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On r-hued coloring of K_4-minor free graphs

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A R T I C L E   I N F O

A B S T R A C T

A list assignment L of G is a mapping that assigns every vertex $v \in V(G)$ a set $L(v)$ of positive integers. For a given list assignment L of G, an (L, r)-coloring of G is a proper coloring c such that for any vertex v with degree $d(v)$, $c(v) \in L(v)$ and v is adjacent to at least min $\{d(v), r\}$ different colors. The r-hued chromatic number of G, $\chi_r(G)$, is the least integer k such that for any $v \in V(G)$ with $L(v) = \{1, 2, \ldots, k\}$, G has an (L, r)-coloring. The r-hued list chromatic number of G, $\chi_{L,r}(G)$, is the least integer k such that for any $v \in V(G)$ and every list assignment L with $|L(v)| = k$, G has an (L, r)-coloring. Let $K(r) = r + 3$ if $2 \le r \le 3$, and $K(r) = \lfloor 3r/2 \rfloor + 1$ if $r \ge 4$. We proved that if G is a K_4-minor free graph, then

 (i) $\chi_r(G) \le K(r)$, and the bound can be attained;
(ii) $\chi_{L,r}(G) \le K(r) + 1$.

This extends a former result in Lih et al. (2003).

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1. Introduction

Graphs in this paper are simple and finite. Undefined terminologies and notations are referred to [4]. Thus for a graph G, $\Delta(G)$, $\delta(G)$, $\chi(G)$ and $\chi_L(G)$ denote the maximum degree, the minimum degree, the chromatic number and the list chromatic number of G, respectively. For $v \in V(G)$, let $N_G(v)$ denote the set of vertices adjacent to v in G, and $d_G(v) = |N_G(v)|$. When G is understood from the context, we often use $N(v)$ and $d(v)$ for $N_G(v)$ and $d_G(v)$, respectively.

Let k, r be integers with $k > 0$ and $r > 0$, and let $\bar{k} = \{1, 2, \ldots, k\}$. If $c : V(G) \mapsto \bar{k}$, and if $V' \subseteq V(G)$, then define $c(V') = \{c(v) | v \in V'\}$. A (k, r)-coloring of a graph G is a mapping $c : V(G) \mapsto \bar{k}$ satisfying both the following.

(C1) $c(u) \ne c(v)$ for every edge $uv \in E(G)$;
(C2) $|c(N_G(v))| \ge \min\{d_G(v), r\}$ for any $v \in V(G)$.

For a fixed integer $r > 0$, the r-*hued chromatic number* of G, denoted by $\chi_r(G)$, is the smallest k such that G has a (k, r)-coloring. The concept was first introduced in [14,10], where $\chi_2(G)$ was called *the dynamic chromatic number* of G. Later in [9], a referee suggested the name of conditional chromatic number of G. Recently, we received several comments on the name of conditional coloring, suggesting that it does not reveal the nature of the coloring. Therefore, we decided to use the name hued coloring to reflect the use of many colors near a vertex.

By the definition of $\chi_r(G)$, it follows immediately that $\chi(G) = \chi_1(G)$, and $\chi_\Delta(G) = \chi(G^2)$, where G^2 is the square graph of G. Thus r-hued colorings are a generalization of the classical vertex coloring. For any integer $i > j > 0$, any (k, i)-coloring

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of G is also a (k, j)-coloring of G, so

$$\chi(G) \leq \chi_2(G) \leq \cdots \leq \chi_r(G) \leq \cdots \leq \chi_\Delta(G) = \chi_{\Delta+1}(G) = \cdots = \chi(G^2).$$

In [11], it is shown that $(3, 2)$-colorability remains NP-complete when restricted to planar bipartite graphs with maximum degree at most 3 and with arbitrarily high girth. This differs considerably from the well-known result that the classical 3-colorability is polynomially solvable for graphs with maximum degree at most 3.

The r-hued chromatic numbers of some classes of graphs are known. For example, the result on complete graphs, cycles, trees and complete bipartite graphs can be found in [9]. In [10], an analogue of Brooks Theorem for χ_2 is proved. It is shown in [5] that $\chi_2(G) \leq 5$ holds for any planar graph G. In [9], it is further showed that for $r \geq 2$, $\chi_r(G) \leq \Delta + r^2 - r + 1$ if $\Delta \leq r$. A *Moore graph* is a regular graph with diameter d and girth $2d + 1$. Ding et al. [6] proved that $\chi_r(G) \leq \Delta^2 + 1$, where the equality holds if and only if G is a Moore graph. This is also improved in [13] where it is shown that $\chi_r(G) \leq r\Delta + 1$.

A list assignment L of G is a mapping that assigns to every vertex v of G a set $L(v)$ of positive integers. For a given list assignment L of G, an (L, r)-coloring of G is a proper coloring c such that for any vertex v with degree $d(v)$, $c(v) \in L(v)$ and v is adjacent to at least $\min\{d(v), r\}$ different colors. The *r-hued list chromatic number* of G, denoted as $\chi_{L,r}(G)$, is the least integer k such that for any $v \in V(G)$ and every list assignment L with $|L(v)| = k$, G has an (L, r)-coloring.

Similarly, $\chi_L(G) = \chi_{L,1}(G)$ and $\chi_{L,\Delta}(G) = \chi_L(G^2)$. As for any integer $i > j > 0$, any (L, i)-coloring of G is also an (L, j)-coloring of G, it follows

$$\chi_L(G) \leq \chi_{L,2}(G) \leq \cdots \leq \chi_{L,r}(G) \leq \cdots \leq \chi_{L,\Delta}(G) = \chi_{L,\Delta+1}(G) = \cdots = \chi_L(G^2).$$

For positive integers k and r, let $L(v) = \bar{k}$, for any $v \in V(G)$. Then every (k, r)-coloring of G is also an (L, r)-coloring of G, and so

$$\chi_r(G) \leq \chi_{L,r}(G).$$

Some recent results are published for the case $r = 2$. Akbari et al. [1] proved that $\chi_{L,2}(G) \leq \Delta(G) + 1$ if G has no component isomorphic to C_5 and if $\Delta(G) \geq 3$. Later in [8], Esperet disproved a conjecture $\chi_{L,2}(G) = \max\{\chi_L(G), \chi_2(G)\}$ made in [1]. Chen et al. [5] showed that $\chi_{L,2}(G) \leq 6$ if G is a planar graph.

A graph G has a graph H as *minor* if H can be obtained from a subgraph of G by contracting edges, and G is called *H-minor free* if G does not have H as a minor. A graph G is called a *series–parallel* graph if each component can be obtained from K_2 by iteratively using the following two operations: replace an edge with a path of length 2 and duplicate an edge. A graph G is K_4-minor free if and only if each block of G is a series–parallel graph. Wegner [16] conjectured that if G is a planar graph, then

$$\chi_\Delta(G) \leq \begin{cases} \Delta(G) + 5, & \text{if } 4 \leq \Delta(G) \leq 7; \\ \lfloor 3\Delta(G)/2 \rfloor + 1, & \text{if } \Delta(G) \geq 8. \end{cases}$$

Define

$$K(r) = \begin{cases} r + 3, & \text{if } 2 \leq r \leq 3; \\ \lfloor 3r/2 \rfloor + 1, & \text{if } r \geq 4. \end{cases}$$

Lih et al. proved the following towards Wegner's conjecture.

Theorem 1.1 (*K.-W. Lih, W.-F. Wang and X. Zhu [12]*)**.** *Let G be a K_4-minor free graph. Then*

$$\chi_\Delta(G) \leq K(\Delta(G)).$$

In this paper, we will extend Theorem 1.1 as the following.

Theorem 1.2. *Let G be a K_4-minor free graph with $\Delta = \Delta(G)$, and $r \geq 2$ be an integer. Then*

(i) $\chi_r(G) \leq K(r)$.
(ii) $\chi_{L,r}(G) \leq K(r) + 1$.

Examples given in [12] show that Theorem 1.2(i) is best possible when $r = \Delta$.

2. Proof of Theorem 1.2

Define $S_G(u) = \{x : d_G(x) \geq 3 \text{ with } ux \in E(G) \text{ or there exists a 2-vertex } w \text{ with } uw, wx \in E(G)\}$. Let $D_G(u) = |S_G(u)|$. See Fig. 1 for the case of $D_G(u) = 2$. It is well known [7] that every K_4-minor free graph contains a vertex of degree at most two. Lih et al. [12] proved the following lemma.

Lemma 2.1 (*K.-W. Lih, W.-F. Wang and X. Zhu [12]*)**.** *Let G be a K_4-minor free graph. Then one of the following conditions holds:*

(i) $\delta(G) \leq 1$;
(ii) *There exists two adjacent 2-vertices;*
(iii) *There exists a vertex u with $d_G(u) \geq 3$ such that $D_G(u) \leq 2$.*
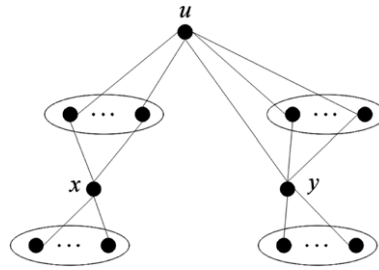
Fig. 1. $S_G(u) = \{x, y\}$.

We will use Lemma 2.1 to prove our result. Before that, we introduce some notations. Let G be a graph with the vertex set V, $V' \subseteq V$ be a vertex subset and $G[V']$ be the induced subgraph of G on V'. A mapping $c : V' \to \cup_{v \in V'} L(v)$ is a partial coloring if c is a proper (L, r)-coloring of $G[V']$. Let c be a partial coloring of G on V'. For the uncolored vertex $v \in V - V'$, let $\{c(v)\} = \emptyset$. For every vertex $v \in V'$, define $c[v]$ as follows.

$$c[v] = \begin{cases} \{c(v)\}, & \text{if } |c(N_G(v))| \geq r; \\ \{c(v)\} \cup c(N_G(v)), & \text{otherwise.} \end{cases} \tag{1}$$

Thus, when a partial coloring c is given, $c[v]$ consists of the set of colors that cannot be used for uncolored neighbors of v. By (1), $|c[v]| \leq r$.

Proof of Theorem 1.2. As it is shown in [5] that $\chi_2(G) \leq 5$ and $\chi_{L,2}(G) \leq 6$ if G is a planar graph, Theorem 1.2 holds for $r = 2$. In the following, we assume that $r \geq 3$.

We argue by contradiction to prove Theorem 1.2. Assume that

$$G \text{ is a counterexample to Theorem 1.2 with } |V(G)| \text{ minimized.} \tag{2}$$

Then for some list assignment $\{L(v) : v \in V(G)\}$, G has no (L, r)-coloring. We may assume that for every $v \in V(G)$, $L(v) = \{1, 2, \ldots, K(r)\}$ if G is a counterexample of Theorem 1.2(i), and that $|L(v)| = K(r) + 1$ if G is a counterexample of Theorem 1.2(ii). As $r \geq 3$ implies $K(r) \geq 6$, we may assume $|V(G)| \geq 7$. By (2), G must be connected.

In the following proof, we will obtain a K_4-minor free graph H by making local modifications of G such that $|V(H)| < |V(G)|$. By (2), H has an (L, r)-coloring c. To obtain a contradiction, we shall extend and modify c to an (L, r)-coloring of G.

Claim 2.2. $\delta(G) = 2$.

If G has a vertex x of degree 1, then let $H = G - x$. As H is a K_4-minor free graph with $|V(H)| < |V(G)|$, it follows by (2) that H has an (L, r)-coloring c. Let $N_G(x) = \{u\}$. By (1) and the definition of $K(r)$, $|c[u]| \leq r < K(r)$, and so the number of colors that cannot be used for the uncolored neighbor x of the vertex u in G is less than $K(r)$. Therefore, we can extend c to an (L, r)-coloring of G by defining $c(x) \in L(x) - c[u]$, contrary to (2). $\square$

Claim 2.3. Any two 2-vertices of G are not adjacent.

If G has two adjacent 2-vertices x and y, then denote $N_G(x) = \{u, y\}$ and $N_G(y) = \{v, x\}$. Let $H = G - x + uy$. As H is K_4-minor free with $|V(H)| < |V(G)|$, by (2), H has an (L, r)-coloring c. For such a coloring c, it follows that $c[y] = \{c(y), c(v)\}$. Therefore, $|c[u] \cup c[y]| \leq |c[u]| + |c[y]| \leq r + 2 < K(r)$, and so the number of colors that cannot be used for the uncolored neighbor x of the vertices u, y in G is less than $K(r)$. Thus, we can extend c to an (L, r)-coloring of G by defining $c(x) \in L(x) - (c[u] \cup c[y])$, contrary to (2). $\square$

By Lemma 2.1, Claims 2.2 and 2.3, G has a vertex u with $d_G(u) \geq 3$ such that $D_G(u) \leq 2$. In the rest of the proof, we always assume that u is such a vertex. For $x \in S_G(u)$, define

$$M_G(u, x) = \{w : w \in N_G(u) \cap N_G(x), d_G(w) = 2\} \quad \text{and} \quad m_G(x) = |M_G(u, x)|. \tag{3}$$

Without loss of generality, we may assume $m_G(x) \geq 1$, and we have the following claim.

Claim 2.4. $D_G(u) = 2$.

By the definition of $D_G(u)$, Claims 2.2 and 2.3, $D_G(u) \geq 1$. Assume that $D_G(u) = 1$ and $S_G(u) = \{x\}$. Then all the neighbors of u are either x or some neighbors of x. Since $m_G(x) \geq 1$, pick $w \in M_G(u, x)$ and define $H = G - w$. As H is also a K_4-minor free graph with $|V(H)| < |V(G)|$, by (2), H has an (L, r)-coloring c. Since $d_G(u) \geq 3$, we have $m_G(x) \geq 2$, and so $c(u) \neq c(x)$. Note that $N_G(u) \subseteq \{x\} \cup N_G(x)$. If $d_G(x) \geq r + 1$, then $|c[x]| = 1$. It follows that $|c(u) \cup c[x]| \leq 1 + r < K(r)$, and so the number of colors that cannot be used for $w \in M_G(u, x)$ is less than $K(r)$. Therefore, as $c(u) \neq c(x)$, we can extend c to an (L, r)-coloring of G by defining $c(w) \in L(w) - (c[u] \cup c[x])$, contrary to (2). $\square$

Claim 2.5. Let $w \in M_G(u, x)$ and c be an (L, r)-coloring of $G - w$ with $c(u) \neq c(x)$. Then $\max\{d_G(u), d_G(x)\} \leq r$.

We argue by contradiction and assume that $\max\{d_G(u), d_G(x)\} = d_G(u) > r$. Since $w \in M_G(u, x)$, then $d_{G-w}(u) \geq r$. Hence by (1), for any (L, r)-coloring c of $G - w$, $|c[u]| = 1$. As $|c[u] \cup c[x]| \leq |c[u]| + |c[x]| \leq r + 1 < K(r)$, the number of colors that cannot be used for the uncolored w in G is less than $K(r)$. Therefore, by $c(u) \neq c(x)$, c can be extended to an (L, r)-coloring of G by choosing $c(w) \in L(w) - (c[u] \cup c[x])$, contrary to (2). $\quad\square$

By Claim 2.4, $D_G(u) = 2$. Let $S_G(u) = \{x, y\}$. Then by the definition of $S_G(u)$, it follows that (see Fig. 1)

$$N_G(u) \subseteq N_G(x) \cup N_G(y) \cup \{x, y\}.$$

Without loss of generality, we assume that $m_G(x) \geq m_G(y)$. Since $d_G(u) \geq 3$, we have $m_G(x) \geq 1$. Pick $w \in M_G(u, x)$ and define

$$H = G - w.$$

Then H is also a K_4-minor free graph with $|V(H)| < |V(G)|$. By (2), H has an (L, r)-coloring c.

Case 1. $xu \in E(G)$.

As $xu \in E(H)$, $c(u) \neq c(x)$. By Claim 2.5, we have $\max\{d_G(u), d_G(x)\} \leq r$. Since x is adjacent to u, we have $|c[u] \cup c[x]| \leq d_G(u) + d_G(x) - m_G(x) - 1$. By $m_G(x) + m_G(y) \geq d_G(u) - 2$ and by $m_G(x) \geq m_G(y)$, we conclude that $m_G(x) \geq \lceil (d_G(u)-2)/2 \rceil = \lceil d_G(u)/2 \rceil - 1$. Hence

$$
\begin{aligned}
|c[u] \cup c[x]| &\leq d_G(u) + d_G(x) - m_G(x) - 1 \\
&\leq d_G(u) + d_G(x) - \lceil d_G(u)/2 \rceil \\
&\leq \lfloor d_G(u)/2 \rfloor + d_G(x) \\
&\leq \lfloor 3r/2 \rfloor \\
&\leq K(r) - 1.
\end{aligned}
$$

It follows that the number of colors cannot be used for the uncolored neighbor w of the vertices u, x in G is less than $K(r)$. Therefore, as $c(u) \neq c(x)$, c can be extended to an (L, r)-coloring of G by choosing $c(w) \in L(w) - (c[u] \cup c[x])$, contrary to (2). This proves Case 1.

Case 2. $xu \notin E(G)$, $yu \notin E(G)$.

Since $xu, yu \notin E(G)$ and $m_G(x) \geq m_G(y)$, we conclude that $m_G(x) \geq \lceil d_G(u)/2 \rceil \geq 2$. Then there exists a 2-vertex w' with $w'x, w'u \in E(H)$, and so $c(u) \neq c(x)$. By Claim 2.5, we have $\max\{d_G(u), d_G(x)\} \leq r$. Since x is not adjacent to u, we have $|c[u] \cup c[x]| \leq d_G(u) + d_G(x) - m_G(x) + 1$. Hence

$$
\begin{aligned}
|c[u] \cup c[x]| &\leq d_G(u) + d_G(x) - m_G(x) + 1 \\
&\leq d_G(u) + d_G(x) - \lceil d_G(u)/2 \rceil + 1 \\
&\leq \lfloor d_G(u)/2 \rfloor + d_G(x) + 1 \\
&\leq \lfloor 3r/2 \rfloor + 1 \\
&\leq \begin{cases} K(3) - 1, & \text{if } r = 3; \\ K(r), & \text{if } r \geq 4. \end{cases}
\end{aligned}
$$

If $|c[u] \cup c[x]| < K(r)$ (the case when $r = 3$ is included), or if $|c[u] \cup c[x]| = K(r)$ and $|L(w)| = K(r) + 1$, then the number of colors that cannot be used for the uncolored neighbor w of the vertices u, x in G is less than $|L(w)|$. Therefore, as $c(u) \neq c(x)$, we can extend c to an (L, r)-coloring of G by defining $c(w) \in L(w) - (c[u] \cup c[x])$, contrary to (2).

Therefore, we assume that $r \geq 4$ and $|c[u] \cup c[x]| = K(r)$. Since $|c[u] \cup c[x]| = |c(N_H(u) \cup N_H(x) \cup \{u, x\})| = K(r)$ and $N_H(u) \cup N_H(x) \cup \{u, x\}$ has exactly $K(r)$ vertices, each vertex in $N_H(u) \cup N_H(x) \cup \{u, x\}$ is colored differently by c. As $r \geq 4$, $d_H(u) + 2 \leq (r - 1) + 2 \leq K(r) - 2$, and so there are at least two choices to color u properly in H (as an (L, r)-coloring). By changing the color of u and preserving the colors of other vertices, we obtain a new coloring (also denoted by c) satisfying $|c[u] \cup c[x]| < K(r)$. Therefore, as $c(u) \neq c(x)$, we can extend c to an (L, r)-coloring of G by defining $c(w) \in L(w) - (c[u] \cup c[x])$, contrary to (2). This proves case 2.

Case 3. $xu \notin E(G)$, $yu \in E(G)$.

If $m_G(x) = m_G(y)$, we may interchange x and y, and it falls under Case 1. Hence we may assume that $m_G(x) > m_G(y)$.

Case 3.1. $d_G(u)$ is odd.

Since $d_G(u)$ is odd, $m_G(x) + m_G(y) = d_G(u) - 1$ is even, and so $m_G(x) \geq m_G(y) + 2 \geq 2$.

Case 3.1.1. $m_G(x) \geq m_G(y) + 4$.

Since $m_G(x) \geq m_G(y) + 4 \geq 4$, $M_H(u, x) \neq \emptyset$, and so $c(u) \neq c(x)$. By Claim 2.5, we have $\max\{d_G(u), d_G(x)\} \leq r$. Hence,

$$
\begin{aligned}
|c[u] \cup c[x]| &\leq d_G(u) + d_G(x) - m_G(x) + 1 \\
&\leq d_G(u) + d_G(x) - (d_G(u) + 3)/2 + 1 \\
&= \lfloor d_G(u)/2 \rfloor + d_G(x) \\
&\leq \lfloor 3r/2 \rfloor \\
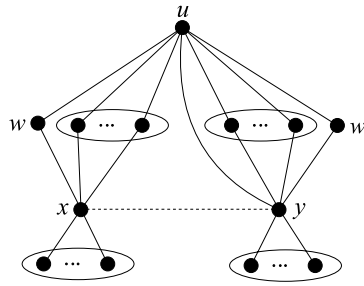&\leq K(r) - 1.
\end{aligned}
$$

Fig. 2. $H' = G - w - w' + xy$.

It follows that the number of colors that cannot be used for the uncolored neighbor w of the vertices u, x in G is less than $K(r)$. Therefore, as $c(u) \neq c(x)$, we can extend c to an (L, r)-coloring of G by defining $c(w) \in L(w) - (c[u] \cup c[x])$, contrary to (2).

Case 3.1.2. $m_G(x) = m_G(y) + 2$.

If $m_G(x) = m_G(y) + 2 \geq 2$, then $M_H(u, x) \neq \emptyset$, and so $c(u) \neq c(x)$. By Claim 2.5, we have $\max\{d_G(u), d_G(x)\} \leq r$. If $d_G(u) < r$, then

$$
\begin{aligned}
|c[u] \cup c[x]| &\leq d_G(u) + d_G(x) - m_G(x) + 1 \\
&\leq d_G(u) + d_G(x) - (d_G(u) + 1)/2 + 1 \\
&= (d_G(u) + 1)/2 + d_G(x) \\
&\leq \lfloor r/2 \rfloor + d_G(x) \\
&\leq \lfloor 3r/2 \rfloor \\
&\leq K(r) - 1.
\end{aligned}
$$

Thus the number of colors that cannot be used for the uncolored neighbor w of the vertices u, x in G is less than $K(r)$. Therefore, as $c(u) \neq c(x)$, we can extend c to an (L, r)-coloring of G by defining $c(w) \in L(w) - (c[u] \cup c[x])$, contrary to (2).

So we assume that $d_G(u) = r$. If $xy \in E(G)$, then

$$
\begin{aligned}
|c[u] \cup c[x]| &\leq d_G(u) + d_G(x) - 1 - (m_G(x) - 1) \\
&\leq d_G(u) + d_G(x) - (d_G(u) + 1)/2 \\
&= (d_G(u) - 1)/2 + d_G(x) \\
&\leq \lfloor r/2 \rfloor + d_G(x) \\
&\leq \lfloor 3r/2 \rfloor \\
&\leq K(r) - 1.
\end{aligned}
$$

It follows that the number of colors that cannot be used for the uncolored neighbor w of the vertices u, x in G is less than $K(r)$. Therefore, as $c(u) \neq c(x)$, we can extend c to an (L, r)-coloring of G by defining $c(w) \in L(w) - (c[u] \cup c[x])$, contrary to (2).

Thus we assume that $d_G(u) = r$ and $xy \notin E(G)$. In this case,

$$
\begin{aligned}
|c[u] \cup c[x]| &\leq d_G(u) + d_G(x) - (m_G(x) - 1) \\
&\leq d_G(u) + d_G(x) - (d_G(u) + 1)/2 + 1 \\
&= (d_G(u) - 1)/2 + d_G(x) + 1 \\
&\leq \lfloor r/2 \rfloor + d_G(x) + 1 \\
&\leq \lfloor 3r/2 \rfloor + 1 \\
&\leq \begin{cases} K(3) - 1, & \text{if } r = 3; \\ K(r), & \text{if } r \geq 4. \end{cases}
\end{aligned}
$$

If $|c[u] \cup c[x]| < K(r)$ (the case when $r = 3$ is included), or if $|c[u] \cup c[x]| = K(r)$ and $|L(w)| = K(r) + 1$, then the number of colors that cannot be used for the uncolored neighbor w of the vertices u, x in G is less than $|L(w)|$. Therefore, as $c(u) \neq c(x)$, we can extend c to an (L, r)-coloring of G by defining $c(w) \in L(w) - (c[u] \cup c[x])$, contrary to (2).

Therefore, we assume that $r \geq 4$ and $|c[u] \cup c[x]| = K(r)$. Since $d_G(u) = d_G(x) = r \geq 4$ is odd, we have $d_G(u) = r \geq 5$. As $m_G(x) = m_G(y) + 2$ and $d_G(u) \geq 5$, $M_G(u, y) \neq \emptyset$, and so we may choose some $w' \in M_G(u, y)$. Now let $H' = G - w - w' + xy$ (see Fig. 2). Then H' is also a K_4-minor free graph with $|V(H')| < |V(G)|$. By (2), H' has an (L, r)-coloring c in which $c(u) \neq c(x)$. So we can extend c to $V(G - w)$ by letting $c(w') = c(x)$. Then as $c(x) \in c[u]$ in $H = G - w$, the extended coloring c is an (L, r)-coloring of $G - w$ in which $c(u) \neq c(x)$ and $|c[u] \cup c[x]| < K(r)$, and so the number of colors that cannot be used for the uncolored neighbor w of the vertices u, x in G is less than $K(r)$. Therefore, as $c(u) \neq c(x)$, c can be further extended to an (L, r)-coloring of G by defining $c(w) \in L(w) - (c[x] \cup c[u])$, contrary to (2).

Case 3.2. $d_G(u)$ is even.

If $d_G(u)$ is even, then $m_G(x) + m_G(y) = d_G(u) - 1$ is odd and $m_G(x) \geq m_G(y) + 1$.

If $m_G(x) \geq m_G(y) + 3 \geq 3$, then $m_H(x) \geq 2$, and so $c(u) \neq c(x)$. By Claim 2.5, we have $\max\{d_G(u), d_G(x)\} \leq r$. Hence $|c[u] \cup c[x]| \leq d_G(u) + d_G(x) - m_G(x) + 1 \leq d_G(x) + d_G(u)/2$. Then

$$|c[u] \cup c[x]| \leq \begin{cases} r + \lfloor r/2 \rfloor, & \text{if } d_G(u) < r; \\ 3r/2, & \text{if } d_G(u) = r. \end{cases}$$

Since $d_G(u)$ is even, when $d_G(u) = r$, $3r/2 = \lfloor 3r/2 \rfloor$, and so $|c[u] \cup c[x]| \leq \lfloor 3r/2 \rfloor \leq K(r) - 1$. Then the number of colors that cannot be used for the uncolored neighbor w of the vertices u, x in G is less than $K(r)$. Therefore, as $c(u) \neq c(x)$, we can extend c to an (L, r)-coloring of G by defining $c(w) \in L(w) - (c[u] \cup c[x])$, contrary to (2).

Assume that $m_G(x) = m_G(y) + 1$. Since $d_G(u) \geq 4$, $m_G(y) = d_G(u)/2 - 1 \geq 1$. Choose $w' \in M_G(u, y)$ and let $H'' = G - w'$. Then H'' is a K_4-minor free graph with $|V(H'')| < |V(G)|$. As $uy \in E(G)$, $c(u) \neq c(y)$. By Claim 2.5, we have $\max\{d_G(u), d_G(y)\} \leq r$. Hence

$$\begin{aligned} |c[u] \cup c[y]| &\leq d_G(u) + d_G(y) - m_G(y) - 1 \\ &\leq d_G(u) + d_G(y) - d_G(u)/2 \\ &= d_G(u)/2 + d_G(y) \\ &\leq \lfloor 3r/2 \rfloor \\ &\leq K(r) - 1. \end{aligned}$$

Thus the number of colors that cannot be used for the uncolored neighbor w' of the vertices u, y in G is less than $K(r)$. Therefore, as $c(u) \neq c(y)$, we can extend c to an (L, r)-coloring of G by defining $c(w') \in L(w') - (c[u] \cup c[y])$, contrary to (2).

Since in all cases, a contradiction is obtained, this establishes the theorem and completes the proof. □

3. Remark

Motivated by Wegner' conjecture and the result in this paper, it is natural to seek, for each integer $r \geq 1$, the smallest integers $f_1(r)$ and $f_2(r)$ such that for any planar graph G, $\chi_r(G) \leq f_1(r)$ and $\chi_{L,r}(G) \leq f_2(r)$. By the Four Color Theorem [2,3,15] and by the results in [5], we believe that the following holds.

Conjecture 3.1. *Let G be a planar graph. Then we have $\chi_r(G) \leq f_1(r)$, where*

$$f_1(r) = \begin{cases} r + 3, & \text{if } 1 \leq r \leq 2; \\ r + 5, & \text{if } 3 \leq r \leq 7; \\ \lfloor 3r/2 \rfloor + 1, & \text{if } r \geq 8. \end{cases}$$

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