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On Spanning Disjoint Paths in Line Graphs

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Abstract Spanning connectivity of graphs has been intensively investigated in the study of interconnection networks (Hsu and Lin, Graph Theory and Interconnection Networks, 2009). For a graph G and an integer $s > 0$ and for $u, v \in V(G)$ with $u \neq v$, an $(s; u, v)$-path-system of G is a subgraph H consisting of s internally disjoint (u, v)-paths. A graph G is **spanning s-connected** if for any $u, v \in V(G)$ with $u \neq v$, G has a spanning $(s; u, v)$-path-system. The **spanning connectivity** $\kappa^*(G)$ of a graph G is the largest integer s such that G has a spanning $(k; u, v)$-path-system, for any integer k with $1 \leq k \leq s$, and for any $u, v \in V(G)$ with $u \neq v$. An edge counter-part of $\kappa^*(G)$, defined as the supereulerian width of a graph G, has been investigated in Chen et al. (Supereulerian graphs with width s and s-collapsible graphs, 2012). In Catlin and Lai (Graph Theory, Combinatorics, and Applications, vol. 1, pp. 207–222, 1991) proved that if a graph G has 2 edge-disjoint spanning trees, and if $L(G)$ is the line graph of

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G, then $\kappa^*(L(G)) \geq 2$ if and only if $\kappa(L(G)) \geq 3$. In this paper, we extend this result and prove that for any integer $k \geq 2$, if G_0, the core of G, has k edge-disjoint spanning trees, then $\kappa^*(L(G)) \geq k$ if and only if $\kappa(L(G)) \geq \max\{3, k\}$.

Keywords Connectivity · Spanning connectivity · Hamiltonian linegraph · Hamiltonian-connected line graph · Supereulerian graphs · Collapsible graphs

1 Introduction

Graphs in this paper are finite and may have multiple edges but no loops. Terminology and notations not defined here are referred to [1]. In particular, for a graph G, $\delta(G)$, $\kappa(G)$ and $\kappa'(G)$ represent the minimum degree, the connectivity and the edge connectivity of the graph G, respectively. A path with initial vertex u and terminal vertex v will be referred as a (u, v)-path. We use $O(G)$ to denote the set of all odd degree vertices in G, and $D_i(G)$ the set of all vertices of degree i in G. A graph G is **Eulerian** if $O(G) = \emptyset$ and G is connected, and is **supereulerian** if G has a Eulerian subgraph H with $V(H) = V(G)$. The maximum number of edge-disjoint spanning trees in a graph G is denoted by $\tau(G)$.

For an integer $s > 0$ and for $u, v \in V(G)$ with $u \neq v$, an $(s; u, v)$-path-system of G is a subgraph H consisting of s internally disjoint (u, v)-paths, and such an H is called a spanning (s, u, v)-path-system if $V(H) = V(G)$. A graph G is **spanning s-connected** if for any $u, v \in V(G)$ with $u \neq v$, G has a spanning $(s; u, v)$-path-system. The **spanning connectivity** $\kappa^*(G)$ of a graph G is the largest integer s such that for any integer k with $1 \leq k \leq s$ and for any $u, v \in V(G)$ with $u \neq v$, G has a spanning $(k; u, v)$-path-system. A graph G is hamiltonian connected if for any $u, v \in V(G)$ with $u \neq v$ G has a path P from u to v such that $V(P) = V(G)$. Thus $\kappa^*(G) \geq 1$ if and only if G is hamiltonian-connected. The hamiltonian connectedness of graphs has been intensively studied, as shown in [8]. The spanning connectivity of a graph has also been studied, as can be seen in Chapters 14 and 15 of [11].

The **line graph** of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent in $L(G)$. Many interesting structure properties of a graph are closely related to the same properties of its line graph. Cai and Corneil [2] proved that the cycle double conjecture [18,20] holds for all 2-edge-connected graphs if and only if it holds for all 2-edge-connected line graphs. Chen et al. [6] proved that to solve Tutte's flow conjectures [13,22] in graphs, one only needs to prove the truth of these conjectures in line graphs. Thomassen's conjecture [21] that "every 4-connected line graph is hamiltonian" had attracted many researchers working on properties of line graphs. Catlin and Lai in [5] characterized line graphs $L(G)$ with $\kappa^*(L(G)) \geq 2$ for graphs G with $\tau(G) \geq 2$.

Theorem 1.1 (Catlin and Lai [5]) *Let G be a graph with $\tau(G) \geq 2$. Then $\kappa^*(L(G)) \geq 2$ if and only if $\kappa(L(G)) \geq 3$.*

By the well known spanning tree packing theorem of Nash-Williams [17] and Tutte [23], every $2k$-edge-connected graph must have k edge-disjoint spanning trees. Therefore, Theorem 1.1 implies the next theorem.

Theorem 1.2 (Zhan [24]) *If* $\kappa'(G) \geq 4$, *then* $\kappa^*(L(G)) \geq 2$.

Huang and Hsu [12] proved the following theorem, which extends Theorem 1.2 from $k = 2$ to all integers $k \geq 2$.

Theorem 1.3 (Huang and Hsu [12]) *For any integer* $k \geq 2$, *if* $\kappa'(G) \geq 2k \geq 4$, *then* $\kappa^*(L(G)) \geq k$.

In this paper, using a modified Catlin's reduction technique [3], we prove a new theorem which includes the theorems mentioned above as special cases.

Let G be a graph such that $\kappa(L(G)) \geq 3$. The **core** of G, denoted by G_0, is defined as follows (see [19]). For each $v \in V(G)$, let $E_G(v)$ denote the set of edges incident with v in G. For any $u, v \in D_2(G)$, $E_G(u) \cap E_G(v) = \emptyset$. For each $v \in D_2(G)$, denote $E_G(v) = \{e'_v, e''_v\}$. Let

$$X_2(G) = \{e''_v : v \in D_2(G)\}.$$

We define the **core** of G by

$$G_0 = (G - D_1(G))/X_2(G).$$

Here is our main theorem:

Theorem 1.4 *Let* $k \geq 2$ *be an integer, and* G *be a connected graph with a core* G_0 *such that* $\tau(G_0) \geq k$. *Then* $\kappa^*(L(G)) \geq k$ *if and only if* $\kappa(L(G)) \geq \max\{3, k\}$.

Remark When $k = 2$, Theorem 1.4 implies Theorem 1.1. Noting that $\kappa'(G_0) \geq \kappa'(G)$ and applying the spanning tree packing theorem of Nash-Williams [17] and Tutte [23], Theorem 1.3 follows from Theorem 1.4 immediately. In [9], infinitely many graphs G satisfying $\kappa'(G) = \tau(G)$ with minimum possible edges have been constructed, and all such graphs are characterized. For any of such graph G, we have $\kappa^*(L(G)) \geq k$ by Theorem 1.4; but we cannot make the same conclusion by Theorem 1.3. Hence, Theorem 1.4 is stronger than Theorem 1.3. The following examples give additional evidences that even edge-connectivity condition in Theorem 1.3 can be relaxed.

Let $k \geq 2$ and H be any graph with k edge-disjoint spanning trees. Obtain G from H by

(i) subdividing every edge of H exactly once, and
(ii) attaching a pendent edge at every vertex of H (not including the new vertices resulting from the subdivision operation).

Since the core of G is H, by Theorem 1.4, $\kappa^*(L(G)) \geq k$. But since $\kappa'(G) < 2$, such a conclusion cannot be made by using Theorem 1.3.

In the next section, we will prove a characterization of a graph G whose line graph $L(G)$ satisfying $\kappa^*(L(G)) \geq k$, analogous to the characterization of Harary and Nash-Williams on hamiltonian line graphs [10]. A reduction method involving s-collapsible graphs will be presented in Sect. 3. In Sect. 4, we review some properties of the core of a graph. The results in Sects. 2, 3, and 4 will be applied to prove the main result in Sect. 5.

2 Spanning Connectivity in Line Graphs

In this section, we shall follow the idea of Harary and Nash-Williams in [10] to determine a relationship between dominating $(k; e', e'')$-trail systems in G and spanning $(k; e', e'')$-path-systems in $L(G)$. We view a trail of G as a vertex-edge alternating sequence

$$v_0, e_1, v_1, e_2, \ldots, e_k, v_k \tag{1}$$

such that all the e_i are distinct and for each $i = 1, 2, \ldots, k e_i$ is incident with both v_{i-1} and v_i. When the edge-vertex incidence is understood from the context for such a trail, we often use an edge sequence $e_1 e_2 \ldots e_k$ to denote the same trail in (1). All the vertices in $\{v_1, v_2, \ldots, v_{k-1}\}$ are **internal vertices** of the trail in (1). For edges $e', e'' \in E(G)$, an (e', e'')-**trail** of G is a trail of G whose first edge is e' and whose last edge is e''. As an example, the trail in (1) is an (e_1, e_k)-trail. An (e', e'')-trail T of G is **dominating in** G if every edge of G is incident with an internal vertex of T; and a **spanning** $(e', \mathbf{e}'')$**-trail** T of G is a dominating (e', e'')-trail T of G such that $V(T) = V(G)$. A **dominating** $(k; e', e'')$-**trail systems** in G is a subgraph H consisting of k edge-disjoint (e', e'')-trail $(T_1, T_2, \ldots, T_k)$ such that every edge of G is incident with an internal vertex of T_i for some $i (1 \leq i \leq k)$.

Theorem 2.1 *Let $s \geq 1$ be an integer, and G a graph with $|E(G)| \geq 3$. The following are equivalent.*

(i) $\kappa^*(L(G)) \geq s$;
(ii) *For any edge e', $e'' \in E(G)$, G has a dominating $(k; e', e'')$-trail-system, for all $1 \leq k \leq s$.*

Proof Assume that $\kappa^*(L(G)) \geq s$. By the definition of κ^*, for any positive integer $k \leq s$, and for any e' and e'' in $E(G)$, $L(G)$ has a spanning $(k; e', e'')$-path-system $(P_1, P_2, \ldots, P_k)$.

Denote $P_i = e^i_1 e^i_2 \ldots e^i_{n_i}$, where each $e^i_j \in E(G) = V(L(G))$, and where $e^i_1 = e'$ and $e^i_{n_i} = e''$, for $i = 1, 2, \ldots, k$. By the definition of a line graph, G has a longest (e', e'')-trail $T_i = e^i_{i_1} e^i_{i_2} \ldots e_{i_{n(i)}}$ such that $e^i_{i_1} = e'$, $e_{i_{n(i)}} = e''$ and $i_1, i_2, \ldots, i_{n(i)}$ is a subsequence of $1, 2, \ldots, n_i$. Since $P_i = e^i_1 e^i_2 \ldots e^i_{n_i}$ is a path in $L(G)$, by the definition of a line graph and by the maximality of $|V(T_i)|$, for any j with $1 \leq j < n(i)$, if $i_{j+1} > i_j + 1$ and if $v_j \in V(G)$ is the vertex in the trail $e^i_{i_1} e^i_{i_2} \ldots e_{i_{n(i)}}$ incident with both e_{i_j} and $e_{i_{j+1}}$, then any edge e^i_t with $i_j < t < i_{j+1}$ must be incident with v_j in G. It follows that $(T_1, T_2, \ldots, T_k)$ is a dominating $(k; e', e'')$-trail-system of G.

Conversely, we assume that (ii) holds to prove (i). Suppose $\{T_1, T_2, \ldots, T_k\}$ is a dominating $(k; e', e'')$-trail-system of G for any k with $1 \le k \le s$. By the definition of dominating $(k; e', e'')$-trail-systems, for any edge $e \in E(G) - \bigcup_{i=1}^{k} E(T_i)$, there exists an i such that e is incident with an internal vertex of T_i. Therefore, we can partition $E(G) - \bigcup_{i=1}^{k} E(T_i)$ into a disjoint union of subsets $X_1, X_2, \ldots, X_k$ such that edges in X_i are incident with internal vertices of T_i. It follows by the definition of line graphs that in $L(G)$, the vertex subset $E(T_i) \cup X_i$ induces a subgraph in $L(G)$ which contains an (e', e'')-path P_i of $L(G)$. Since every edge of G must be in an $E(T_i) \cup X_i$, $(P_1, P_2, \ldots, P_k)$ is a spanning (e', e'')-path system of $L(G)$. □

3 Reductions and s-Collapsible Graphs

Throughout this paper, we shall adopt the convention that any graph G is 0-edge-connected, and always assume that $s \ge 1$ is an integer.

Definition 3.1 A graph G is s-**collapsible** if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning subgraph Γ_R such that

(i) both $O(\Gamma_R) = R$ and $\kappa'(\Gamma_R) \ge s - 1$, and
(ii) $G - E(\Gamma_R)$ is connected.

Thus a collapsible graph defined in [3] is a 1-collapsible graph in Definition 3.1. A spanning subgraph Γ_R of G with both properties in Definition 3.1 is an (s, R)-**subgraph** of G. Let $\mathcal{C}_s$ denote the collection of all s-collapsible graphs. Then $\mathcal{C}_1$ is the collection of all collapsible graphs [3]. By definition, for $s \ge 1$, any $(s+1, R)$-subgraph of G is also an (s, R)-subgraph of G. This implies that

$$\mathcal{C}_{s+1} \subseteq \mathcal{C}_s, \text{ for any positive integer } s. \tag{2}$$

For a graph G, and for $X \subseteq E(G)$, the **contraction** G/X is obtained from G by identifying the two ends of each edge in X and then by deleting the resulting loops. If H is a subgraph of G, then we write G/H for $G/E(H)$, and we use v_H to denote the vertex in G/H onto which H is contracted.

Proposition 3.2 ([7,15]) *Let $s \ge 1$ be an integer. Then $\mathcal{C}_s$ satisfies the following.*

(C1) $K_1 \in \mathcal{C}_s$
(C2) *If $G \in \mathcal{C}_s$ and if $e \in E(G)$, then $G/e \in \mathcal{C}_s$.*
(C3) *If H is a subgraph of G and if $H, G/H \in \mathcal{C}_s$, then $G \in \mathcal{C}_s$.*

Let G be a graph, and $s > 0$ be an integer. For any distinct $u, v \in V(G)$, an **($\mathbf{s}$; $\mathbf{u}$, $\mathbf{v}$)-trail-system** of G is a subgraph H consisting of s edge-disjoint (u, v)-trails. A graph is **supereulerian with width s** if for any $u, v \in V(G)$ with $u \ne v$, G has a spanning $(s; u, v)$-trail-system. The **supereulerian width** $\mu'(G)$ of a graph G is the largest integer s such that G is supereulerian with width k for any integer k with $1 \le k \le s$. A reduction method on applying s-collapsible graphs to study $\mu'(G)$ has been developed in [7,15].

Lemma 3.3 ([7,15]) *Let $s \geq 1$ be an integer. If a graph $G \in \mathcal{C}_s$, then $\mu'(G) \geq s+1$.*

A graph is $\mathcal{C}_\mathbf{s}$**-reduced** if it contains no nontrivial subgraph in $\mathcal{C}_s$. It is shown in [7] that every graph G has a unique collection of maximally s-collapsible subgraphs $H_1, H_2, \ldots, H_c$, and the graph $G'_s = G/(\cup_{i=1}^{c} E(H_i))$ is $\mathcal{C}_s$-reduced, which is called the $\mathcal{C}_\mathbf{s}$**-reduction** of G.

Lemma 3.4 ([7,15]) *Let $s \geq 1$ be an integer, G be a graph and H be a subgraph of G such that $H \in \mathcal{C}_s$. Each of the following holds.*

- (i) *$G \in \mathcal{C}_s$ if and only if $G/H \in \mathcal{C}_s$.*
- (ii) *$\mu'(G) \geq s+1$ if and only if $\mu'(G/H) \geq s+1$.*

Let $F(G, s)$ denote the minimum number of additional edges that must be added to G to result in a graph Γ with $\tau(\Gamma) \geq s$. The quantity of $F(G, s)$ has been determined in [16], whose matroidal versions are proved in [14,15].

Theorem 3.5 ([7,15]) *Let $s \geq 1$ be an integer. If $F(G, s+1) \leq 1$, then $G \in \mathcal{C}_s$ if and only if $\kappa'(G) \geq s+1$.*

Theorem 3.6 (Catlin et al. Theorem 1.3 of [4]) *Let G be a connected graph and t an integer. If $F(G, 2) \leq 2$, then $G \in \mathcal{C}_1$ if and only if the $\mathcal{C}_1$-reduction of G is not a member in $\{K_2\} \cup \{K_{2,t} : t \geq 1\}$.*

4 Facts on the Core of a Graph

Throughout this section, we assume that G is a connected graph satisfying $\kappa(L(G)) \geq 3$. For any $e', e'' \in E(G)$, let $G(e', e'')$ be the graph obtained from G by replacing $e' = u'v'$ by a path $u'v_{e'}v'$ and by replacing $e'' = u''v''$ by a path $u''v_{e''}v''$, where $v_{e'}$ and $v_{e''}$ are new vertices added to the graph when subdividing e' and e'', respectively.

Proposition 4.1 (Shao, Lemma 1.4.1 and Proposition 1.4.2 of [19]) *Let G be a connected graph with $\kappa(L(G)) \geq 3$, and let G_0 denote the core of G. Each of the following holds.*

- (i) *G_0 is uniquely defined.*
- (ii) *$\delta(G_0) \geq \kappa'(G_0) \geq 3$.*
- (iii) *If G_0 is supereulerian, then $L(G)$ is hamiltonian.*
- (iv) *If for any $e', e'' \in E(G_0)$, $G_0(e', e'')$ has a spanning $(v_{e'}, v_{e''})$-trail, then $L(G)$ is hamiltonian-connected.*

In this section, we extend some of Shao's results above for later applications in our proofs. For any integer $k > 0$, and for any $e', e'' \in E(G_0)$, define $G_0^k(e', e'')$ be the graph obtained from $G_0(e', e'')$ by, for any $v \in \{v_{e'}, v_{e''}\}$, replacing each edge incident with v in $G_0(e', e'')$ by a set of $\lceil k/2 \rceil$ parallel edges. As examples, $G_0^1(e', e'') = G_0^2(e', e'') = G_0(e', e'')$.

Lemma 4.2 *Let k, l and s be integers such that $s \geq 1 l \geq 2$ and $k \geq 2$.*

(i) ([7,15]) *Let lK_2 is the loopless connected graph with two vertices and l edges. Then $lK_2 \in \mathcal{C}_s$ if and only if $l \geq s+1$. More generally, if T is a tree with $|E(T)| \geq 2$ and if lT is the graph obtained from T by replacing every edge of T by a set of l parallel edges. Then $lT \in \mathcal{C}_s$ if and only if $l \geq s+1$.*

(ii) *If $G_0 - \{e', e''\} \in \mathcal{C}_{k-1}$, then $G_0^k(e', e'') \in \mathcal{C}_{k-1}$.*

Proof (ii). Let $G' = G_0 - \{e', e''\}$. By the definition of $G_0^k(e', e'')$, $G_0^k(e', e'')/G' = lK_{1,2}$ with $l \geq k$. By Lemma 4.2(i), $G_0^k(e', e'')/G' = lK_{1,2} \in \mathcal{C}_{k-1}$. Since $G' \in \mathcal{C}_{k-1}$, it follows by Proposition 3.2 (C3) that $G_0^k(e', e'') \in \mathcal{C}_{k-1}$. □

Theorem 4.3 *Let G be a graph with core G_0, and let $k \geq 3$ be an integer. Each of the following holds.*

(i) *If for any $e', e'' \in E(G_0)$ with $e' \neq e''$, $G_0(e', e'') \in \mathcal{C}_1$, then $\kappa^*(L(G)) \geq 2$.*

(ii) *If for any $e', e'' \in E(G_0)$ with $e' \neq e''$, $G_0^k(e', e'')$ has a spanning $(k; v_{e'}, v_{e''})$-trail system, then G_0 has a spanning $(k; e', e'')$-trail system.*

(iii) *If for any distinct edges $e' = u'v'$ and $e'' = u''v''$ in $E(G_0)$, G_0 has a spanning $(k; e', e'')$-trail system $(T_1, T_2, \ldots, T_k)$ such that for any $v \in \{u', v', u'', v''\}$, there exists an i with $1 \leq i \leq k$, and such that T_i contains v as an internal vertex, then $\kappa^*(L(G)) \geq k$.*

(iv) *If for any $e', e'' \in E(G_0)$ with $e' \neq e''$, $G_0 - \{e', e''\} \in \mathcal{C}_{k-1}$, then $\kappa^*(L(G)) \geq k$.*

Proof (i) Since $G_0(e', e'') \in \mathcal{C}_1$, by Lemma 3.3, $G_0(e', e'')$ has a spanning $(v_{e'}, v_{e''})$-trail. Thus by Proposition 4.1(iv), $\kappa^*(L(G)) \geq 2$.

(ii) Let H'' be a spanning $(k; v_{e'}, v_{e''})$-trail system of $G_0^k(e', e'')$. Then H'' is an edge disjoint union of $(v_{e'}, v_{e''})$-trails $T_1', T_2', \ldots, T_k'$. For each $i = 1, 2, \ldots, k$, let

$$T_i = G_0[E(T_i' - \{v_{e'}, v_{e''}\}) \cup \{e', e''\}].$$

Then each T_i is an (e', e'')-trail, and $(T_1, T_2, \ldots, T_k)$ is a spanning $(k; e', e'')$-trail system of G_0.

(iii) By Theorem 2.1, it suffices to show that for any $e', e'' \in E(G)$ with $e' \neq e'' G$ has a dominating $(k; e', e'')$-trail system. By the assumption of (iii),

$$\text{for any } e', e'' \in E(G_0)\ (e' \neq e''),\ G_0 \text{ has a spanning } (k; e', e'') - \text{trail system with the property stated in } (iii). \tag{3}$$

Let $e', e'' \in E(G)$ be two distinct edges. Let $e \in \{e', e''\}$. If $e \in E(G - D_1(G))$ and e is not incident with a vertex $z \in D_2(G)$, then let $f(e) = e$, which is an edge in $E(G_0)$. If $e \in E(G - D_1(G))$ and e is incident with a vertex $z \in D_2(G)$, then we may assume that $e \in E(G_0)$ and that the edge in $E_G(z) - \{e\}$ has been contracted in obtaining G_0, and define $f(e) = e$, which is again an edge in $E(G_0)$. If e is incident with a vertex $z \in D_1(G)$, then denote $e = zw$, where $w \notin D_1(G)$. Define $f(e) \in E_G(w) - \{e\}$ so that $f(e') \neq f(e'')$. This

can be done as $\kappa(L(G)) \geq 3$, when $z \in D_1(G)$, w must be incident with at least 4 edges in G. In any case, $f(e) \in E(G_0)$. Since $f(e'), f(e'') \in E(G_0)$ and $f(e') \neq f(e'')$, by (3), G_0 has a spanning $(k; f(e'), f(e''))$-trail system $(T_1', T_2'', \ldots, T_k')$ satisfying the assumption of (iii).
For each $i \in \{1, 2, \ldots, k\}$, let $X_2(T_i')$ be the set of all edges $e \in E(T_i')$ such that for some vertex $z \in D_2(G)$, $X_e := E_G(z) = \{e, f\}$. Define

$$T_i = G[(E(T_i') - X_2(T_i')) \bigcup \left(\bigcup_{e \in X_2(T_i')} X_e \right) \bigcup \{e', e''\}].$$

In other words, T_i is obtained from T_i' by replacing each $e \in E(T_i')$ that is incident with a vertex $z \in D_2(G)$ by the path consisting with both edges incident with z in G, and then extending the resulting trail to an (e', e'')-trail. It follows that $(T_1, T_2, \ldots, T_k)$ is a $(k; e', e'')$-trail system that contains all vertices of degree at least 3 in G, such that for any $v \in \{u', v', u'', v''\}$, there exists an i, $(1 \leq i \leq k)$, such that T_i contains v as an internal vertex. Thus every edge not in $\bigcup_{i=1}^{k} E(T_i)$ must be incident with an internal vertex of some T_i, and so $(T_1, T_2, \ldots, T_k)$ is a dominating $(k; e', e'')$-trail system.

(iv) By Theorem 4.3(iii), it suffices to show that the hypothesis of Theorem 4.3(iii) will hold.
Suppose that for any $e', e'' \in E(G_0)$ with $e' \neq e''$, $G_0 - \{e', e''\} \in \mathcal{C}_{k-1}$. By Lemma 4.2, $G_0^k(e', e'') \in \mathcal{C}_{k-1}$. It follows by Lemma 3.3 that $G_0^k(e', e'')$ has a $(k; v_{e'}, v_{e''})$-trail system $(T_1', T_2', \ldots, T_k')$. Denote $e' = u'v'$, $e'' = u''v''$ in $E(G_0)$. By the definition of $G_0^k(e', e'')$, there are at most $\lceil k/2 \rceil$ of these $(v_{e'}, v_{e''})$-trails that contain the one of the $\lceil k/2 \rceil$ edges parallel to $v_{e'}v'$. This implies that for any $v \in \{u', v'\}$, at least one T_i' will use v as an internal vertex. Similarly, for any $v \in \{u'', v''\}$, at least one T_i' will use v as an internal vertex. Define

$$T_i = G_0[E(T_i' - \{v_{e'}, v_{e''}\}) \cup \{e', e''\}], \ (1 \leq i \leq k).$$

Then $(T_1, T_2, \ldots, T_k)$ is a spanning $(k; e', e'')$-trail system satisfying the hypothesis of Theorem 4.3(iii), and so $\kappa^*(L(G)) \geq k$. This completes the proof of the theorem. □

5 Proof of Theorem 1.4

In this section, we shall prove the following slightly stronger result, which implies Theorem 1.4.

Proof By the Menger's theorem (Theorem 9.1 of [1]), for any graph G, we always have

$$\kappa(G) \geq \kappa^*(G). \tag{4}$$

By the definition of hamiltonian-connectivity, we know that every hamiltonian-connected graph G with at least 4 vertices must have connectivity at least 3. This, together with (4), implies that if $\kappa^*(L(G)) \geq k \geq 2$, then $\kappa(L(G)) \geq \max\{3, k\}$.

It remains to prove that for $k \geq 2$, if $\tau(G) \geq k$ and if $\kappa(L(G)) \geq \max\{3, k\}$, then $\kappa^*(L(G)) \geq k$.

First assume that $k = 2$. By Theorem 4.3, it suffices to show that for any pair of distinct edges $e', e'' \in E(G_0)$, $G_0(e', e'') \in \mathcal{C}_1$. We argue by contradiction and assume that $G_0(e', e'') \notin \mathcal{C}_1$. Let G'_0 denote the $\mathcal{C}_1$-reduction of $G_0(e', e'')$. Since $\tau(G_0) \geq 2$, it follows that $F(G_0(e', e''), 2) \leq 2$, and so $F(G'_0, 2) \leq 2$. By Theorem 3.6, $G'_0 \in \{K_2, K_{2,t}, (t \geq 1)\}$. Since $G_0(e', e'')$ has no cut edges, neither does G'_0. Hence $G'_0 = K_{2,t}$ for some $t \geq 2$. By Proposition 4.1, $\kappa'(G_0) \geq 3$, and so we must have $t = 2$, and $v_{e'}$ and $v_{e''}$, the two vertices newly added when subdividing e' and e'', are two nonadjacent vertices of G'_0. By the definition of a core, the other two vertices in $V(G'_0) - \{v_{e'}, v_{e''}\}$ must be nontrivial vertices, and so $\{e', e''\}$ must be an essential edge cut of G, contrary to the assumption that $\kappa(L(G)) \geq 3$. This settles the case when $k = 2$.

Now we assume that $k \geq 3$. By Theorem 4.3, it suffices to show that the hypothesis of Theorem 4.3(ii) or (iii) holds. We shall assume that

$$G_0 - \{e', e''\} \notin \mathcal{C}_{k-1}, \tag{5}$$

to prove Theorem 4.3(ii) holds.

Let $e', e'' \in E(G_0)$ be two distinct edges such that $e' = u'v'$ and $e'' = u''v''$. Let G' denote the $\mathcal{C}_{k-1}$-reduction of $G_0 - \{e', e''\}$. Since $\tau(G_0) \geq k$, $F(G_0 - \{e', e''\}, 2) \leq 2$.

Case 1 $F(G_0 - \{e', e''\}, k) \leq 1$.

Since G' is a contraction of $G_0 - \{e', e''\}$, $F(G', k) \leq F(G_0 - \{e', e''\}, k) \leq 1$. By (5) and by Lemma 3.4, $G' \notin \mathcal{C}_{k-1}$. By Theorem 3.5, $G' \in \mathcal{C}_{k-1}$ if and only if $\kappa'(G') \geq k$. As $G' \notin \mathcal{C}_{k-1}\kappa'(G') \leq k-1$. Since $F(G', k) \leq 1$, there must be an edge $f \notin E(G')$ such that $\tau(G' + f) \geq k$, and so $\kappa'(G' + f) \geq \tau(G' + f) \geq k$. This implies that G' must have an edge-cut X consisting of $k-1$ edges $\{e_1, e_2, \ldots, e_{k-1}\}$, which is also an edge cut of $G_0 - \{e', e''\}$.

Since $\tau(G_0) \geq k$, G_0 has k spanning trees, denoted as $T_1, T_2, \ldots, T_k$. Since $F(G_0 - \{e', e''\}, k) \leq 1$ and $G_0 - \{e', e''\}$ has an edge cut X with $|X| = k-1$, we may assume that $e' \in E(T_k)$, and $e'' \notin \bigcup_{i=2}^{k-1} E(T_i)$. Hence we may assume that $e_i \in E(T_i)$, for $1 \leq i \leq k-1$. Since T_k is a spanning tree of G_0, we may assume that $T_k - e'$ has a (u', u'')-path P_k. Since $G_0[\bigcup_{i=1}^{k-1} E(T_i)]$ is a spanning subgraph of $G_0 - \{e, e''\}$ that has $k-1$ edge-disjoint spanning trees, it follows by Theorem 3.5 and Lemma 3.3 with $s = k-2$ that $G_0 - (\{e', e''\} \cup E(P_k))$ has a spanning $(k-1; u', u'')$-trail system $(P_1, P_2, \ldots, P_{k-1})$. Let $P'_i = G_0[E(P_i) \cup \{e', e''\}]$, $1 \leq i \leq k$. Then $(P'_1, P'_2, \ldots, P'_k)$ is a spanning $(k; e', e'')$-trail system of G_0. Hence Theorem 4.3(ii) holds.

Case 2 $F(G_0 - \{e', e''\}, k) = 2$.

Let $T_1, T_2, \ldots, T_k$ denote k edge-disjoint spanning trees of G_0. Since $F(G_0 - \{e', e''\}, 2) = 2$, we must have $e', e'' \in \bigcup_{i=1}^{k} E(T_i)$. Choose $(T_1, T_2, \ldots, T_k)$, among all such choices of edge-disjoint spanning trees, such that

$$|\{T_i : E(T_i) \cap \{e', e''\} \neq \emptyset\}| \text{ is minimized.} \tag{6}$$

Subcase 2.1 For some i, $e', e'' \in E(T_i)$.

We may assume that $e', e'' \in E(T_k)$. Since T_k is a spanning tree of G_0, we may assume (renaming the end vertices of e' and e'' if needed) that $T_k - \{e', e''\}$ has a (u', u'')-path P_k. Since $G_0[\bigcup_{i=1}^{k-1} E(T_i)]$ is a spanning subgraph of $G_0 - \{e, e''\}$ that has $k-1$ edge-disjoint spanning trees, by Theorem 3.5 and Lemma 3.3 with $s = k-2$ that $G_0[\bigcup_{i=1}^{k-1} E(T_i)]$ has a spanning $(k-1; u', u'')$-trail system $(P_1, P_2, \ldots, P_{k-1})$. Let $P_i' = G_0[E(P_i) \cup \{e', e''\}]1 \le i \le k$. Then $(P_1', P_2', \ldots, P_k')$ is a spanning $(k; e', e'')$-trail system of G_0. Hence Theorem 4.3(ii) holds.

Subcase 2.2 For any i, $|\{e', e''\} \cap E(T_i)| \le 1$.

We may assume that $e' \in E(T_{k-1})$ and $e'' \in E(T_k)$. Let T_{k-1}' and T_{k-1}'' be the two components of $T_{k-1} - e'$, and let T_k' and T_k'' be the two components of $T_k - e''$. We may further assume that $V(T_{k-1}') \cap V(T_k'') \neq \emptyset$.

We shall show first that $V(T_{k-1}') = V(T_k')$ and so $V(T_{k-1}'') = V(T_k'')$. Since T_{k-1}' and T_{k-1}'' are the two components of $T_{k-1} - e'$, we may assume that $u' \in V(T_{k-1}')$ and $v' \in V(T_{k-1}'')$. Similarly, we may assume that $u'' \in V(T_k')$ and $v'' \in V(T_k'')$. Since T_{k-1} and T_k are spanning trees of G_0, if $V(T_{k-1}') \neq V(T_k')$, then either $v'' \in V(T_{k-1}')$ or $u'' \in V(T_{k-1}'')$. Arguing by contradiction, we assume that $V(T_{k-1}') \neq V(T_k')$, and by symmetry, we assume further that $v'' \in V(T_{k-1}')$. It then follows that $T_{k-1}' + e''$ has a unique cycle C'' which contains at least one edge $e''' \in E(T_{k-1}')$. Redefine $L_{k-1} = T_{k-1} + e'' - e'''$ and $L_k = T_k - e'' + e'''$. Then $(T_1, \ldots, T_{k-2}, L_{k-1}, L_k)$ is a set of k edge-disjoint spanning trees violating (6). Hence we must have

$$V(T_{k-1}') = V(T_k') \text{ and } V(T_{k-1}'') = V(T_k''). \tag{7}$$

By (7), both $(T_{k-1} \cup T_k)[V(T_{k-1}')]$ and $(T_{k-1} \cup T_k)[V(T_{k-1}'')]$ are graphs with 2 edge-disjoint spanning trees. By Theorem 3.5 with $s = 1$, both are in $\mathcal{C}_1$, and so by Lemma 3.3, $(T_{k-1} \cup T_k)[V(T_{k-1}')]$ has a spanning (u', u'')-trail P_{k-1} and $(T_{k-1} \cup T_k)[V(T_{k-1}'')]$ has a spanning (v', v'')-trail P_k. Since $T_1, T_2, \ldots, T_{k-2}$ are spanning trees of G_0, each T_i has a (u', u'')-path $P_i (1 \le i \le k-2)$. Let $P_i' = G_0[E(P_i) \cup \{e', e''\}]$, $1 \le i \le k$. Then $(P_1', P_2', \ldots, P_k')$ is a spanning $(k; e', e'')$-trail system of G_0. Hence Theorem 4.3(ii) holds.

Since in all the cases, either Theorem 4.3(ii) or Theorem 4.3(iii) must hold, and so by Theorem 4.3, $\kappa^*(L(G)) \ge k$. This completes the proof. □

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