we construct a 3-regular weighted graph G^3 from G.

Let G^3 be the graph obtained from G by replacing each maximal path whose internal vertices have degree 2 in G by an edge, and, for $e \in E(G^3)$, let $q(e)$, the weight of e, be the number of internal vertices in the corresponding maximal path in G. Then G^3 is 3-regular, and $d_2(G) = \sum_{e\in E(G^3)} q(e)$, $d_3(G) = |V(G^3)|$, and $n = d_2(G) + d_3(G) = \sum_{e\in E(G^3)} q(e) + |V(G^3)|$. By the hypotheses of Theorem 20, and by the definition of G^3, G^3 is 2-edge-connected.

Now, we present the following claim.

Claim 3. *G^3 is simple.*

Proof of Claim 3. Suppose, to the contrary, that G^3 contains loops or multiple edges.

First, suppose that G^3 has a loop l. Let v be the vertex incident with l. Note that G^3 is 3-regular. The other edge incident with v is a cut edge of G^3 (see Fig. 4), contrary to the fact that G^3 is 2-edge-connected.

Next, suppose that G^3 has multiple edges. If G^3 has three multiple edges between one pair of vertices, then, since G^3 is 3-regular, and by the construction of G^3, we have $G \cong K_{2,3}(k_1, k_2, k_3)$. Note that G is a counterexample. We may assume that $k_1 < k_2 \le k_3$. Then G is $(k_1 + 1)$-supereulerian. Since $k_1 + 1 \le \lceil \frac{k_1+k_2+k_3}{3} \rceil = \lceil \frac{n-2}{3} \rceil$, G is $\lceil \frac{n-2}{3} \rceil$-supereulerian, a contradiction. So G^3 has at most two multiple edges between any pair of vertices. Hence, we can find a pair of vertices u, v in G^3 with multiple edges $e_1 = uv$, $e_2 = uv$, by the assumption that G^3 has multiple edges.

In the following, let $N_{G^3}(u) \setminus \{v\} = w$, and let P_1, P_2, and P_3 be the maximal paths in G corresponding to e_1, e_2, and $e_3 = uw$, respectively, depicted in Fig. 5 (the number of internal vertices of P_i may not be accurate).

Claim 3.1. *Both P_1 and P_2 have internal vertices in G.*

Proof of Claim 3.1. Suppose, to the contrary, that P_1 has no internal vertex. Denote $P_1 = e = uv$ and $G_1 = G - e$. Then, we claim that G_1 is 2-edge-connected. By way of contradiction, suppose that G_1 contains a cut edge e'. If u and v belong to the same component of $G_1 - e'$, then e' is also a cut edge of G, a contradiction; if u and v belong to two distinct components of $G_1 - e'$, then u is a cut vertex of G, contrary to Claim 1.

Hence, G_1 is 2-edge-connected. Note that $|V(G_1)| = |V(G)| = n$ and $|E(G_1)| = |E(G)| - 1 < |E(G)|$. By the minimality of G, either G_1 is $\lceil \frac{n-2}{3} \rceil$-supereulerian, and hence G is $\lceil \frac{n-2}{3} \rceil$-supereulerian, a contradiction; or $G_1 \cong K_{2,3}(\frac{n-2}{3}, \frac{n-2}{3}, \frac{n-2}{3})$, and hence G is $\lceil \frac{n-2}{3} \rceil$-supereulerian by (b) of Lemma 23, a contradiction. □

By Claim 3.1, for $i = 1, 2$, we may assume that $x_i \in V(\mathring{P}_i)$ such that $ux_i \in E(G)$, i.e., x_i is the neighbor of u in P_i. To finish the proof of Claim 3, it suffices to consider the following two cases.

Case 1. P_3 has internal vertices.

Let $x_3 \in V(\mathring{P}_3)$ such that $ux_3 \in E(G)$, $G^* = G/\{ux_1, ux_2, ux_3\}$, $P_i^* = P_i/\{ux_i\}$ the path in G^* ($i = 1, 2, 3$), and u^* the resulting vertex (of degree 3) obtained by contracting $\{ux_1, ux_2, ux_3\}$, depicted in Fig. 6. Then $n^* = |V(G^*)| = n - 3$ and $|E(G^*)| = |E(G)| - 3$. By the minimality of G, we can obtain that G^* is either $\lceil \frac{n^*-2}{3} \rceil$-supereulerian or the graph $K_{2,3}(\frac{n^*-2}{3}, \frac{n^*-2}{3}, \frac{n^*-2}{3})$. The latter case does not hold; otherwise, $G \cong K_{2,3}(\frac{n-2}{3}, \frac{n-2}{3}, \frac{n-2}{3})$, a contradiction. So we need to consider the former case.

Let L^* be a spanning even subgraph of G^* with the least number of components. Then $n_c(L^*) \le \lceil \frac{n^*-2}{3} \rceil$. Let L_1^* be the component of L^* containing u^*. Then, we may assume that L_1^* is nontrivial; otherwise, the vertices in $V(P_1^*) \cup V(P_2^*)$ are all trivial in L^*, and then we can replace these trivial components by $u^*P_1^*vP_2^*u^*$ to obtain a spanning even subgraph of G^* with fewer components than L^*, contrary to the choice of L^*.

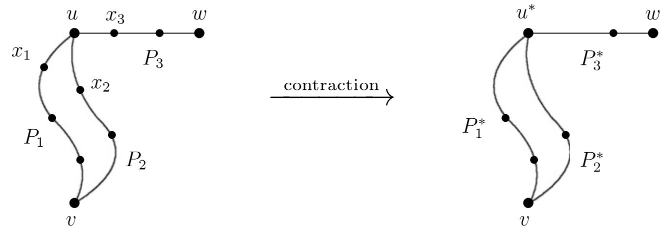

Fig. 6. The demonstration of contraction when P_3 has internal vertices.

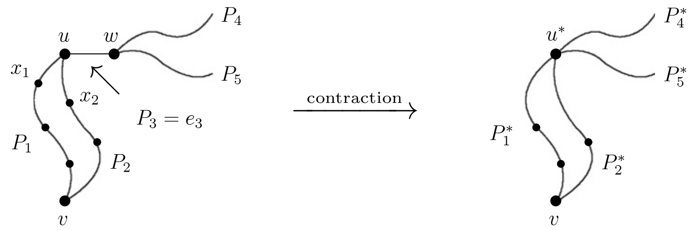

Fig. 7. The demonstration of contraction when P_3 is an edge.

Since L_1^* is nontrivial and $d_{G^*}(u^*) = 3$, we may assume that $P_i^*, P_j^* \subseteq L_1^*$, and that the internal vertices of P_k^* are trivial components in L^*, where $\{i, j, k\} = \{1, 2, 3\}$. Then, let L_1 be the even subgraph of G obtained from L_1^* by replacing P_i^* and P_j^* by P_i and P_j, respectively, and let $L = (L^* - L_1^*) \cup L_1 \cup \{x_k\}$. Then L is a spanning even subgraph of G with $n_c(L) = n_c(L^*) + 1 \le \lceil \frac{n^*-2}{3} \rceil + 1 = \lceil \frac{n-2}{3} \rceil$ since $n^* = n - 3$. Hence, G is $\lceil \frac{n-2}{3} \rceil$-supereulerian, a contradiction.

Case 2. P_3 has no internal vertex.

Then, we can denote $P_3 = e_3 = uw$. Let e_4, e_5 be the two edges incident with w excepting e_3, and P_4, P_5 the maximal paths in G corresponding to e_4, e_5, respectively. Let $G^* = G/\{ux_1, ux_2, e_3\}$, $P_i^* = P_i/\{ux_i\}$ the path in G^* ($i = 1, 2$), P_j^* the path in G^* corresponding to P_j in G ($j = 4, 5$), and u^* the resulting vertex (of degree 4) obtained by contracting $\{ux_1, ux_2, e_3\}$, depicted in Fig. 7. Then $n^* = n(G^*) = n - 3$ and $|E(G^*)| = |E(G)| - 3$. Since $d_{G^*}(u^*) = 4$ and $\Delta(K_{2,3}(k, k, k)) = 3$, and by the minimality of G, G^* is $\lceil \frac{n^*-2}{3} \rceil$-supereulerian.

Let L^* be a spanning even subgraph of G^* with the least number of components. Then $n_c(L^*) \le \lceil \frac{n^*-2}{3} \rceil$. Let L_1^* be the component of L^* containing u^*. Then, by arguing similarly as Case 1, we may assume that L_1^* is nontrivial. Hence, $d_{L_1^*}(u^*) = 2, 4$.

Subcase 2.1. $d_{L_1^*}(u^*) = 2$.

Then, exactly two of $\{P_1^*, P_2^*, P_4^*, P_5^*\}$ belong to L_1^*. By symmetry, we may assume that $P_1^*, P_2^* \subseteq L_1^*$, or $P_1^*, P_4^* \subseteq L_1^*$, or $P_4^*, P_5^* \subseteq L_1^*$.

Subcase 2.1.1. $P_1^*, P_2^* \subseteq L_1^*$.

In this case, the internal vertices of P_4^* and P_5^* are trivial components in L^*, and $L_1^* = u^*P_1^*vP_2^*u^*$. Let $L_1 = uP_1vP_2u$, and $L = (L^* - L_1^*) \cup L_1 \cup \{w\}$. Then L is a spanning even subgraph of G with $n_c(L) \le \lceil \frac{n-2}{3} \rceil$. Hence, G is $\lceil \frac{n-2}{3} \rceil$-supereulerian, a contradiction.

Subcase 2.1.2. $P_1^*, P_4^* \subseteq L_1^*$.

In this case, the internal vertices of P_2^* and P_5^* are trivial components in L^*. Let L_1 be the graph obtained from L_1^* by replacing $vP_1^*u^*P_4^*$ by vP_1uwP_4, and $L = (L^* - L_1^*) \cup L_1 \cup \{x_2\}$. Then L is a spanning even subgraph of G with $n_c(L) \le \lceil \frac{n-2}{3} \rceil$. Hence, G is $\lceil \frac{n-2}{3} \rceil$-supereulerian, a contradiction.

Subcase 2.1.3. $P_4^*, P_5^* \subseteq L_1^*$.

In this case, the internal vertices of P_1^*, P_2^* and v are trivial components in L^*. Let $\tilde{L_1^*} = L_1^* \cup u^*P_1^*vP_2^*u^*$. Then, we can replace L_1^* and the corresponding trivial components by $\tilde{L_1^*}$ in L^*, to reduce its number of components, contrary to the choice of L^*.

Subcase 2.2. $d_{L_1^*}(u^*) = 4$.

In this case, we can construct two even subgraphs L_1' and L_1'' of G from L_1^*: $L_1' = uP_1vP_2u$, and L_1'' is obtained from L_1^* by deleting the vertices in $V(\mathring{P}_1^*) \cup V(\mathring{P}_2^*) \cup \{v\}$, and then replacing P_4^*, P_5^* by P_4, P_5, respectively. Let $L = (L^* - L_1^*) \cup L_1' \cup L_1''$. Then L is a spanning even subgraph of G with $n_c(L) \le \lceil \frac{n-2}{3} \rceil$. Hence, G is $\lceil \frac{n-2}{3} \rceil$-supereulerian, a contradiction.

This completes the proof of Claim 3. □

Now, we continue to prove Theorem 20. Note that G^3 is 2-edge-connected. By Theorem 24, there exists a constant p and $3p$ perfect matchings $M_1, M_2, \ldots, M_{3p}$ such that each edge of G^3 is in p of them. For $1 \le i \le 3p$, let $q(M_i) = \sum_{e \in M_i} q(e)$ be the weight of M_i. Without loss of generality, we can assume that $q(M_1) \le q(M_2) \le \cdots \le q(M_{3p})$. By Theorem 24, $\sum_{i=1}^{3p} q(M_i) = p \sum_{e \in E(G^3)} q(e) = pd_2(G)$. Hence, $q(M_1) \le \lfloor d_2(G)/3 \rfloor$.

Since M_1 is a perfect matching, $G^3 - M_1$ is a 2-factor of G^3. By Claim 3, each component (i.e., cycle) of $G^3 - M_1$ contains at least three vertices. So $n_c(G^3 - M_1) \leq \lfloor n(G^3)/3 \rfloor = \lfloor d_3(G)/3 \rfloor$.

Now, we come back to consider the graph G. Let L_1 be the set of cycles (in G) which are the preimages of the cycles in $G^3 - M_1$, L_2 the set of vertices (in G) which are the internal vertices of the preimages of the edges in M_1, and let $L = L_1 \cup L_2$. Then L is a spanning even subgraph of G with

$$n_c(L) = n_c(L_1) + n_c(L_2) = n_c(G^3 - M_1) + q(M_1) \leq \left\lfloor \frac{d_3(G)}{3} \right\rfloor + \left\lfloor \frac{d_2(G)}{3} \right\rfloor.$$

Note that

$$\left\lfloor \frac{d_3(G)}{3} \right\rfloor + \left\lfloor \frac{d_2(G)}{3} \right\rfloor \leq \left\lceil \frac{d_2(G) + d_3(G) - 2}{3} \right\rceil = \left\lceil \frac{n-2}{3} \right\rceil.$$

This implies that G is $\lceil \frac{n-2}{3} \rceil$-supereulerian, a contradiction.

This completes the proof of Theorem 20. □

Acknowledgments

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