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ORIGINAL PAPER

On Group Choosability of Total Graphs

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Abstract In this paper, we study the group and list group colorings of total graphs and present group coloring versions of the total and list total colorings conjectures. We establish the group coloring version of the total coloring conjecture for the following classes of graphs: graphs with small maximum degree, two-degenerate graphs, planner graphs with maximum degree at least 11, planner graphs without certain small cycles, outerplanar graphs and near outerplanar graphs with maximum degree at least 4. In addition, the group version of the list total coloring conjecture is established for forests, outerplanar graphs and graphs with maximum degree at most two.

Keywords Total coloring · List total coloring · Group choosability

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1 Introduction

Throughout this paper, we consider simple graphs, and follow [6] for terminology and notations not defined here. For a graph G, the vertex set, edge set, maximum degree and minimum degree of G are denoted by $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ (or simply V, E, Δ, δ), respectively. If $v \in V(G)$, we use $\deg_G(v)$ (or simply $\deg(v)$) and $N_G(v)$ to denote the degree and neighborhood of v in G, respectively.

A *proper coloring* of G is a coloring of the vertices of G such that no two adjacent vertices are assigned the same color. The minimum number of colors in any proper coloring of G, $\chi(G)$, is called the *chromatic number* of G. A *k-list assignment* for a graph G is a function L which assigns to each vertex v of G a list of colors $L(v)$ such that $|L(v)| = k$. An *L-coloring* of G is a proper coloring c such that $c(v) \in L(v)$ for each vertex v. If for every k-list assignment L, a proper L-coloring of G exists, then G is said to be *k-choosable* and the *choice number*, $\chi_l(G)$, of G is the smallest k such that G is k-choosable. The concepts of *chromatic index*, $\chi'(G)$, and *choice index*, $\chi'_l(G)$, can be defined similarly in terms of coloring the edges of G.

Recall that the *total graph* of a graph G, denoted by $T(G)$, is a graph whose vertices are the edges and vertices of G, and adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G. The *total chromatic number* of G, $\chi''(G)$, is the chromatic number of $T(G)$. Clearly $\chi''(G) = \chi(T(G)) \geq \Delta(G) + 1$. Behzad [1] and Vizing [18] posed independently the following famous conjecture, which is known as the *total coloring conjecture*.

Conjecture 1 *For any graph G, $\chi''(G) \leq \Delta(G) + 2$.*

The *total choice number* of G, $\chi''_l(G)$, is the choice number of $T(G)$. It follows directly from the definition that $\chi''_l(G) \geq \chi''(G)$. The notation of total choosability was first introduced by Borodin et al. [2]. They proposed the following conjecture, known as the *list total coloring conjecture*.

Conjecture 2 *For any graph G, $\chi''_l(G) = \chi''(G)$.*

The concept of group coloring of graphs was first introduced by Jaeger et al. [11]. Assume that A is a group and $F(G, A)$ denotes the set of all functions $f : E(G) \longrightarrow A$. Consider an arbitrary orientation of G. Graph G is called *A-colorable* if for every $f \in F(G, A)$, there is a vertex coloring $c : V(G) \longrightarrow A$ such that $c(x) - c(y) \neq f(xy)$ for each directed edge from x to y. The *group chromatic number* of G, $\chi_g(G)$, is the minimum k such that G is A-colorable for any group A of order at least k. In [15], the concept of group choosability is introduced as an extension of list coloring and group coloring. Let A be a group of order at least k and $L : V(G) \longrightarrow 2^A$ be a list assignment of G. For $f \in F(G, A)$, an *(A, L, f)-coloring* of G is an L-coloring $c : V(G) \longrightarrow A$ such that $c(x) - c(y) \neq f(xy)$ for each directed edge from x to y. If for each $f \in F(G, A)$ there exists an (A, L, f)-coloring for G, then we say that G is *(A, L)-colorable*. If for any group A of order at least k and any k-list assignment $L : V(G) \longrightarrow 2^A$, G is (A, L)-colorable, then we say that G is *k-group choosable*.

The *group choice number* of G, $\chi_{gl}(G)$, is the smallest k such that G is k-group choosable. Clearly group choosability of a graph is independent of the orientation on G. The concept of group choosability is also studied in [5]. The authors used the concept of *D-group choosability* to establish the group version of the Brooks' Theorem. A graph G is called *D-group choosable* if it is (A, L)-colorable for each group A with $|A| \geq \Delta(G)$ and every list assignment $L : V(G) \longrightarrow 2^A$ with $|L(v)| = \deg(v)$. They proved the following theorem, which is a characterization of D-group choosable graphs.

Theorem 1.1 ([5]) *A connected graph G is D-group choosable if and only if G has a block which is neither a complete graph nor a cycle.*

For a graph G, let $\overline{\delta}(G) = \max\{\delta(H) : H$ is a subgraph of $G\}$, and let $\mathrm{col}(G) = \overline{\delta}(G) + 1$. For an integer $d > 0$, a graph G is called *d-degenerate* if $d \geq \overline{\delta}(G)$. The following result is the group version of the Brooks' Theorem.

Theorem 1.2 ([5]) *Let G be a connected simple graph.*

(i) (*Lemma* 2.3 *of* [5]) $\chi_{gl}(G) \leq col(G)$.

(ii) (*Theorem* 1.1 *of* [5]) $\chi_{gl}(G) \leq \Delta(G) + 1$, *where equality holds if and only if G is either a cycle or a complete graph.*

We extend the concepts of total and list total colorings to total group and list total group colorings of graphs. We let $\chi''_g(G) = \chi_g(T(G))$ (resp., $\chi''_{gl}(G) = \chi_{gl}(T(G))$) and we call it the *total group chromatic number* (resp. *total group choice number*) of G. Clearly the following inequality holds for the above mentioned chromatic numbers of G.

$$\chi''_{gl}(G) \geq \max\{\chi''_g(G), \chi''_l(G)\} \geq \chi''(G).$$

Now we extend the total coloring and list total coloring conjectures as follows:

Conjecture 3 *For every graph G, $\chi''_{gl}(G) \leq \Delta(G) + 2$.*

Conjecture 4 *For every graph G, $\chi''_{gl}(G) = \chi''_g(G)$.*

Note that it is not the case that $\chi''_{gl}(G) = \chi''_l(G)$ for every graph G. Juvan et al. [12] proved that $\chi''_l(C_n) = 3$ if $n \equiv 0 \pmod 3$, but in this paper, we will show in Theorem 3.1 that $\chi''_{gl}(C_n) = 4$ for every cycle C_n on n vertices. The following conjecture express a weaker version of Conjecture 3.

Conjecture 5 *For every graph G, $\chi''_g(G) \leq \chi''(G) + 1$.*

In this paper, we are interested in Conjectures 3 and 4 and we will establish Conjecture 3 for certain classes of graphs such as planar graphs with maximum degree at least 11, two-degenerate graphs, planar graphs without certain cycles, outerplanar graphs and near-outerplanar graphs with maximum degree at least 4. Also we show that Conjecture 4 holds for graphs with maximum degree at most two, forests and outerplanar graphs. Subsequently, it will be shown that Conjecture 5 holds for the above mentioned classes of graphs.

2 Some Upper Bounds

In this section, we give some upper bounds for $\chi''_{gl}(G)$ of a graph G and use these bounds to verify Conjectures 3 and 4 for certain classes of graphs. The *group choice index* of a graph G, $\chi'_{gl}(G)$, is defined as the group choice number of its line graph i.e. $\chi'_{gl}(G) = \chi_{gl}(L(G))$. Clearly $\chi'_{gl}(G) \geq \chi'(G) \geq \Delta(G)$. This concept is studied in [13] where the authors conjectured that every graph with maximum degree Δ is $(\Delta + 1)$-edge group choosable. Moreover, they gave infinite families of graphs G with $\chi'_{gl}(G) = \Delta(G)$. The following lemma shows that Conjecture 3 holds for these graphs.

Lemma 2.1 *For every graph G we have* $\chi''_{gl}(G) \leq \chi'_{gl}(G) + 2$.

Proof Let G be a graph and A be a group of order at least $\chi'_{gl}(G) + 2$. Also let $L : V \cup E \mapsto 2^A$ be any $(\chi'_{gl}(G)+2)$-list assignment of $V \cup E$ and $f \in F(T(G), A)$ be arbitrary. First we color the vertices of G from their lists. Since $\chi'_{gl}(G) \geq \Delta(G)$ and $\chi_{gl}(G) \leq \Delta(G) + 1$ by Theorem 1.2, such a coloring $c : V(G) \mapsto A$ exists. For each edge $e = uv$ of G, without loss of generality, let the edge eu be directed from e to u and also the edge ev be directed from e to v in $T(G)$. For each edge $e = uv$ of G, remove $f(eu) + c(u)$ and $f(ev) + c(v)$ from $L(e)$. Since for each edge e of G, $|L(e)| \geq \chi'_{gl}(G) + 2$, each edge of G retains at least $\chi'_{gl}(G)$ admissible colors in its list and so, by the definition of $\chi'_{gl}(G)$, it is possible to color the edges of G from their lists. So we can color the vertices of $T(G)$ from their lists and this yields an (A, L, f)-coloring of $T(G)$, which shows that $\chi''_{gl}(G) \leq \chi'_{gl}(G) + 2$. □

Since $T(G)$ contains a $K_{\Delta+1}$ as a subgraph where $\Delta = \Delta(G)$, and so

$$\Delta(G) + 1 \leq \chi''_{gl}(G) \leq \mathrm{col}(T(G)) \quad \text{for any graph } G. \tag{1}$$

Let G be a non-regular graph with $\Delta = \Delta(G) \geq 2$ and $d = \overline{\delta}(G)$. Since G is non regular, $d < \Delta$. Let $v_1, v_2, \ldots, v_n$ be an ordering of $V(G)$ such that in G, every v_i is adjacent to at most d vertices in $\{v_{i+1}, \ldots, v_n\}$. Let $E_1, E_2, \ldots, E_n$ be disjoint edge subsets of G such that E_1 is the set of edges in $E(G)$ incident with v_1 in G, and for $1 \leq i < n$, E_{i+1} is the set of all edges in $E(G) - \cup_{j=1}^{i} E_j$ incident with v_{i+1} in G. Then in $T(G)$, every v_{i+1} has degree at most $\Delta + d$ in $T(G) - (\{v_1, \ldots, v_i\} \cup_{j-1}^{i} E_i)$, and every $e \in E_{i+1}$ has degree at most $\Delta + d$ in $T(G) - (\{v_1, \ldots, v_i, v_{i+1}\} \cup_{j-1}^{i} E_i)$. Thus $\mathrm{col}(T(G)) \leq \Delta + \mathrm{col}(G) - 1$. If G is a Δ-regular graph, then $T(G)$ is 2Δ-regular which is not complete nor a cycle. By Theorem 1.2, we have $\chi''_{gl}(G) \leq 2\Delta = \Delta + \mathrm{col}(G) - 1$. Hence, by (1), we have the following result.

Theorem 2.2 *Let G be a graph with maximum degree* $\Delta \geq 2$. *Then*

$$\chi''_{gl}(G) \leq \Delta + col(G) - 1.$$

Since forests are 1-degenerate, the total graph of a forest with $\Delta \geq 2$ is Δ-degenerate. It follows that if G is a forest with $\Delta \geq 2$, then

$$\mathrm{col}(T(G)) = \Delta + 1. \tag{2}$$

By Theorem 2.2, and by the fact that forests are 1-degenerate, Conjectures 3 and 4 hold for all forests.

Corollary 2.3 *Let G be a forest with maximum degree* $\Delta \geq 2$. *Then*

$$\chi''_{gl}(G) = \chi''_g(G) = \Delta + 1.$$

A graph H is a *minor* of a graph K if H can be obtained from a subgraph of K by contracting some edges. A graph G is called K_4*-minor free* if it has no subgraph isomorphic to a minor of K_4. It is well-known [7,8] that every K_4-minor free graph has a vertex of degree at most two. A planar graph is called *outerplanar* if it has a drawing in which each vertex lies on the boundary of the outer face. It is well-known that a graph is outerplanar if and only if it contains neither K_4 nor $K_{2,3}$ as a minor (see, for example [6]).

It is well known [9] that a connected graph G is 2-choosable if and only if G is obtained by successively removing vertices of degree 1 until what remains, is isomorphic to either K_1, C_{2m+2} or $\Theta_{2,2,2m}$ for some m, where $\Theta_{2,2,2m}$ is the graph consisting of two distinguished vertices v_0, v_{2m} connected by three paths P_1, P_2 and P_3 of lengths 2, 2 and $2m$, respectively. So the class of 2-degenerate graphs properly contains 2-choosable graphs, outerplanar graphs, non-regular subcubic graphs, planar graphs of girth at least six and *unicycle graphs*, graphs with exactly one cycle, as subclasses. Using Theorem 2.2, we have the following corollary.

Corollary 2.4 Conjecture 3 holds for every two-degenerate graph. In particular, Conjecture 3 holds for planar graphs of girth at least six, K_4-minor free, outerplanar and 2-choosable graphs.

3 Graphs with Bounded Degrees

In this section, we prove that Conjectures 3 and 4 hold for graphs with maximum degree at most two and wheel graphs with maximum degree at least six. Also we prove that Conjecture 3 holds for any planar graph with maximum degree at least 11.

Theorem 3.1 *Let* P_n *and* C_n *be the path and cycle on* $n \geq 2$ *vertices, respectively. Then* $\chi''_g(P_n) = \chi''_{gl}(P_n) = 3$ *and* $\chi''_g(C_n) = \chi''_{gl}(C_n) = 4$.

Proof Clearly $\chi''_g(P_n) \geq \chi''(P_n) = 3$. On the other hand, $\chi''_{gl}(P_n) \leq \mathrm{col}(T(P_n)) = 3$. It follows that $\chi''_g(P_n) = \chi''_{gl}(P_n) = 3$. Denote by $v_1, v_2, \ldots, v_n$ the vertices of C_n in the order they appear in C_n with $u_i = v_i v_{i+1}$, where $1 \leq i < n$, and $u_n = v_n v_1$. Let $T = T(C_n)$. By Theorem 1.2, we have $\chi''_{gl}(C_n) \leq 4$, for any n. Also it is easy to see [17] that $\chi''(C_n) = 3$ if $n = 3t$ and $\chi''(C_n) = 4$, otherwise. Consequently, if n is not a multiple of three, we obtain the desired result. So let $n = 3t$ and $A = (Z_3, +)$ be the group with elements 0,1,2 where "+" is the addition modulo 3. Define $f \in F(T, Z_3)$ with $f(u_{n-1}u_n) = 1$, $f(u_n u_1) = 2$ and $f(e) = 0$ otherwise, where the edge $u_{n-1}u_n$ in T is directed from u_{n-1} to u_n and the edge $u_n u_1$ is directed from u_n to u_1. Suppose that $c : V(T) \longrightarrow Z_3$ is an (Z_3, f)-coloring of the vertices of $T(C_n)$ with $c(v_1) = i, c(v_2) = j$ and $c(u_1) = k$, where $0 \leq i \neq j \neq k \leq 2$.

Since $f(u_1u_2) = f(v_2u_2) = 0$, $c(u_2)$ must be different from $c(u_1) = k$ and $c(v_2) = j$ and hence $c(u_2) = i$. By the same reasoning, $c(v_3) = k$ and since $n = 3t$ we have $c(v_{n-1}) = j$, $c(v_n) = k$ and finally $c(u_{n-1}) = i$. Consequently, for any choices of i, j, k, since $f(u_{n-1}u_n) \equiv 1 \pmod 3$ and $f(u_nu_1) \equiv -1 \pmod 3$, and because the neighbors v_1, v_2, u_{n-1}, u_1 of u_n have colors i, k, i, k, respectively, it follows that u_n cannot be given any of the colors $i, k, i-1, k-1$. Accordingly, as $i \neq k$, there is no admissible color available for u_n. Hence $\chi''_g(C_n) \geq 4$. Therefore for any n, we obtain $\chi''_g(C_n) = 4$. Now the inequality $\chi''_g(C_n) \leq \chi''_{gl}(C_n) \leq 4$, implies that $\chi''_g(C_n) = \chi''_{gl}(C_n) = 4$ which completes the proof. □

Note that if G is a graph with components $G_1, G_2, \ldots, G_t$, then we have: $\chi''_g(G) = \max\{\chi''_g(G_1), \ldots, \chi''_g(G_t)\}$, $\chi''_{gl}(G) = \max\{\chi''_{gl}(G_1), \ldots, \chi''_{gl}(G_t)\}$. Combining these facts and Theorem 3.1, we obtain the following corollary.

Corollary 3.2 *Let G be a graph with maximum degree at most two. Then we have $\chi''_g(G) = \chi''_{gl}(G)$.*

Lemma 3.3 *Let G be a graph with maximum degree Δ. If $v_0 \in V(G)$ is the only vertex in G with $\deg(v_0) = \Delta$ and if for any vertex $v \in V(G) - \{v_0\}$, $1 \leq \deg(v) \leq \frac{\Delta}{2}$, then $col(T(G)) = \Delta + 1$.*

Proof Let X be the set of edges of G that are not incident with v_0. Then $\deg_{T(G)}(e) \leq \Delta$, for each edge $e \in X$. By the definition of X, $G - X$ consists of a star centered on v_0, together possibly with some isolated vertices. It follows that $\overline{\delta}(T(G)) \leq \Delta$, and so $\mathrm{col}(T(G)) \leq \Delta + 1$. □

The *wheel graph*, W_n, is the graph obtained from C_n by adjoining a vertex to all vertices of C_n. When $n \geq 6$, Lemma 3.3 and (1) can be applied to obtain the next result.

Corollary 3.4 *Let $n \geq 6$ be an integer.*

$$\chi''_{gl}(W_n) \leq col(T(W_n)) \leq n + 1.$$

Theorem 3.5 ([14]) *For every planar graph G with minimum degree at least 3 there is an edge $e = uv$ with* $\deg(u) + \deg(v) \leq 13$.

Theorem 3.6 *Let $k \geq 11$ and G be a planar graph with maximum degree at most k. Then $col(T(G)) \leq k + 2$.*

Proof Let G be a minimum counterexample to Theorem 3.6 for the given value of k, so that $\mathrm{col}(T(G)) > k+2$. If G contains a vertex u with $\deg(u) \leq 2$, then by the minimality of G we have $\mathrm{col}(T(G-u)) \leq k+2$. Since the degree of each edge incident to u in $T(G)-u$ is at most $k+1$ we have $\mathrm{col}(T(G)) \leq k+2$, contrary to the assumption that G is a counterexample. So we may assume that $\delta \geq 3$. By Theorem 3.5, there exists an edge $e = uv$ with $\deg_{T(G)}(e) \leq 13$. We may assume that $\deg(u) \leq 6$. By minimality of G we have $\mathrm{col}(T(G-e)) \leq k + 2$. Since $\max\{\deg_{T(G)-u}(e), \deg_{T(G)}(u)\} \leq 12$ and $k \geq 11$ we obtain that $\mathrm{col}(T(G)) \leq k + 2$, which is a contradiction. □

Using Theorem 3.6, we obtain the following corollary, which states that Conjecture 3 holds for planar graphs with maximum degree at least 11.

Corollary 3.7 *Let G be a planar graph with maximum degree Δ. Then*

$$\chi''_{gl}(G) \leq col(T(G)) \leq \max\{13, \Delta + 2\}.$$

4 Outerplanar and Near Outerplanar Graphs

In this section, we give some upper bounds for the total group choice number of outerplanar and near outerplanar graphs, which establish Conjectures 3 and 4 for outerplanar graphs and Conjecture 3 for near outerplanar graphs with maximum degree at least 4. We need the following lemma of Borodin and Woodall [3].

Lemma 4.1 *Let G be an outerplanar graph. Then at least one of the following holds.*

(a) $\delta(G) = 0$ *or* 1.
(b) *There exists an edge uv such that* $\deg(u) = \deg(v) = 2$.
(c) *There exists a 3-face uxy such that* $\deg(u) = 2$ *and* $\deg(x) = 3$.
(d) *There exist two 3-faces xu_1v_1 and xu_2v_2 such that* $\deg(u_1) = \deg(u_2) = 2$ *and* $\deg(x) = 4$ *and these five vertices are all distinct.*

The following theorem implies that Conjectures 3 and 4 hold for outerplanar graphs with maximum degree at least 5.

Theorem 4.2 *Let $k \geq 5$ and G be an outerplanar graph with maximum degree $\Delta \leq k$. Then $col(T(G)) \leq k + 1$.*

Proof Let G be a minimum counterexample to the theorem. So $\mathrm{col}(T(G)) > k + 1$ for some $k \geq 5$. If G contains a vertex v of degree one with neighborhood u, then by minimality of G we have $\mathrm{col}(T(G - v)) \leq k + 1$. Since $\deg_{T(G)-v}(uv) \leq k$ and $\deg_{T(G)}(v) \leq 2$ we have $\mathrm{col}(T(G)) \leq k + 1$, a contradiction. So by Lemma 4.1, G contains an edges uv such that $\deg(u) = 2$ and $\deg(u) + \deg(v) \leq 6$. By the minimality of G, $\mathrm{col}(T(G - uv)) \leq k + 1$. Again since $k \geq 5$, $\deg_{T(G)-u}(uv) \leq 5$ and $\deg_{T(G)}(u) = 4$, we obtain that $\mathrm{col}(T(G)) \leq k + 1$, a contradiction. □

By (1) and Theorem 4.2, we have the following corollary.

Corollary 4.3 *Let G be an outerplanar graph with maximum degree $\Delta \geq 5$ then $\chi''_{gl}(G) = \chi''_{g}(G) = \Delta + 1$.*

By a *near outerplanar* graph we mean one that is either K_4-minor free or $K_{2,3}$-minor free. Near outerplanar graphs are an extension of outerplanar graphs. Theorem 4.5, will show that Conjecture 3 holds for the class of $K_{2,3}$-minor free graphs. In fact, in Theorem 4.5 we will replace the class of $K_{2,3}$-minor free graphs by the slightly larger class of $\bar{K}_2 + (K_1 \cup K_2)$-minor free graphs, where $\bar{K}_2 + (K_1 \cup K_2)$ is the graph obtained from $K_{2,3}$ by adding an edge joining two vertices of degree 2. Before we proceed, we need the following lemma.

Lemma 4.4 ([10]) *Let G be a connected $\bar{K}_2 + (K_1 \cup K_2)$-minor free graph. Then each block of G is either K_4-minor free or isomorphic to K_4.*

Theorem 4.5 *Let $k \geq 4$ and G be a $\bar{K}_2 + (K_1 \cup K_2)$-minor free graph with maximum degree $\Delta \leq k$. Then $\chi''_{gl}(G) \leq k + 2$.*

Proof Let G be a minimum counterexample to Theorem 4.5 and also let $k \geq 4$ be such that $\chi''_{gl}(G) > k + 2$. Then there is a group A with $|A| \geq k + 2$, a $(k + 2)$-list assignment $L : V(T(G)) \longrightarrow 2^A$ and $f \in F(T(G), A)$, such that $T(G)$ is not (A, L, f)-colorable. Since the total group choice number of a graph is the maximum total group choice number of its components, we may assume that G is connected. Clearly $G \neq K_4$ since $\chi''_{gl}(K_4) \leq 6$ by Corollary 2.2. If G is 2-connected, then by Lemma 4.4 G is K_4-minor free and so by Corollary 2.4, $\chi''_{gl}(G) \leq k + 2$, contrary to the assumption that G is a counterexample. Hence G is not 2-connected and so G has an end-block B with cut-vertex v. Let $B \cong K_4$ with $V(B) = \{v, u, w, x\}$. Then by the minimality of G the vertices and edges of $G - \{u, w, x\}$ can be properly colored from their lists. Each edge $e \in B$ incident with v has a residual list containing at least $(k+2)-(k-2) = 4$ usable colors. Coloring all edges $e \in N_{T(G)}(v) \cap E(B)$, retains at least 4 usable colors in the list of each other vertex and edge of B. By Theorem 3.1, this coloring can be extended to all the remaining vertices and edges of B. This shows that $\chi''_{gl}(G) \leq k+2$, again contradicting the choice of G. Hence B is K_4-minor free and so B contains at least two vertices of degree at most 2. Let $z \neq v$ be a vertex of degree 2 in B such that $\deg_G(z) \leq 2$. By the minimality of G we have $\chi''_{gl}(G - z) \leq k + 2$. Since $\deg_{T(G)-z}(e) \leq k + 1$ for each edge e incident to z and $\deg_{T(G)}(z) = 4$, coloring the edges incident to z and then z yields an (A, L, f)-coloring for $T(G)$, again contrary to the assumption that G is a counterexample. □

Corollary 4.6 *Let G be a $K_{2,3}$-minor free graph with maximum degree at least 4. Then G is $(\Delta + 2)$-total group choosable.*

Using Corollaries 2.4 and 4.6 we conclude that Conjecture 3 holds for near-outer-planar graphs with maximum degree at least 4.

5 Planar Graphs Without Small Cycles

In this section, we prove that Conjecture 3 holds for some planar graphs without certain small cycles. Our proofs are based on some structural lemmas and discharging method.

Lemma 5.1 *Let $k \geq 3$ and G be a graph with $\delta \geq 2$ and $\Delta \geq k$ such that, for every $e \in E(G)$, $col(T(G - e)) \leq k + 2$. Then $col(T(G)) \leq k + 2$.*

Proof Let v be a vertex with $d(v) = \delta \leq 2$ and e be an edge incident with v. Then $\mathrm{col}(T(G - e)) \leq k + 2$. Since $\deg_{T(G)-v}(e) \leq k + 1$ and $\deg_{T(G)}(v) \leq 4$, we have $\mathrm{col}(T(G)) \leq k + 2$. □

A *plane graph* is a particular drawing of a planar graph on the plane. A 2-*alternating cycle* in a graph G is a cycle of even length in which alternate vertices have degree 2 in G.

Theorem 5.2 ([16]) *Let G be a connected planar graph with $\delta \geq 2$. If G contains neither 5-cycles nor 6-cycles, then G contains a 2-alternating cycle or an edge uv such that* $\deg(u) + \deg(v) \leq 9$.

The following theorem establishes Conjecture 3 for a planar graph with maximum degree at least 7 that contains no 5-cycles or 6-cycles.

Theorem 5.3 *Let $k \geq 7$ and G be a planar graph with maximum degree $\Delta \leq k$. If G contains neither 5-cycles nor 6-cycles, then col$(T(G)) \leq k+2$.*

Proof Let $G = (V, E)$ be a minimum counterexample to Theorem 5.3 for the given value of k so that $\mathrm{col}(T(G)) > k+2$. By Lemma 5.1, we may assume that $\delta \geq 3$. First suppose that G has an edge $e = uv$ with $\deg(u) + \deg(v) \leq 9$. Without loss of generality, we assume that $\deg(u) \leq 4$. Then $\mathrm{col}(T(G-e)) \leq k+2$. Since $\deg_{T(G)-u}(e) \leq 8$ and $\deg_{T(G)}(u) \leq 8$, we have $\mathrm{col}(T(G)) \leq k+2$, a contradiction. Hence for every edge $e = uv$ of G, $\deg(u) + \deg(v) \geq 10$. By Theorem 5.2, G must contain a 2-alternating cycle C. Let U be the set of the vertices of C that have degree 2 in G, and let $H = G - U$. By the minimality of G and as $|V(H)| < |V(G)|$, we have $\mathrm{col}(T(H)) \leq k+2$. For each $e \in E(C)$ and $v \in U$ of C in G we have $\deg_{T(G)-U}(e) \leq k+1$ and $\deg_{T(G)}(v) \leq 4$ and so $\mathrm{col}(T(G)) \leq k+2$, a contradiction. This contradiction completes the proof of the theorem. □

A cycle C of length k in a graph G is called a *k-net* if C has at least one chord in G. The following is a structural lemma for plane graphs without 5-nets.

Lemma 5.4 ([4]) *Let G be a planar graph with $\delta \geq 3$ and without 5-nets. Then G contains an edge xy such that* $\deg(x) + \deg(y) \leq 9$.

By Theorems 1.2 and 5.5 below establishes Conjecture 3 for every planar graph without 5-nets and maximum degree at least 7.

Theorem 5.5 *Let $k \geq 7$ and G be a planar graph with maximum degree $\Delta \leq k$. If G contains no 5-nets, then col$(T(G)) \leq k+2$.*

Proof Let $G = (V, E)$ be a minimum counterexample to Theorem 5.5 for the given value of k such that $\mathrm{col}(T(G)) > k+2$. By Lemma 5.1, we may assume that $\delta \geq 3$. By Lemma 5.4, G contains an edge $e = xy$ such that $\deg(x)+\deg(y) \leq 9$ and $\deg(x) \leq 4$. By the minimality of G we have $\mathrm{col}(T(G-e)) \leq k+2$. Since $\deg_{T(G)-x}(e) \leq 8$ and $\deg_{T(G)}(x) \leq 8$, we have $\mathrm{col}(T(G)) \leq k+2$, a contradiction. □

We denote the set of faces of a plane graph G by $F(G)$ or simply by F. For a plane graph G and $f \in F(G)$, we write $f = u_1u_2\ldots u_n$ where $u_1, u_2, \ldots, u_n$ are the vertices on the boundary walk of f enumerated clockwise. Let $\delta(f)$ denote the minimum degree of vertices incident with f. The *degree of a face* f, denoted by $\deg(f)$, is the number of edge steps in the boundary walk. A *k-vertex* (resp., *k^+-vertex*) is a vertex of degree k (resp. a vertex of degree at least k). The following theorem establishes Conjecture 3 for planar graphs without 4-cycles and maximum degree at least 6.

Theorem 5.6 *Let $k \geq 6$ and G be a planar graph with maximum degree $\Delta \leq k$ such that G has no cycle of length 4. Then $\chi''_{gl}(G) \leq k+2$.*

Proof Let G be a minimum counterexample for the given value of k. Then there is a group A with $|A| \geq k+2$, a $(k+2)$-list assignment $L : V(T(G)) \mapsto 2^A$ and $f \in F(T(G), A)$, such that $T(G)$ is not (A, L, f)-colorable. Then we have the following claims.

(a) G is connected.
(b) Any vertex v is incident with at most $\left\lfloor \frac{\deg(v)}{2} \right\rfloor$ 3-faces.
(c) $\delta(G) \geq 3$.
(d) G contains no edge uv with $\min\{\deg(u), \deg(v)\} \leq \frac{k}{2}$ and $\deg(u) + \deg(v) \leq k + 2$.
(e) G does not contain any 3-face $F = uvw$ such that $\deg(u) = \deg(v) = \deg(w) = 4$.

Note that (a) is straightforward, and so is (b), since the absence of 4-cycles implies that G does not have two adjacent 3-cycles. To verify (c), suppose that G contains a vertex v with $d(v) \leq 2$ and pick an edge e incident with v. Then $\chi''_{gl}(G - e) \leq k + 2$, by the minimality of G, and so G has an (A, L, f)-coloring c for $T(G - e)$. Erase the color of vertex v in this coloring. Since there are at least $(k + 2) - (k + 1)$ available colors in the list of edge e, it can be colored by a color in the list. Now since $k \geq 6$, v can be colored from its list, which gives rise to an (A, L, f)-coloring for $T(G)$, contrary to the assumption that G is a counterexample. This establishes (c).

If G contains an edge uv with $\min\{\deg(u), \deg(v)\} \leq \frac{k}{2}$ and $\deg(u) + \deg(v) \leq k + 2$, then any (A, L, f)-coloring of $T(G - uv)$ can be extended to an (A, L, f)-coloring of $T(G)$, contrary to the assumption that G is a counterexample. This proves (d).

We argue by contradiction to prove (e). Let such a face exist. Let $G' = G - \{uv, vw, uw\}$. By the minimality of G, $T(G')$ has an (A, L, f)-coloring c. Erase the colors of u, v, w and, for an element $x \in \{u, v, w, uv, vw, uw\}$, let $L'(x)$ be the set of available colors in the list of x. Since $k \geq 6$ and since u, v, and w are degree 4 vertices, each $|L'(x)| \geq k - 2 \geq 4$. By Theorem 3.1, it is possible to recolor the elements u, v, w, uv, vw, uw in the 3-cycle uvw with available colors from their own lists so that c is indeed an (A, L, f)-coloring of $T(G)$, again contrary to the assumption that G is a counterexample. This completes the verifications for these claims.

Since G is a planar graph, by Euler's Formula, we have: $\sum_{v \in V}(2\deg(v) - 6) + \sum_{f \in F}(\deg(f) - 6) = -6(|V| - |E| + |F|) = -12$.

We define the initial charge function $w(x)$ for each $x \in V \cup F$. Let $w(v) = 2\deg(v) - 6$ if $v \in V$ and $w(f) = \deg(f) - 6$ if $f \in F$. It follows that $\sum_{x \in V \cup F} w(x) < 0$. We construct a new charge $w^*(x)$ on G as follows:

Each 3-face receives $\frac{3}{2}$ from its incident vertices of degree at least 5.
Each 3-face receives $\frac{3}{4}$ from its incident vertices of degree 4.
Each 5-face receives $\frac{1}{3}$ from its incident vertices of degree at least 5.
Each 5-face receives $\frac{1}{4}$ from its incident vertices of degree 4.

Note that $w^*(f) = w(f) \geq 0$ if $\deg(f) \geq 6$. Assume that $\deg(f) = 3$. If $\delta(f) = 3$, then f is incident with two 6^+-vertices by (d). So $w^*(f) \geq w(f) + 2 \times \frac{3}{2} = 0$. Otherwise, f is incident with at least one 5^+-vertex by (e). So $w^*(f) \geq w(f) + \frac{3}{2} + 2 \times \frac{3}{4} = 0$. Let $\deg(f) = 5$. If $\delta(f) = 3$, then f is incident with at most two vertices of degree 3 by (d), and if f is incident with two vertices of degree 3, then f is incident with three 6^+-vertices. Thus $w^*(f) \geq w(f) + \min\{2 \times \frac{1}{3} + 2 \times \frac{1}{4}, 3 \times \frac{1}{3}\} = 0$. Otherwise, $w^*(f) \geq w(f) + 5 \times \frac{1}{4} > 0$. Let v be a vertex of G. Clearly, $w^*(v) = w(v) = 0$

if $\deg(v) = 3$. If $\deg(v) = 4$, then v is incident with at most two 3-faces by (b). So $w^*(v) \geq w(v) - 2 \times \frac{3}{4} - 2 \times \frac{1}{4} = 0$. If $\deg(v) = 5$, then v is incident with at most two 3-faces by (b). So $w^*(v) \geq w(v) - 2 \times \frac{3}{2} - 3 \times \frac{1}{3} = 0$. If $\deg(v) = 6$, then $w^*(v) \geq w(v) - 3 \times \frac{3}{2} - 3 \times \frac{1}{3} > 0$. If $\deg(v) \geq 7$, so $w^*(v) \geq w(v) - \lfloor \frac{\deg(v)}{2} \rfloor \times \frac{3}{2} - \lceil \frac{\deg(v)}{2} \rceil \times \frac{1}{3} > 0$. It follows that $0 > \sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w^*(x) \geq 0$, a contradiction. □

A face f is called *simple* if its boundary is a cycle. If $f = u_1 u_2 \ldots u_n$ is not simple, then f contains at least one cut vertex v. Let $m_v(f)$ denotes the number of times through v of f in clockwise order. We shall show that planar graphs without a cycle of length 4 or 5 and with maximum degree at most 5 are totally $(\Delta+2)$-group choosable.

Theorem 5.7 *Let $k \geq 5$ and G be a planar graph with maximum degree $\Delta \leq k$ such that G contains neither 4-cycles nor 5-cycles. Then $\chi''_{gl}(G) \leq k + 2$.*

Proof Theorem 5.6 implies Theorem 5.7 when $k \geq 6$. Hence it is sufficient to prove the theorem when $k = 5$.

Let G be a minimum counterexample for the given value of k. Then there is a group A with $|A| \geq k + 2$, a $(k + 2)$-list assignment $L : V(T(G)) \longrightarrow 2^A$ and $f \in F(T(G), A)$, such that $T(G)$ is not (A, L, f)-colorable. As G is a minimum counterexample, G has the following properties:

(a) G is connected.
(b) Any vertex v is incident with at most $\lfloor \frac{\deg(v)}{2} \rfloor$ 3-faces.
(c) The minimum degree of G is at least 3.
(d) G contains no edge uv with $\min\{\deg(u), \deg(v)\} = 3$ and $\deg(u) + \deg(v) \leq 7$.

The proofs of (a)–(d) are similar to proofs of claims (a)–(d) in Theorem 5.6. Define the initial charge $w(x) = \deg(x) - 4$ for each $x \in V \cup F$. By Euler's Formula, $\sum_{x \in V \cup F} w(x) = -8$. We construct a new charge $w^*(x)$ on G according to the following rules:

(R1) Each r-face f with $r \geq 6$ gives $(1 - \frac{4}{r})m_v(f)$ to each incident vertex v; note that $1 - \frac{4}{r} \geq \frac{1}{3}$, with equality if and only if $r = 6$.
(R2) Each 3-vertex v receives $\frac{1}{3}$ from u if v is incident with 3-face f and u is a neighbor of v but not incident with f.
(R3) Each 3-face receives $\frac{1}{2}$ from its incident vertex v if $\deg(v) \geq 5$ and receives $\frac{1}{3}$ if $\deg(v) = 4$.

Let f be an r-face of G. If $r \geq 6$, then by (R1), $w^*(f) \geq 0$. If $r = 3$, then let u_1, u_2, u_3 be the vertices of f. Without loss of generality, we assume that $d(u_1) \geq d(u_2) \geq d(u_3)$. By (c), $d(u_3) \geq 3$. If $d(u_2) \geq 5$, or if $d(u_3) \geq 4$, then by (R3), $w^*(f) \geq 0$. Suppose $d(u_3) = 3$. Then by (d), $d(u_2) \geq 5$, and so by (R3), we always have $w^*(f) \geq 0$.

Let v be a vertex of G. Suppose that $\deg(v) = 3$. If v is incident with a 3-face f, then by (R2), v receives at least $\frac{2}{3}$ from its incident faces and $\frac{1}{3}$ from its incident vertex not lying on f. So $w^*(v) \geq w(v) + \frac{2}{3} + \frac{1}{3} = 0$. Otherwise, v receives at least $3 \times \frac{1}{3}$ from its incident faces and hence $w^*(v) \geq w(v) + 1 = 0$. Let $\deg(v) = 4$. The vertex v is incident with at most two 3-faces by (b), so v gives at most $\frac{2}{3}$ to its incident 3-faces. Also v receives at least $\frac{2}{3}$ from its incident faces of degree at least 6. Hence $w^*(v) \geq w(v) + \frac{2}{3} - \frac{2}{3} = 0$.

Finally let $\deg(v) = d \geq 5$. Suppose that v is incident with at most t 3-faces, to which v gives charge $\frac{t}{2}$ by (R3); then v is incident with $d - tr(\geq 6)$-faces, which gives at least $\frac{d-t}{3}$ to v by (R1). By (b), $t \leq \frac{d}{2}$, and v is adjacent to $d - 2t$ vertices not incident with the 3-faces incident with v, to which v may give up to $\frac{d-2t}{3}$ by (R2). It follows that, by $d \geq 5$,

$$w^*(v) \geq w(v) - \frac{t}{2} - \frac{d-2t}{3} + \frac{d-t}{3} = d - 4 - \frac{t}{6} > 0.$$

It follows that $\sum_{x \in V \cup F} w^*(x) = \sum_{x \in V \cup F} w(x) > 0$, a contradiction. This contradiction completes the proof of the theorem. □

Lemma 5.8 ([19]) *Let G be a planar graph with $\delta \geq 3$ and no five cycles. Then there exists an edge xy such that* $\deg(x) = 3$ *and* $\deg(y) \leq 5$.

The following theorem proves Conjecture 3 for planar graphs without 5-cycles and maximum degree at least 6.

Theorem 5.9 *Let $k \geq 6$ and G be a planar graph with maximum degree $\Delta \leq k$. If G contains no 5-cycles, then $\chi''_{gl}(G) \leq k + 2$.*

Proof Let G be a minimum counterexample for the given value of k. Then there is a group A with $|A| \geq k + 2$, a $(k + 2)$-list assignment $L : V(T(G)) \longrightarrow 2^A$ and $f \in F(T(G), A)$, such that $T(G)$ is not (A, L, f)-colorable. By Lemma 5.1, we may assume that $\delta \geq 3$. So by Lemma 5.8, G contains an edge xy such that $\deg(x) = 3$ and $\deg(y) \leq 5$. The graph $G - e$ has an (A, L, f)-coloring by the minimality of G. Now erase the color of x in this coloring and color the edge xy from its list, which is possible since its list has at least $(k + 2) - 7 \geq 1$ usable colors. Since $\deg_{T(G)}(x) = 6$ and $k \geq 6$, the vertex x can be colored from its list. Thus G has an (A, L, f)-coloring, contrary to the assumption that G is a counterexample. □

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