On s-Hamiltonian Line Graphs

Hong-Jian Lai[1] and Yehong Shao[2]

[1] DEPARTMENT OF MATHEMATICS,
WEST VIRGINIA UNIVERSITY,
MORGANTOWN, WV 26506
E-mail: hjlai@math.wvu.edu

[2] ARTS AND SCIENCES,
OHIO UNIVERSITY SOUTHERN,
IRONTON, OH 45638
E-mail: shaoy@ohio.edu

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Abstract: For an integer $s \geq 0$, a graph G is s-hamiltonian if for any vertex subset $S' \subseteq V(G)$ with $|S'| \leq s$, $G - S'$ is hamiltonian. It is well known that if a graph G is s-hamiltonian, then G must be $(s+2)$-connected. The converse is not true, as there exist arbitrarily highly connected nonhamiltonian graphs. But for line graphs, we prove that when $s \geq 5$, a line graph is s-hamiltonian if and only if it is $(s+2)$-connected. J. Graph Theory 74: 344–358, 2013

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1. INTRODUCTION

Graphs considered in this article are finite graphs. Undefined notations and terminology will follow those in [1]. Let G be a graph. As in [1], $\kappa'(G)$ and $\kappa(G)$ denote the edge-connectivity and the connectivity of G, respectively. A graph is *trivial* if it contains no edges. An edge cut X of G is *essential* if $G - X$ has at least two nontrivial components. For an integer $k > 0$, a graph G is *essentially k-edge-connected* if G does not have an essential edge cut X with $|X| < k$. For any $v \in V(G)$ and an integer $i \geq 0$, define

$$E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\},$$
$$D_i(G) = \{u \in V(G) : d_G(u) = i\} \text{ and } d_i(G) = |D_i(G)|.$$

The *line graph* of a graph G, written $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. The following conjecture is still open.

Conjecture 1.1 (Thomassen [12]). *Every 4-connected line graph is hamiltonian.*

Toward this conjecture, Zhan proved:

Theorem 1.2 (Zhan, Theorem 3 in [14]). *If $\kappa(L(G)) \geq 7$, $L(G)$ is hamiltonian-connected.*

A graph G of order $n \geq 3$ is called *s-hamiltonian*, $0 \leq s \leq n-3$, if the removal of any k vertices, $0 \leq k \leq s$, results in a hamiltonian graph. It is well known that if a graph G is s-hamiltonian, then G is $(s+2)$-connected. The converse, on the other hand, is not true, as $K_{m,m+1}$ is m-connected but nonhamiltonian. In this article, we investigate s-hamiltonian line graphs, and prove that this necessary condition is also sufficient among line graphs, when $s \geq 5$.

Theorem 1.3. *Let G be a connected graph and $s \geq 5$ an integer. Then $L(G)$ is s-hamiltonian if and only if $L(G)$ is $(s+2)$-connected.*

Theorem 1.3 is motivated by the following question: what is the smallest positive integer k such that a line graph $L(G)$ is s-hamiltonian if and only if $L(G)$ is $(s+2)$-connected for all integers $s \geq k$? Theorem 1.3 suggests that $k \leq 5$. Let $G(t)$ denote the graph obtained from the Petersen graph by attaching $t > 0$ pendant edges at each vertex of the Petersen graph. Then $L(G(t))$ is 3-connected but not 1-hamiltonian. Therefore, $k \geq 2$. Hence, we know that $k \in \{2, 3, 4, 5\}$ but the exact value of k remains to be determined. Note that if Theorem 1.3 holds for $k = 2$, then it implies Thomassen's conjecture (Conjecture 1.1). If Thomassen's conjecture is true, then there are hamiltonian properties that are polynomial in line graphs (see [7]). As a corollary of Theorem 1.3, 5-hamiltonicity is the first "reasonable" hamiltonian property which is known to be polynomial in line graphs.

On the other hand, Broersma and Veldman proved the following.

Theorem 1.4 (Broersma and Veldman [2]). *Let $k \geq s \geq 0$ be integers and let G be a k-triangular simple graph. Then $L(G)$ is s-hamiltonian if and only if $L(G)$ is $(s+2)$-connected.*

In [2], Broersma and Veldman asked the question if the conclusion of Theorem 1.4 remains valid for other values of s when k is given. Theorem 1.3 settles this problem raised by Broersma and Veldman for larger values of s, without the restriction that G is k-triangular.

Though, it is not known whether Theorem 1.3 can be extended to claw-free graphs. We conjecture that there exists an integer k such that for any $s \geq k$, a claw-free graph G is s-hamiltonian if and only if G is $(s+2)$-connected.

Clearly, if $L(G)$ is a complete graph, then $L(G)$ is s-hamiltonian for any integer s with $0 \leq s \leq |V(L(G))| - 3$. Throughout this article, we assume that $L(G)$ is not complete.

2. MECHANISM

The *spanning tree packing number* of G, written $\tau(G)$, is the maximum number of edge-disjoint spanning trees of G. The following two theorems are well known.

Theorem 2.1 (Nash-Williams [9], see also Theorem 3 and Corollary 18 of [4]). *Let G be a connected graph. If $\frac{|E(G)|}{|V(G)|-1} \geq 2$, then there exists a nontrivial subgraph H with $\tau(H) \geq 2$.*

Theorem 2.2 (Nash-Williams [10] and Tutte [13]). *Let $k \geq 1$ be an integer and G be a connected graph. Then $\tau(G) \geq k$ if and only if for any partition of the vertices of G into c parts, there are at least $k(c-1)$ edges of G whose endpoints are in different parts of the partition.*

Let $X \subseteq E(G)$ be an edge subset. The *contraction*G/X is the graph obtained from G by identifying two ends of each edge of X and then deleting the resulting loops. When $X = \{e\}$, we use G/e for $G/\{e\}$. Given a graph G, one can repeatedly contract all nontrivial subgraphs H of G with $\tau(H) \geq 2$. The resulting graph is called the *τ-reduction* of G.

Theorem 2.3 (Theorem E and Corollary 5 of [8]). *Let H be a subgraph of G with $\tau(H) \geq 2$. Then $\tau(G) \geq 2$ if and only if $\tau(G/H) \geq 2$. In particular, $\tau(G) \geq 2$ if and only if the τ-reduction of G is K_1.*

Let $O(G)$ denote the set of odd degree vertices of a graph G. We say that G is *Eulerian* if G is a connected graph with $O(G) = \emptyset$. A subgraph H of G is a *spanning Eulerian subgraph* if H is an Eulerian graph with $V(H) = V(G)$. A subgraph H' of G is a *dominating Eulerian subgraph* if H' is Eulerian and $G - V(H')$ is edgeless. We use SES to denote a spanning Eulerian subgraph and DES to denote a dominating Eulerian subgraph. Clearly, an SES of G is also a DES of G.

A graph G is *collapsible* if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph H_R such that $O(H_R) = R$. Catlin ([3]) showed that any graph G has a unique subgraph H such that every component of H is a maximally collapsible subgraph of G and every nontrivial collapsible subgraph of G is contained in a component of H. The contraction G/H is called the *c-reduction* of G. A graph G is *c-reduced* if the c-reduction of G is itself. Note that, as K_3 is collapsible [3], the c-reduction of K_3 is K_1; but the τ-reduction of K_3 is K_3 itself. The following summarize some of the former results concerning collapsible graphs (the contraction below is the c-reduction).

Theorem 2.4. *Let G be a connected graph and $F(G)$ denote the minimum number of edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. Each of the following holds.*

(i) (Catlin [3]). If H is a collapsible subgraph of G, then G is collapsible if and only if G/H is collapsible; G has an SES if and only if G/H has an SES.
(ii) (Catlin, Han, and Lai, Theorem 1.5 of [5]). If $F(G) \leq 2$, either G is collapsible or the c-reduction of G is a K_2 or $K_{2,t}$ for some integer $t \geq 1$.

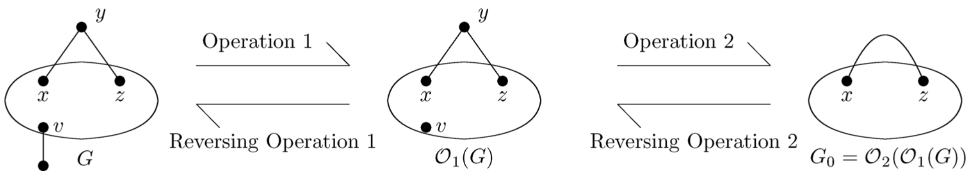

FIGURE 1. The core graph.

Let G be a connected, essentially 3-edge-connected graph such that $L(G)$ is not a complete graph. The *core* of this graph G, written G_0, is obtained by the following two operations (see Fig. 1) repeatedly.

Operation 1. Delete each vertex of degree 1.
Operation 2. For each vertex y of degree 2 with $E_G(y) = \{xy, yz\}$, contract exactly one edge in $E_G(y)$. This amounts to deleting edges xy, yz and vertex y for each path xyz in G with $d_G(y) = 2$ and replacing xy, yz by a new edge xz.

Let $\mathcal{O}_1(G)$ denote the graph obtained from G by applying Operation 1 to each vertex of degree 1, and $\mathcal{O}_2(G)$ the graph obtained from G by applying Operation 2 to each vertex of degree 2. So $G_0 = \mathcal{O}_2(\mathcal{O}_1(G))$.

The main idea of the proof of Theorem 1.3 is to convert a DES of $G_0 - S$ to a DES of $G - S$ for any $S \subseteq E(G)$. Note that some edges of S may not be in G_0 after Operation 2, so we need to define $G_0 - S$. Let $e \in S \subseteq E(G)$. By Operation 1, if e is a pendent edge (an edge incident with a vertex of degree 1), then $G_0 - \{e\} = G_0$; if e is incident with a vertex y of degree 2 with $e \in E_G(y) = \{xy, yz\}$, then we define $G_0 - \{e\} = G_0 - \{xz\}$.

Lemma 2.5. *Let G be an essentially 3-edge-connected graph and $N_G(D_1(G) \cup D_2(G))$ be the set of neighbors of all the vertices of degree 1 or 2 in G.*

(i) (Shao, Lemma 1.4.1 of [11]). The core graph G_0 is uniquely defined and $\kappa'(G_0) \geq 3$.
(ii) Suppose $S \subseteq E(G)$, and all isolated vertices in $G - S$ and $G_0 - S$ resulting from deleting S are deleted. If $G - S$ is connected and $G_0 - S$ has a DES L' containing $N_G(D_1(G) \cup D_2(G))$, then the graph L obtained by reversing Operation 2 on L' is a DES of $G - S$.

Proof. (ii) Since L' is an Eulerian subgraph of $G_0 - S$, and reversing Operation 2 is simply replacing an edge with a path of length 2, L is an Eulerian subgraph of $G - S$ and $V(L') \subseteq V(L)$. It suffices to show that $G - S - V(L)$ is edgeless, or equivalently, for any vertex $v \in V(G - S)$, either $v \in V(L)$ or $N_{G-S}(v) \subseteq V(L)$.

Clearly, $v \in V(G)$. If $v \in D_1(G) \cup D_2(G)$, then $N_G(v) \subseteq N_G(D_1(G) \cup D_2(G)) \subseteq V(L') \subseteq V(L)$, done.

Now we assume that $v \in \cup_{i \geq 3} D_i(G)$ and we first show that $v \in V(G_0 - S)$. As G is essentially 3-edge-connected, $v \in V(G_0)$. If one of the neighbors of v is a vertex of degree 1 or 2 in G, then $v \in N_G(D_1(G) \cup D_2(G)) \subseteq V(L') \subseteq V(L)$, done. Then we may assume that every neighbor of v is a vertex of degree at least 3 in G. As G is essentially 3-edge-connected, $\{v\} \cup N_G(v) \subseteq V(G_0)$ and $E_G(v) = E_{G_0}(v)$. Then $v \in V(G_0 - S)$ otherwise $E_G(v) = E_{G_0}(v) \subseteq S$, which implies $v \notin V(G - S)$, a contradiction.

Since L' is a DES of $G_0 - S$ and $v \in V(G_0 - S)$, either $v \in V(L')$ or $N_{G_0-S}(v) \subseteq V(L')$. Since $V(L') \subseteq V(L)$, $v \in V(L)$ or $N_{G-S}(v) = N_{G_0-S}(v) \subseteq V(L)$. This proves that L is a DES of $G - S$. ■

Theorem 2.6 below reveals the relationship between Hamilton cycles in a line graph $L(G)$ and dominating Eulerian subgraphs in G.

Theorem 2.6 (Harary and Nash-Williams [6]). *Let G be a connected graph with at least three edges. The line graph $L(G)$ is hamiltonian if and only if G has a dominating Eulerian subgraph.*

Let $S \subseteq E(G)$ and S' be the corresponding vertex set in the line graph $L(G)$. By the definition of line graphs, $L(G) - S' = L(G - S)$. Note that deleting vertex set S' in $L(G)$ corresponds to deleting edge set S in G. We may freely discard isolated vertices that arise in $G - S$ by edge deletion, because isolated vertices in a graph will not generate any vertex or edges in its line graph. For simplicity, we use $G - S$ in the discussions instead of $G - S - D_0(G - S)$. Throughout this article, isolated vertices arising from edge deletion will be deleted automatically unless otherwise specified. A relationship between Hamilton cycles in $L(G) - S'$ and dominating Eulerian subgraphs in $G - S$ is stated in Theorem 2.7.

Theorem 2.7. *Let G be a connected graph with at least three edges and $s \geq 0$ an integer. The line graph $L(G)$ is s-hamiltonian if and only if $G - S$ has a dominating Eulerian subgraph for any $S \subset E(G)$ with $|S| \leq s$.*

In Sections 3 and 4, we study the spanning tree packing number of an essentially 7-edge-connected and 3-edge-connected graph G in two cases with respect to the fact whether G is isomorphic to $G_{3,6}$ or not. Using these results and Theorem 2.4(ii), we prove our main result Theorem 1.3 in Section 5.

3. THE CASE OF $G_{3,6}$

Let $G = G_{3,6}$ denote a simple bipartite graph with a vertex bipartition (A, B) where $A = D_3(G)$ and $B = D_6(G)$. In this section, we prove that the core of the graph obtained from $G_{3,6}$ by deleting at most three edges has two edge disjoint spanning trees. In Lemma 3.1, $G - X$ refers to $G - X - D_0(G - X)$ as explained in Section 2. In the proof of Lemma 3.1, our approach is a refinement of the techniques that Zhan used to prove Theorem 1.2 in [14].

Lemma 3.1. *If $G = G_{3,6}$ is an essentially 7-edge-connected graph, and if $X \subseteq E(G)$ with $|X| \leq 3$, then $\tau((G - X)_0) \geq 2$.*

Proof. Let $G_c = (G - X)_0$. The operations involved are:

$$G \to G - X \to_{Operation1} \mathcal{O}_1(G - X) \to_{Operation2} (G - X)_0 = G_c.$$

Since $|X| \leq 3$ and $\delta(G) = 3$, the operation from G to $G - X$ generates at most one vertex of degree 1 in $G - X$ (the extreme case happens when two edges in X are incident with a vertex of degree 3 in G). Thus, at most four edges will be deleted by the operations from G to $\mathcal{O}_1(G - X)$. And by inspection, since $G = G_{3,6}$ and G is essentially 7-edge-connected, we have that $2d_4(\mathcal{O}_1(G - X)) + d_5(\mathcal{O}_1(G - X)) \leq 4$. Since

Operation 2 will not change the degree of vertices of degree 4 or 5, we have $2d_4(G_c) + d_5(G_c) \leq 4$.

Recall that G is bipartite, and $D_3(G)$ is independent in G, but $D_3(G_c)$ may not be independent in G_c. For a vertex u of degree 6 in G, $d_{G-X}(u) \in \{3, 4, 5, 6\}$. And the only possibility of generating a new vertex of degree 3 is that all three edges of X are incident with a vertex of degree 6. In this situation, $d_4(G_c) = d_5(G_c) = 0$. So we have the following claim.

Claim 1.

(i) Either $D_3(G_c)$ is independent in G_c with $|D_3(G) \cap D_3(G_c)| = d_3(G_c)$ or $D_3(G) \cap D_3(G_c)$ is maximally independent in G_c with $|D_3(G) \cap D_3(G_c)| = d_3(G_c) - 1$.
(ii) If $|D_3(G_c) \cap D_3(G)| = d_3(G_c) - 1$, then $2d_4(G_c) + d_5(G_c) = 0$.
(iii) If $|D_3(G_c) \cap D_3(G)| = d_3(G_c)$, then $2d_4(G_c) + d_5(G_c) \leq 4$. ■

For $S \subseteq E(G_c)$, let $L_1, L_2, \ldots, L_{r+m+t}$ be all the components of $G_c - S$, where $L_1, L_2, \ldots, L_r$ are the components, each of which is a single vertex of degree 3 in G_c, $L_{r+1}, \ldots, L_{r+m}$ are the nontrivial components of $G_c - S$, and $L_{r+m+1}, \ldots, L_{r+m+t}$ are the remaining components of $G_c - S$, i.e., each of them is a single vertex of degree at least 4 in G_c. Each set of above three categories is possibly an empty set.

By Theorem 2.2, to prove $\tau(G_c) \geq 2$ for any such S, it suffices to show that

$$|S| \geq 2(r + m + t - 1), \quad \text{or equivalently,} \quad 2|S| \geq 4(r + m + t - 1). \tag{1}$$

For each i, let $\partial_{G_c}(L_i)$ be the set of edges with one end in L_i, another end not in L_i and let $d_{G_c}(L_i) := |\partial_{G_c}(L_i)|$. By the definitions of L_i and $d_{G_c}(L_i)$, we have

$$\text{for any } i \text{ with } 1 \leq i \leq r, \quad d_{G_c}(L_i) = 3, \tag{2}$$

and

$$\text{for } r + m + 1 \leq j \leq r + m + t, \quad d_{G_c}(L_j) \geq 4. \tag{3}$$

Claim 2. *If there exists a j with $r + 1 \leq j \leq r + m$ such that $G_c - L_j$ is edgeless, then (1) holds.*

Proof of Claim 2. If $G_c - L_j$ is edgeless, then each of $L_1, L_2, \ldots, L_{j-1}, L_{j+1}, \ldots, L_{r+m+t}$ is a single vertex component and they are independent in G_c. By Lemma 2.5(i), $\delta(G_c) \geq 3$. So $|S| \geq \sum_{i=1}^{j-1} d_{G_c}(L_i) + \sum_{i=j+1}^{r+m+t} d_{G_c}(L_i) \geq 3d_{G_c}(L_j) \geq 3(r + m + t - 1)$. Thus (1) holds. ■

By Claim 2, we assume that for each j with $r + 1 \leq j \leq r + m$, $G_c - L_j$ is nontrivial. As L_j is also nontrivial, we have that

$$\text{for } r + 1 \leq j \leq r + m, \quad \partial_{G_c}(L_j) \text{ is an essential edge cut of } G_c. \tag{4}$$

Since G_c is the core of $G - X$, by the definition of a core graph, if X is an essential edge cut of G_c, then X is also an essential edge cut of $G - X$ (see Corollary 5.2 and its proof in Section 5). Together with (4), for $r + 1 \leq j \leq r + m$, $\partial_{G_c}(L_j)$ is also an essential edge cut of $G - X$ and so

$$\text{for } r + 1 \leq j \leq r + m, \quad \text{a subset of } \partial_{G_c}(L_j) \cup X \text{ is an essential edge cut of } G. \tag{5}$$

Claim 3.

(i) If $L_1 \cup L_2 \cup \cdots \cup L_r$ is not independent in G_c, then $\sum_{i=r+1}^{r+m+t} d_{G_c}(L_i) \geq 3r-6$. Furthermore, if $r \geq 2m+2t+1$, then (1) holds.
(ii) If $L_1 \cup L_2 \cup \cdots \cup L_r$ is independent in G_c, then $\sum_{i=r+1}^{r+m+t} d_{G_c}(L_i) \geq 3r$. Furthermore, if $r \geq 2m+2t-2$, then (1) holds.

Proof of Claim 3.

(i) By Claim 1(i), the independence number of $G_c[D_3(G_c)]$ is $d_3(G_c)$ or $d_3(G_c)-1$. Without loss of generality, we may assume that $L_1 \cup L_2 \cup \cdots \cup L_{r-1}$ is independent in G_c. Together with (2), there are at most three cross-edges between $G_c[L_1 \cup L_2 \cup \cdots \cup L_{r-1}]$ and $G_c[L_r]$. And as $L_1 \cup L_2 \cup \cdots \cup L_{r-1}$ is independent in G_c, by (2), we have $\sum_{i=r+1}^{r+m+t} d_{G_c}(L_i) \geq \sum_{i=1}^{r-1} d_{G_c}(L_i) - 3 \geq 3(r-1)-3 = 3r-6$. Together with (2), $2|S| = \sum_{i=1}^{r} d_{G_c}(L_i) + \sum_{i=r+1}^{r+m+t} d_{G_c}(L_i) \geq 3r+3r-6 = 6r-6$.
If $r \geq 2m+2t+1$, then $2|S| \geq 6r-6 = 4r+2r-6 \geq 4r+(4m+4t+2)-6 = 4r+4m+4t-4$. Hence (1) holds.
(ii) can be proved similarly. ■

By Claim 3, we can assume that

$$r \leq 2m+2t-y, \quad \text{where}$$
$$y=3 \text{ if } L_1 \cup L_2 \cup \cdots \cup L_r \text{ is independent in } G_c, \quad \text{and} \quad y=0 \text{ otherwise.} \tag{6}$$

Claim 4.

(i) $\sum_{i=r+1}^{r+m} d_{G_c}(L_i) \geq 7m-x$, where

$$x \leq \begin{cases} 6 & : \text{ if } m \geq 2 \\ 3 & : \text{ if } m=1 \\ 0 & : \text{ if } m=0 \end{cases}$$

(ii) $\sum_{i=r+m+1}^{r+m+t} d_{G_c}(L_i) \geq 6t - (2|\{v \in V(L_{r+m+1}) \cup \cdots \cup V(L_{r+m+t}) : d_{G_c}(v) = 4\}| + |\{v \in V(L_{r+m+1}) \cup \cdots \cup V(L_{r+m+t}) : d_{G_c}(v) = 5\}|) \geq 6t - (2d_4(G_c) + d_5(G_c))$.

Proof of Claim 4.

(i) It is trivial if $m=0$. Recall that $|X| \leq 3$. By (5), if $m=1$, $\sum_{i=r+1}^{r+m} d_{G_c}(L_i) \geq 7m-3$ and the equality holds when each of X has exactly one end in L_{r+1}; if $m \geq 2$, $\sum_{i=r+1}^{r+m} d_{G_c}(L_i) \geq 7m-6$ and the equality holds when all six ends of X are in $\cup_{i=r+1}^{r+m} L_{r+i}$ and the two ends of each edge of X lie in different components.
(ii) It follows from (3). So Claim 4 is established. ■

By (2) and Claim 4, $2|S| = \sum_{i=1}^{r} d_{G_c}(L_i) + \sum_{i=r+1}^{r+m} d_{G_c}(L_i) + \sum_{i=r+m+1}^{r+m+t} d_{G_c}(L_i) \geq 3r+7m-x+6t-(2d_4(G_c)+d_5(G_c)) = 3r+[2m+2t-y]+4m+4t+(m+y-x)-(2d_4(G_c)+d_5(G_c))$. Then by (6),

$$2|S| \geq 4r+4m+4t+(m+y-x)-(2d_4(G_c)+d_5(G_c)). \tag{7}$$

If $L_1 \cup \cdots \cup L_r$ is not independent in G_c, then by (6), (7), Claim 1(ii) and Claim 4(i), $2|S| \geq 4r + 4m + 4t + (m - x) \geq 4r + 4m + 4t - 4$, and (1) holds.

If $L_1 \cup \cdots \cup L_r$ is independent in G_c and $m \neq 2$, then by (6), (7), Claim 1(iii) and Claim 4(i), $2|S| \geq 4r + 4m + 4t + (m + 3 - x) - 4 = 4r + 4m + 4t + (m - 1 - x) \geq 4r + 4m + 4t - 4$, and (1) holds. So we assume that $L_1 \cup \cdots \cup L_r$ is independent in G_c and $m = 2$. If $x \leq 5$, then by (6), (7) and Claim 4, $2|S| \geq 4r + 4m + 4t + (2 + 3 - 5) - 4 = 4r + 4m + 4t - 4$. So by Claim 4(i) that $x \leq 6$, we may assume that $x = 6$. By the proof of Claim 4(i), $\sum_{i=r+1}^{r+m} d_{G_c}(L_i) = 7m - 6$ if and only if all six ends of X are in $\cup_{i=r+1}^{r+m} L_{r+i}$ and the two ends of each edge of X lie in different components. So each vertex in $\cup_{i=r+m+1}^{r+m+t} L_{r+i}$ has degree equal to 6 in G_c. Thus, by (2), (6), and Claim 4, $2|S| \geq 3r + 7m - x + 6t = 3r + [2m + 2t - y] + 4m + 4t + (m + y - x) \geq 4r + 4m + 4t + (2 + 3 - 6) = 4r + 4m + 4t - 1$.

Hence (1) is established, and so is Lemma 3.1.

4. AN ASSOCIATE RESULT

Again as explained in Section 2, the graph $G_1 = \mathcal{O}_1(G - S)$ in Theorem 4.1(ii) refers to $G_1 = \mathcal{O}_1(G - S - D_0(G - S))$ unless otherwise specified. Throughout this section, let $d_i = |D_i(G)|$.

Theorem 4.1. *Let G be a 3-edge-connected and essentially 7-edge-connected graph such that G is not isomorphic to $G_{3,6}$.*

(i) If

$$d_5 + \sum_{i \geq 7} \left(\frac{2i}{3} - 4 \right) d_i < 2, \tag{8}$$

then $2 + d_3 - \sum_{i \geq 5} (i - 4) d_i \leq 0$.

(ii) Let $S \subseteq E(G)$. If $|S| \leq 3$, then $G_1 = \mathcal{O}_1(G - S)$ has two edge disjoint spanning trees.

Proof. Claim 5 below follows from the assumption that G is essentially 7-edge-connected.

Claim 5. *For any edge $uv \in E(G)$, $d_G(u) + d_G(v) \geq 9$. In particular, for any $v \in D_3(G)$, any neighbor u of v has degree at least 6.* ■

Claim 6. *If $d_3 = 2d_6$ and $\Delta(G) \leq 6$, then G is isomorphic to $G_{3,6}$.*

Proof of Claim 6. Since $\Delta(G) \leq 6$, it follows by Claim 5 that every vertex in $D_3(G)$ must be and only be adjacent to vertices in $D_6(G)$. As $d_3 = 2d_6$, by counting the incidences, every vertex in $D_6(G)$ must also be adjacent to vertices in $D_3(G)$. So G must be isomorphic to $G_{3,6}$ and this proves Claim 6. ■

Proof of Theorem 4.1(i). Since G is not isomorphic to $G_{3,6}$, by Claim 6, we may assume that

$$\text{if } \Delta(G) \leq 6, \quad \text{then } d_3 \neq 2d_6. \tag{9}$$

The condition (8) immediately implies that

$$d_5 \le 1, \quad \Delta(G) \le 8, \quad d_7 \le 2, \quad \text{and} \quad d_8 \le 1. \tag{10}$$

By Claim 5, $3d_3 \le \sum_{i\ge 6} id_i$, or $d_3 \le \sum_{i\ge 6} \frac{i}{3}d_i$. Together with $\Delta(G) \le 8$, we have that

$$d_3 \le 2d_6 + \frac{7}{3}d_7 + \frac{8}{3}d_8, \quad \text{or} \quad 0 \le 2d_6 + \frac{7}{3}d_7 + \frac{8}{3}d_8 - d_3. \tag{11}$$

We argue by way of contradiction, and assume that $\sum_{i\ge 5}(i-4)d_i - d_3 < 2$. By $\Delta(G) \le 8$ again,

$$\sum_{i\ge 5}(i-4)d_i - d_3 = d_5 + 2d_6 + 3d_7 + 4d_8 - d_3 < 2, \quad \text{or}$$

$$2d_6 + \frac{7}{3}d_7 + \frac{8}{3}d_8 - d_3 < 2 - \left(d_5 + \frac{2}{3}d_7 + \frac{4}{3}d_8\right). \tag{12}$$

Combining (11) and (12), we have

$$0 \le 2d_6 - d_3 + \frac{7}{3}d_7 + \frac{8}{3}d_8 < 2 - \left(d_5 + \frac{2}{3}d_7 + \frac{4}{3}d_8\right). \tag{13}$$

■

Claim 7. $d_8 = 0$ *and* $\Delta(G) \le 7$.

Proof of Claim 7. Assume that $d_8 \ne 0$. By (10), $d_8 = 1$. Then by (13), $0 < \frac{2}{3} - (d_5 + \frac{2}{3}d_7)$, which implies that $d_5 = d_7 = 0$. Plugging them into (13), we have $0 \le 2d_6 - d_3 + \frac{8}{3} < \frac{2}{3}$, or $-\frac{8}{3} \le 2d_6 - d_3 < -2$, contrary to the fact that $2d_6 - d_3$ is an integer. So we must have $d_8 = 0$. Together with (10), $\Delta(G) \le 7$.

Plug $d_8 = 0$ into (13) to get

$$0 \le 2d_6 - d_3 + \frac{7}{3}d_7 < 2 - \left(d_5 + \frac{2}{3}d_7\right). \tag{14}$$

■

Claim 8. $d_7 = 1$.

Proof of Claim 8. First, we assume that $d_7 \ge 2$. By (10), $d_7 = 2$. Together with (8) and $\Delta(G) \le 7$, $d_5 = 0$. Together with Claim 7 and (14), $-14/3 \le 2d_6 - d_3 < -4$, a contradiction. So we may exclude the case $d_7 \ge 2$. If $d_7 = 0$, then by (14), $0 \le 2d_6 - d_3 < 2 - d_5$. By (9) that $2d_6 - d_3 \ne 0$, we have $1 \le 2d_6 - d_3 < 2 - d_5$. So $d_5 = 0$ and $2d_6 - d_3 = 1$. It follows that $d_3 \equiv 1 \pmod 2$. Since $d_5 = 0$ and $d_i = 0 (i \ge 7)$, the total number of odd degree vertices of G is an odd number d_3, contrary to the fact that in every graph, the number of odd degree vertices must be even. Hence we must have $d_7 = 1$. ■

Claim 9. $d_5 = 0$ *and* $d_3 \equiv 0$ *(mod 2).*

Proof of Claim 9. Plug $d_7 = 1$ into (14) to get

$$0 \le 2d_6 - d_3 + \frac{7}{3} < \frac{4}{3} - d_5, \quad \text{or} \quad -\frac{7}{3} \le 2d_6 - d_3 < -1 - d_5. \tag{15}$$

Since $2d_6 - d_3$ is an integer, (15) implies that $d_5 = 0$ and $2d_6 - d_3 = -2$. Then Claim 9 is established. ■

It follows from Claims 7, 8, and 9 that the number of odd degree vertices of G is an odd number $d_3 + d_5 + d_7 = d_3 + 1$, a contradiction. So we establish Theorem 4.1(i).

Proof of Theorem 4.1(ii). Since $\kappa'(G) \geq 3$, $|S| \leq 3$, G is essentially 7-edge-connected, and isolated vertices of $G - S$ are deleted, we have that $G_1 = \mathcal{O}_1(G - S)$ is connected and $\delta(G_1) \geq 2$. We argue by contradiction, and assume that

$$G \text{ is a counterexample of Theorem 4.1}(ii) \tag{16}$$

with

$$|E(G)| \text{ minimized.} \tag{17}$$

Next, we show that every nontrivial subgraph of G_1 does not have two edge disjoint spanning trees. Suppose not. Then G_1 has a nontrivial proper subgraph H with $\tau(H) \geq 2$. Note that H is also a subgraph of G. If $G/H \cong K_1$, then $G_1/H \cong K_1$, and so $\tau(G_1) \geq 2$, contrary to (16). So we can assume that $G/H \neq K_1$. As the contraction will not decrease the edge connectivity, G/H is 3-edge-connected and essentially 7-edge-connected. If G/H is isomorphic to $G_{3,6}$, then by Lemma 3.1, $\tau(G/H) = \tau(G_{3,6}) \geq 2$. By Theorem 2.3, $\tau(G) \geq 2$ and so $\tau(G_1) \geq 2$, contrary to (16). So G/H is not isomorphic to $G_{3,6}$ and thus G/H satisfies the conditions of Theorem 4.1. By (16), (17), and $|E(G/H)| < |E(G)|$, $G_1/H = \mathcal{O}_1(G/H - S)$ has two edge disjoint spanning trees. This, together with Theorem 2.3, implies that $\tau(G_1) \geq 2$, contrary to (16). Hence,

$$\text{if } H \text{ is a nontrivial proper subgraph of } G_1, \text{ then } \tau(H) < 2. \tag{18}$$

Note that by (18), G_1 must be simple. Clearly, $|V(G_1)| > 1$. By Theorem 2.1, if $\frac{|E(G_1)|}{|V(G_1)|-1} \geq 2$, then a violation to (18) will be found. Thus we may assume

$$\frac{|E(G_1)|}{|V(G_1)| - 1} < 2, \quad \text{or equivalently,} \quad 2|V(G_1)| - |E(G_1)| - 2 > 0. \tag{19}$$

Since $\delta(G) \geq \kappa'(G) \geq 3$, $|V(G)| = \sum_{i \geq 3} d_i$ and $2|E(G)| = \sum_{i \geq 3} i d_i$. Thus

$$4|V(G)| - 2|E(G)| = d_3 - \sum_{i \geq 5} (i - 4) d_i. \tag{20}$$

By Claim 5, $d_3 \leq \sum_{i \geq 6} \frac{i}{3} d_i$. Together with (20), we have that

$$4|V(G)| - 2|E(G)| \leq -d_5 - \sum_{i \geq 7} \left(\frac{2i}{3} - 4 \right) d_i. \tag{21}$$

■

Claim 10. $4|V(G_1)| - 2|E(G_1)| - 4 \leq 2 + 4|V(G)| - 2|E(G)|$.

Proof of Claim 10. We assume that $|S| = 3$, and the case when $|S| \leq 2$ can be proved similarly. In the proof below, $G - S$ is the graph obtained by deleting all edges of S from G and keep all resulting isolated vertices.

If $D_0(G - S) \cup D_1(G - S) = \emptyset$, then $|V(G_1)| = |V(G)|$ and $|E(G_1)| = |E(G)| - 3$. Hence $4|V(G_1)| - 2|E(G_1)| - 4 = 2 + 4|V(G)| - 2|E(G)|$ and Claim 10 holds. Next we assume that $D_0(G - S) \cup D_1(G - S) \neq \emptyset$.

Case 1: If two edges in S are incident with a vertex of degree 3 or three in S are incident with a vertex of degree 4, then $D_0(G - S) = 0, D_1(G - S) =$

1 and $|V(G_1)| = |V(G)| - 1, |E(G_1)| = |E(G)| - 4$. So $4|V(G_1)| - 2|E(G_1)| - 4 = 4|V(G)| - 2|E(G)|$ and Claim 10 holds.

Case 2: If three edges in S are incident with a vertex of degree 3 in G, then $D_0(G - S) = 1, D_1(G - S) = 0$ and $|V(G_1)| = |V(G)| - 1, |E(G_1)| = |E(G)| - 3$. So $4|V(G_1)| - 2|E(G_1)| - 4 = 4|V(G)| - 2|E(G)| - 2$ and Claim 10 is holds. ■

By Claim 10, (20), and (21), we have

$$4|V(G_1)| - 2|E(G_1)| - 4 \leq 2 + 4|V(G)| - 2|E(G)| = 2 + d_3 - \sum_{i\geq 5}(i-4)d_i, \tag{22}$$

and

$$4|V(G_1)| - 2|E(G_1)| - 4 \leq 2 + 4|V(G)| - 2|E(G)| \leq 2 - d_5 - \sum_{i\geq 7}\left(\frac{2i}{3} - 4\right) d_i. \tag{23}$$

By (19) and (23), $0 < 2 - d_5 - \sum_{i\geq 7}(\frac{2i}{3} - 4)d_i$, which is the condition (8) in Theorem 4.1(i). Then $2 + d_3 - \sum_{i\geq 5}(i-4)d_i \leq 0$ follows from Theorem 4.1(i). By (22), $4|V(G_1)| - 2|E(G_1)| - 4 \leq 0$, contrary to (19). Hence, Theorem 4.1(ii) is established.

5. THE PROOF OF THEOREM 1.3

In this section, we first show that Theorem 1.3 holds for $s = 5$ by proving Theorem 5.3 below. The proof of Theorem 5.3 involves a lot of edge contractions. We will repeatedly use Proposition 5.1 below and Lemma 2.5(ii) in the proof.

Proposition 5.1. *Let G be a graph and H a subgraph of G. If X is an edge cut (or essential edge cut, respectively) of G/H, then X is also an edge cut (or essential edge cut, respectively) of G.*

Proof. Let $e \in E(H)$, X be an essential edge cut of G/e, G'_1 and G'_2 be the two sides of $(G/e) - X$, and let v_e denote the vertex of G/e onto which e is contracted. We may assume that $v_e \in V(G'_1)$. Then $G_1 = G[E(G'_1) \cup e]$ and $G_2 = G'_2$ are the two sides of $G - X$, and so X is an edge-cut of G. Both G'_1 and G'_2 have edges, so do G_1 and G_2. Thus we proved that if X is an essential edge cut of G/e, then X is an essential edge cut of G. Thus, Proposition 5.1 can be proved by applying induction on $E(H)$. ■

Corollary 5.2. *Let G be an essentially 3-edge-connected graph and G_0 the core of G. If X is an edge cut (or essential edge cut, respectively) of G_0, then X is also an edge cut (or essential edge cut, respectively) of G.*

Proof. It follows from Proposition 5.1 as the core of an essentially 3-edge-connected graph G can be viewed as a contraction of G (contracting all the pendent edges and one from the two edges of degree 2 vertices). ■

Theorem 5.3. *Let G be a connected graph. Then $L(G)$ is 5-hamiltonian if and only if $L(G)$ is 7-connected.*

Proof. By Theorem 2.7, it suffices to show that for any $S \subseteq E(G)$ with $|S| \le 5$, $G - S$ has a DES. ■

Let $X \subseteq S$ be a subset with $|X| = \min\{|S|, 3\}$ and $Y = S - X$. Then $|X| \le 3$ and $|Y| \le 2$. In order to show that $G - X - Y$ has a DES, we use Lemma 3.1 and Theorem 4.1 to prove Lemma 5.5(i)-(iii), showing that $(G_0 - X)_0 - Y$ has a DES, which can be extended to a DES of $G - X - Y$.

Lemma 5.4. *Let G be an essentially 7-edge-connected graph and G_0 be its core. If $X \subseteq S \subseteq E(G)$ is a subset with $|S| \le 5$, $|X| = min\{|S|, 3\}$ and $Y = S - X$, then $F((G_0 - X)_0 - Y) \le 2$.*

Proof of Lemma 5.4. Since G is essentially 7-edge-connected, by Lemma 2.5(i) and Corollary 5.2, the core graph G_0 is 3-edge-connected and essentially 7-edge-connected.

If G_0 is not $G_{3,6}$, by Theorem 4.1(ii), $F(\mathcal{O}_1(G_0 - X)) = 0$. As each incident edge of a degree 2 vertex belongs to exactly one spanning tree of $\mathcal{O}_1(G_0 - X))$, simply deleting such edges in each spanning tree generates two edge disjoint spanning trees of $(G_0 - X)_0$. Since $|Y| \le 2$, $F((G_0 - X)_0 - Y) \le 2$.

If G_0 is $G_{3,6}$, then $G = G_{3,6}$ otherwise G has essential edge cuts of size 3 or 6, a contradiction. By Lemma 3.1, $F((G - X)_0) = 0$. Since $|Y| \le 2$, $F((G - X)_0 - Y) \le 2$. As $G_0 = G = G_{3,6}$, $F((G_0 - X)_0 - Y) \le 2$. This completes the proof of Lemma 5.4. ■

Lemma 5.5. *Let G be an essentially 7-edge-connected graph and G_0 be its core. Let $X \subseteq S \subseteq E(G)$ be a subset with $|S| \le 5$, $|X| = min\{|S|, 3\}$ and $Y = S - X$. Each of the following holds*

- *(i) Let $H = G_0 - X$ and H_0 be the core of H. Then $H_0 - Y$ has a DES, written L''.*
- *(ii) Let L' be the graph obtained by reversing Operation 2 on L'' in H. Then L' is a DES of $H - Y = G_0 - X - Y = G_0 - S$.*
- *(iii) Let L be the graph obtained by reversing Operation 2 on L' in G. Then L is a DES of $G - S$.*

Proof of Lemma 5.5. Let R_1 be the c-reduction of $H_0 - Y$. By Lemma 5.4 and Theorem 2.4(ii), $R_1 \in \{K_1, K_2, K_{2,l}\}$. We prove Lemma 5.5(i)–(iii) by considering each case, respectively, and repeatedly using Lemma 2.5(ii).

Case 1: $R_1 \cong K_1$ or $K_{2,l}$ where l is an even integer.

Since R_1 has an SES, by Theorem 2.4(i), $H_0 - Y$ has an SES, written L''. So (i) is established.

And since $\kappa'(H_0) \ge 3$ and $|Y| \le 2$, we have that

$$V(L'') = V(H_0 - Y) = V(H_0). \tag{24}$$

Since H is essentially 4-edge-connected, $N_H(D_1(H) \cup D_2(H)) \subseteq V(H_0) = V(L'')$. By Lemma 2.5(ii), L' is a DES of $H - Y$ and so Lemma 5.5(ii) is established. By the definition of L', we also have

$$V(L'') \subseteq V(L'). \tag{25}$$

Let $w \in D_1(G) \cup D_2(G)$. Then $d_G(w) \le 2$. Let $wt \in E(G)$. Since G is essentially 7-edge-connected, $d_G(t) \ge 7$. So $t \in V(G_0)$ and $d_{G_0}(t) \ge 7$, which implies $d_{G_0-X}(t) \ge 4$. By the definition of core graphs, (24), and (25), $t \in V((G_0 - X)_0) = V(H_0) = V(L'') \subseteq V(L')$. This proves that $N_G(D_1(G) \cup D_2(G)) \subseteq V(L')$. By Lemma 5.5(ii) and Lemma 2.5(ii), Lemma 5.5(iii) is established.

Case 2: $R_1 \cong K_2$ or $K_{2,1}$.

Then let $D_1(R_1) = \{a, b\}$. Notice the operations involved are the following:
$G \Rightarrow G_0 \Rightarrow G_0 - X = H \Rightarrow \mathcal{O}_2(\mathcal{O}_1(H)) = H_0 \Rightarrow H_0 - Y \Rightarrow (H_0 - Y)_c = R_1$.

Claim 11.

(i) If $R_1 \cong K_2$, then at most one of $\{a, b\}$ is contracted from a nontrivial subgraph of $H_0 - Y$.
(ii) If $R_1 \cong K_{2,1}$, then neither a nor b is a vertex contracted from a nontrivial subgraph of $H_0 - Y$.

Proof of Claim 11.

(i) By way of contradiction, we assume that both a and b are contracted from a nontrivial subgraph of $H_0 - Y$. Then $\{ab\}$ is an essential edge cut of $H_0 - Y$, and so $\{ab\} \cup Y$ contains an essential edge cut of size at most 3 in H_0, which by Corollary 5.2, corresponds to an essential edge cut of size at most 3 in H_0, contrary to the fact that H_0 is essentially 4-edge-connected.
(ii) Let $D_2(K_{2,1}) = \{c\}$. By way of contradiction, we assume that either a or b is a vertex contracted from a nontrivial subgraph in H_0. Then $\{ac\} \cup Y$ or $\{bc\} \cup Y$ contains an essential edge cut of size at most 3 in H_0, which by Corollary 5.2, corresponds to an essential edge cut of size at most 3 in H_0, contrary to the fact that H is essentially 4-edge-connected. Hence Claim 11 is established. ■

By Claim 11, we may assume that

$$b \text{ (neither } a \text{ nor } b) \text{ is not contracted from a subgraph of } H_0 - Y \text{ if } R_1 \cong K_2\ (R_1 \cong K_{2,1}). \tag{26}$$

So b is a vertex of $H_0 - Y$ if $R_1 \cong K_2$, and a and b are vertices of $H_0 - Y$ if $R_1 \cong K_{2,1}$. By Theorem 2.4(i),

$$H_0 - Y \text{ has a DES, written } L'', \text{ containing all vertices of } H_0 - Y - \{b\} \text{ if } R_1 \cong K_2, \tag{27}$$

and

$$H_0 - Y \text{ has a DES, written } L'', \text{ containing all vertices of } H_0 - Y - \{a, b\} \text{ if } R_1 \cong K_{2,1}. \tag{28}$$

Thus Lemma 5.5(i) is established.

Let $h \in D_1(H) \cup D_2(H)$. Then $d_H(h) \leq 2$. Let $hg \in E(H)$. Since H is essentially 4-edge-connected, $d_H(g) \geq 4$. So $g \in V(H_0)$ and $d_{H_0}(g) \geq 4$, which implies $d_{H_0-Y}(g) \geq 2$. By (26), $d_{H_0-Y}(b) = 1$ when $R_1 \cong K_2$ ($d_{H_0-Y}(a) = d_{H_0-Y}(b) = 1$ when $R_1 \cong K_{2,1}$ respectively). Thus $g \neq b$ when $R_1 \cong K_2$ ($g \neq a, b$ when $R_1 \cong K_{2,1}$ respectively). Together with (27) and (28), we have that $g \in V(L'')$. This proves that $N_H(D_1(H) \cup D_2(H)) \subseteq V(L'')$. By Lemma 5.5(i) and Lemma 2.5(ii), Lemma 5.5(ii) is established.

Let $w \in D_1(G) \cup D_2(G)$ and $wt \in E(G)$. By the same argument as in Case 1, $d_{H_0-Y}(t) \geq 2$. Since $d_{H_0-Y}(b) = 1$ when $R_1 \cong K_2$ ($d_{H_0-Y}(a) = d_{H_0-Y}(b) = 1$ when $R_1 \cong K_{2,1}$ respectively), $t \notin \{a, b\}$. By (27) and (28), $t \in V(L'') \subseteq V(L')$. This proves that

$N_G(D_1(G) \cup D_2(G)) \subseteq V(L')$. By Lemma 5.5(ii) and Lemma 2.5(ii), Lemma 5.5(iii) is established.

Case 3: $R_1 \cong K_{2,l}$ where l is odd and $t \geq 3$.

Let $D_2(K_{2,l}) = \{w_1, w_2, \ldots, w_l\}$ and $D_t(K_{2,l}) = \{v_1, v_2\}$. If w_i $(1 \leq i \leq l)$ is a vertex contracted from a nontrivial subgraph T of $H_0 - Y$, then $\{w_iv_1, w_iv_2\}$ is an essential 2-edge-cut of $H_0 - Y$. Since H_0 is essentially 4-edge-connected, $|Y| \leq 2$ and $l \geq 3$, by counting the incidence, we have the following claim.

Claim 12. *At least one vertex in $D_2(K_{2,l})$, say w_k, is not a vertex contracted from a nontrivial subgraph of $H_0 - Y$, and w_k is incident with at most one edge of Y.*

Since l is odd, $K_{2,l} - w_k$ is a DES of $K_{2,l}$. By Theorem 2.4(i),

$$H_0 - Y \text{ has a DES, written } L'', \text{ containing all vertices of } H_0 - Y - \{w_k\}. \tag{29}$$

This completes the proof of Lemma 5.5(i).

Let $h \in D_1(H) \cup D_2(H)$. Then $d_H(h) \leq 2$. Let $hg \in E(H)$. Since H is essentially 4-edge-connected, $d_H(g) \geq 4$. Then $g \in V(H_0)$ and $d_{H_0}(g) \geq 4$. It implies that $d_{H_0-Y}(g) \geq 2$ and the equality holds only if all edges of Y are incident with g. By Claim 12, $g \neq w_k$. Together with (29), we have that $g \in V(L'')$. This proves that $N_H(D_1(H) \cup D_2(H)) \subseteq V(L'')$. By Lemma 5.5(i) and Lemma 2.5(ii), Lemma 5.5(ii) is established.

Let $w \in D_1(G) \cup D_2(G)$ and $wt \in E(G)$. Since G is essentially 7-edge-connected, $d_G(t) \geq 7$. Then $t \in V(G_0)$ and $d_{G_0-X}(t) \geq 4$ and the equality holds only if all edges of X are incident with g. It follows that $t \in V((G_0 - X)_0) = V(H_0)$ and so $d_{H_0-Y}(t) \geq 2$ and the equality holds only if all edges of $X \cup Y$ are incident with g. By Claim 12, $t \neq w_k$ and thus by (29), $t \in V(L'') \subseteq V(L')$. This proves that $N_G(D_1(G) \cup D_2(G)) \subseteq V(L')$. By Lemma 5.5(ii) and Lemma 2.5(ii), Lemma 5.5(iii) is established.

Proof of Theorem 1.3. By Theorem 5.3, we may assume that $s \geq 6$. Let $S \subseteq V(L(G))$ with $|S| \leq s$.

Pick a subset $S_1 \subseteq S$ such that $|S_1| = 5$ and let $S_2 = S - S_1$. Then $S_2 \in V(L(G))$. Since $\kappa(L(G)) \geq s+2$, $\kappa(L(G) - S_2) \geq s + 2 - |S_2| = (s - |S_2|) + 2 = 5 + 2 \geq 7$. Let $S_2' \subseteq E(G)$ be the corresponding edge set of $S_2 \subseteq V(L(G))$. Since $L(G) - S_2 = L(G - S_2')$, we have $\kappa(L(G - S_2')) \geq 7$ and so by Theorem 5.3, $L(G) - (S_1 \cup S_2) = (L(G) - S_2) - S_1 = L(G - S_2') - S_1$ is hamiltonian. Thus $L(G)$ is s-hamiltonian. ■

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