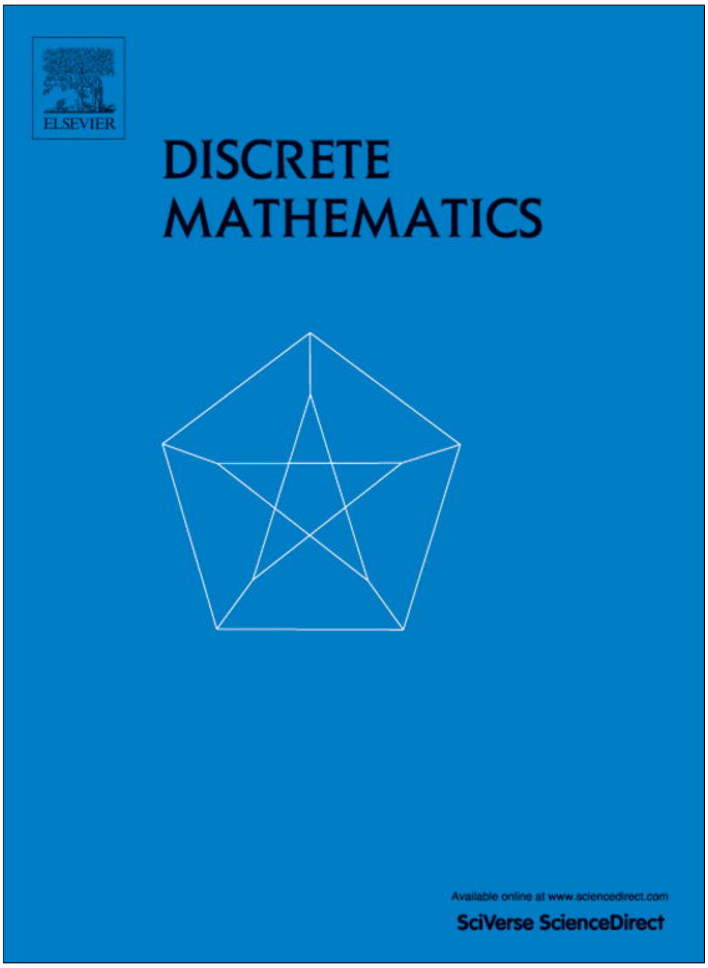
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Realizing degree sequences with k-edge-connected uniform hypergraphs

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ABSTRACT

An integral sequence $d = (d_1, d_2, \ldots, d_n)$ is hypergraphic if there is a simple hypergraph H with degree sequence d, and such a hypergraph H is a realization of d. A sequence d is r-uniform hypergraphic if there is a simple r-uniform hypergraph with degree sequence d. Similarly, a sequence d is r-uniform multi-hypergraphic if there is an r-uniform hypergraph (possibly with multiple edges) with degree sequence d. In this paper, it is proved that an r-uniform hypergraphic sequence $d = (d_1, d_2, \ldots, d_n)$ has a k-edge-connected realization if and only if both $d_i \geq k$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^n d_i \geq \frac{r(n-1)}{r-1}$, which generalizes the formal result of Edmonds for graphs and that of Boonyasombat for hypergraphs. It is also proved that a nonincreasing integral sequence $d = (d_1, d_2, \ldots, d_n)$ is the degree sequence of a k-edge-connected r-uniform hypergraph (possibly with multiple edges) if and only if $\sum_{i=1}^n d_i$ is a multiple of r, $d_n \geq k$ and $\sum_{i=1}^n d_i \geq \max\{\frac{r(n-1)}{r-1}, rd_1\}$.

1. Introduction

This paper focuses on the study of degree sequences in hypergraphs. Undefined terms can be found in [1] for hypergraphs and [3] for graphs. A *hypergraph* H is a pair $(V, \mathcal{E})$ where V is the vertex set of H and $\mathcal{E}$ is a collection of not necessarily distinct nonempty subsets of V. Note that we allow a hypergraph to have isolated vertices, which differs slightly from [1]. An element in V is a *vertex* of H, and an element in $\mathcal{E}$ is a *hyperedge* or simply an *edge* of H. The *degree* of a vertex v in H, denoted by $d_H(v)$ or $d(v)$, is the number of edges in H containing v. Let $\mathcal{E} = \{E_1, E_2, \ldots, E_m\}$. A hypergraph H is *simple* if $E_i \subseteq E_j$ implies that $i = j$ for any i, j with $1 \leq i, j \leq m$. Let $r \geq 2$ be an integer. A hypergraph H is an *r-uniform hypergraph* if $|E_i| = r$ for each i with $1 \leq i \leq m$. Thus a simple graph is a simple 2-uniform hypergraph, and vice versa. Let G and H be hypergraphs with $V(G) \cap V(H) = \emptyset$. Then $G \cup H$ is the hypergraph with vertex set $V(G) \cup V(H)$ and edge set $\mathcal{E}(G) \cup \mathcal{E}(H)$. If X is a collection of nonempty subsets of $V(H)$ and $X \cap \mathcal{E}(H) = \emptyset$, then $H + X$ is the hypergraph with vertex set $V(H)$ and edge set $\mathcal{E}(H) \cup X$.

If a hypergraph H has vertices $v_1, v_2, \ldots, v_n$, then the sequence $(d(v_1), d(v_2), \ldots, d(v_n))$ is a *degree sequence* of H. A sequence $d = (d_1, d_2, \ldots, d_n)$ is *hypergraphic* if there is a simple hypergraph H with degree sequence d, and such a hypergraph H is a *realization* of d, or a *d-realization*. A sequence d is *r-uniform hypergraphic* if there is a simple r-uniform hypergraph H with degree sequence d. Similarly, a sequence d is *multi-hypergraphic* if there is a hypergraph (possibly with multiple edges) with degree sequence d. A sequence d is *r-uniform multi-hypergraphic* if there is a r-uniform hypergraph (possibly with multiple edges) with degree sequence d. A 2-uniform hypergraphic sequence is also referred to as a *graphic sequence*.

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Let H be a hypergraph and $V_1, V_2, \ldots, V_k$ be subsets of $V(H)$. A hyperedge $E \in \mathcal{E}(H)$ is $(V_1, V_2, \ldots, V_k)$-*crossing* if $E \cap V_i \neq \emptyset$ for $1 \leq i \leq k$. If in addition, $E \subseteq \cup_{i=1}^{k} V_i$, then E is *exact*-$(V_1, V_2, \ldots, V_k)$-*crossing*. When $k = 1$, E is said to be V_1-*crossing* and *exact*-V_1-*crossing*, respectively. The set of all exact-$(V_1, V_2, \ldots, V_k)$-crossing edges of H is denoted by $\mathcal{E}^H_{V_1V_2\cdots V_k}$. A *walk* in a hypergraph H is a finite alternating sequence $W = (v_0, E_1, v_1, E_2, \ldots, E_k, v_k)$, where v_i is a vertex for $i = 0, 1, \ldots, k$ and E_j is an edge such that $v_{j-1}, v_j \in E_j$ for $j = 1, 2, \ldots, k$. A walk W is a *path* if all the vertices v_i for $i = 0, 1, \ldots, k$ and all the edges in W are distinct. A hypergraph is *connected* if for each pair of distinct vertices there exists a path from one to the other. Let X be a nonempty proper subset of V and $\overline{X} = V - X$. The set of all $(X, \overline{X})$-crossing hyperedges of a hypergraph H is an *edge-cut* of H between X and $\overline{X}$, denoted by $[X, \overline{X}]_H$, or $[X, \overline{X}]$. The number of hyperedges in $[X, \overline{X}]_H$ is denoted by $|[X, \overline{X}]_H|$ or $d_H(X)$. For a positive integer k, a hypergraph $H = (V, \mathcal{E})$ is k-*edge-connected* if $d_H(X) \geq k$ holds for every nonempty proper subset $X \subset V$. The *edge connectivity* of H is the maximum k such that H is k-edge-connected.

Edmonds gave the following characterization for a graphic sequence to have a k-edge-connected realization.

Theorem 1.1 (*Edmonds [8]*). *A graphic sequence $d = (d_1, d_2, \ldots, d_n)$ has a k-edge-connected realization if and only if*

(i) $d_i \geq k$ *for* $i = 1, 2, \ldots, n$;
(ii) $\sum_{i=1}^{n} d_i \geq 2(n-1)$ *if* $k = 1$.

Characterizations of uniform hypergraphic sequences or uniform multi-hypergraphic sequences to have connected realizations have been obtained by Boonyasombat [4] and Tusyadej, respectively.

Theorem 1.2 (*Boonyasombat, Theorem 4.1 of [4]*). *An r-uniform hypergraphic sequence $d = (d_1, d_2, \ldots, d_n)$ has a connected realization if and only if*

(i) $d_i \geq 1$ *for* $i = 1, 2, \ldots, n$;
(ii) $\sum_{i=1}^{n} d_i \geq \frac{r(n-1)}{r-1}$.

Theorem 1.3 (*Tusyadej, Page 4 of Berge [1]*). *A nonincreasing integer sequence $d = (d_1, d_2, \ldots, d_n)$ is the degree sequence of a connected r-uniform hypergraph (possibly with multiple edges) if and only if each of the following holds*

(i) $\sum_{i=1}^{n} d_i$ *is a multiple of* r;
(ii) $d_n \geq 1$; *and*
(iii) $\sum_{i=1}^{n} d_i \geq \max\{\frac{r(n-1)}{r-1}, rd_1\}$.

Degree sequence problems of hypergraphs are much harder than those of graphs. Actually the characterizations of hypergraphic sequences is still open for $r \geq 3$ (see [1,2,6,7,9]). The problem seems to be difficult even for $r = 3$. In [5], only the necessary condition for a hypergraphic sequence was given for $r = 3$. In fact, in [6], the authors reported that they were neither able to give a polynomial time algorithm nor able to prove that the problem is NP-complete even for $r = 3$.

In this paper, we investigate necessary and sufficient conditions for an r-uniform hypergraphic sequence to have a k-edge-connected realization. Our main results, Theorems 1.4 and 1.5 below, generalize Theorems 1.1–1.3, respectively.

Theorem 1.4. *An r-uniform hypergraphic sequence $d = (d_1, d_2, \ldots, d_n)$ has a k-edge-connected realization if and only if*

(i) $d_i \geq k$ *for* $i = 1, 2, \ldots, n$;
(ii) $\sum_{i=1}^{n} d_i \geq \frac{r(n-1)}{r-1}$ *if* $k = 1$.

Theorem 1.5. *A nonincreasing integer sequence $d = (d_1, d_2, \ldots, d_n)$ is the degree sequence of a k-edge-connected r-uniform hypergraph (possibly with multiple edges) if and only if each of the following holds*

(i) $\sum_{i=1}^{n} d_i$ *is a multiple of* r;
(ii) $d_n \geq k$; *and*
(iii) $\sum_{i=1}^{n} d_i \geq \max\{\frac{r(n-1)}{r-1}, rd_1\}$.

In Sections 2 and 3, we will present the proofs of Theorems 1.4 and 1.5 respectively. A further conjecture will be proposed in Section 4.

2. The proof of Theorem 1.4

The main effort will be the proof for the sufficiency. We will first show that d has an h-edge connected realization H for some $h \geq 1$. If $h < k$, then we will show that it is possible to perform some edge switching to find a d-realization with higher edge connectivity.

The following lemmas hold for any possibly nonsimple hypergraph.

Lemma 2.1. *Let H be an r-uniform hypergraph on n vertices. If H is connected, then $|\mathcal{E}(H)| \geq \frac{n-1}{r-1}$. Moreover, the equality holds if and only if for any edge $E \in \mathcal{E}(H)$, $H - E$ has r components.*

Proof. We establish the inequality by induction on n. If $n = r$, then it has an edge containing all vertices and so $|\mathcal{E}(H)| \geq 1$ ($|\mathcal{E}(H)| = 1$ for simple hypergraphs). Assume that $n \geq r+1$ and that the inequality holds for smaller values of n. We remove edges from H one by one until there are at least 2 components. Let $H_1, H_2, \ldots, H_t$ be these components. Removing a single edge can only create at most r components, thus $2 \leq t \leq r$. Suppose that the number of vertices in H_i is n_i for $1 \leq i \leq t$. Then $\sum_{i=1}^{t} n_i = n$. By the inductive hypothesis, $|\mathcal{E}(H_i)| \geq \frac{n_i-1}{r-1}$. Thus $|\mathcal{E}(H)| \geq \sum_{i=1}^{t} |\mathcal{E}(H_i)| + 1 = \frac{n-t}{r-1} + 1 \geq \frac{n-r}{r-1} + 1 = \frac{n-1}{r-1}$.

Now suppose that the equality holds. If there exists an edge $E_0 \in \mathcal{E}(H)$ such that $H - E_0$ has less than r components, denoted by $H_1, H_2, \ldots, H_t$, where $1 \leq t < r$. Let n_i be the number of vertices in H_i for $1 \leq i \leq t$. Then $\sum_{i=1}^{t} n_i = n$. Since each H_i is a connected r-uniform hypergraph, $|\mathcal{E}(H_i)| = \frac{n_i-1}{r-1}$. Then $|\mathcal{E}(H)| = \sum_{i=1}^{t} |\mathcal{E}(H_i)| + 1 = \frac{n-t}{r-1} + 1 > \frac{n-r}{r-1} + 1 = \frac{n-1}{r-1}$, contrary to $|\mathcal{E}(H)| = \frac{n-1}{r-1}$. Hence for any edge $E \in \mathcal{E}(H)$, $H - E$ has r components.

To prove the sufficiency of the second part, we argue by induction on n. If $n = r$, then $|\mathcal{E}(H)| = 1 = \frac{n-1}{r-1}$, and so we assume that $n > r$ and it holds for smaller values of n. Pick $E \in \mathcal{E}(H)$. Let $H_1, H_2, \ldots, H_r$ be the components of $H - E$ and $n_i = |V(H_i)|$ for $i = 1, 2, \ldots, r$. We claim that for each i and any edge $E' \in \mathcal{E}(H_i)$, $H_i - E'$ has r components. If not, then there exist j with $1 \leq j \leq r$ and an edge $E'' \in \mathcal{E}(H_j)$ such that $H_j - E''$ has less than r components. Then $H - E'' = (H_j - E'') \cup (\cup_{i \neq j} H_i) + \{E\}$ has less than r components, contrary to the assumption. Hence the claim holds and by induction, $|\mathcal{E}(H_i)| = \frac{n_i-1}{r-1}$. Thus $|\mathcal{E}(H)| = \sum_{i=1}^{r} |\mathcal{E}(H_i)| + 1 = \frac{n-r}{r-1} + 1 = \frac{n-1}{r-1}$, completing the proof. □

Lemma 2.2. *Let H be an r-uniform h-edge-connected hypergraph and $[X, \overline{X}]$ be an edge-cut of size h. Then for any vertex $u \in X$ with $d_H(u) > h$ and for any vertex $v \in \overline{X}$, there exist vertices $u_2, u_3, \ldots, u_r \in X$ such that $\{u, u_2, \ldots, u_r\} \in \mathcal{E}(H)$ and $\{v, u_2, \ldots, u_r\} \notin \mathcal{E}(H)$.*

Proof. Let $d_H(u) = k$ and k' be the number of $(X, \overline{X})$-crossing edges containing u. Then $k' \leq h < k$, and there are $k - k'$ exact-X-crossing edges containing u. That is, there exist distinct $(r-1)$-subsets $U_1, U_2, \ldots, U_{(k-k')}$ of X such that for each $i = 1, 2, \ldots, k - k'$, $U_i \cup \{u\} \in \mathcal{E}(H)$. Let v be any vertex in $\overline{X}$. If for each $i = 1, 2, \ldots, k - k'$, $U_i \cup \{v\} \in \mathcal{E}(H)$, then $|[X, \overline{X}]| \geq k' + (k-k') > h$, contrary to $|[X, \overline{X}]| = h$. Thus there exists a set U_j where $1 \leq j \leq k-k'$ such that $U_j \cup \{v\} \notin \mathcal{E}(H)$. Let $U_j = \{u_2, u_3, \ldots, u_r\}$. Then $\{u, u_2, \ldots, u_r\} \in \mathcal{E}(H)$ but $\{v, u_2, \ldots, u_r\} \notin \mathcal{E}(H)$. □

Lemma 2.3. *Let d be a sequence satisfying Theorem 1.4*(i) *and* (ii). *Then for any disconnected d-realization H with components $H_1, H_2, \ldots, H_l$, there exists an edge $E \in \mathcal{E}(H_j)$ such that the number of components of $H_j - E$ is at most $r - 1$, for some j with $1 \leq j \leq l$.*

Proof. Suppose that there is no such edge $E \in \mathcal{E}(H_i)$ for $i = 1, 2, \ldots, l$. Let $|V(H)| = n$ and $|V(H_i)| = n_i$ for each $i = 1, 2, \ldots, l$. By Lemma 2.1, $|\mathcal{E}(H_i)| = \frac{n_i-1}{r-1}$. Thus $|\mathcal{E}(H)| = \sum_{i=1}^{l} |\mathcal{E}(H_i)| = \frac{n_1+n_2+\cdots+n_l-l}{r-1} = \frac{n-l}{r-1} < \frac{n-1}{r-1}$, and so $\sum_{i=1}^{n} d_i = r|\mathcal{E}(H)| < \frac{r(n-1)}{r-1}$, contrary to Theorem 1.4(ii). □

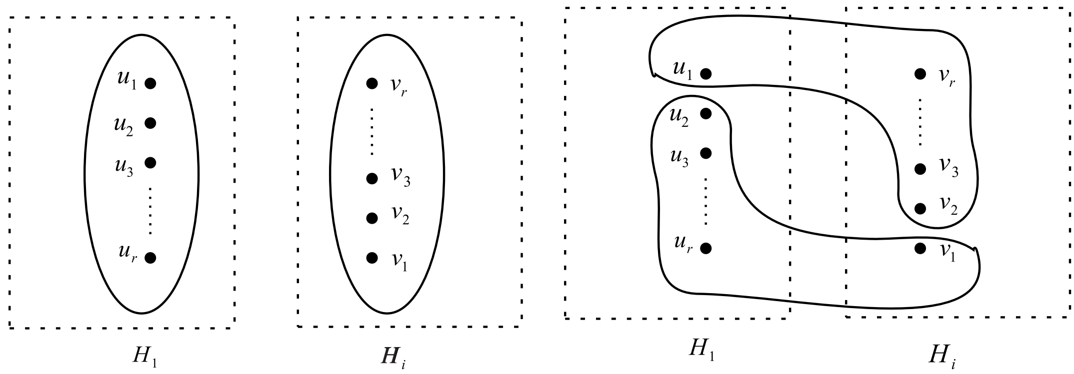
Lemma 2.4. *Suppose that H is an r-uniform hypergraph with edges $E_0 = \{u, x_2, x_3, \ldots, x_r\}$ and $F_0 = \{v, y_2, y_3, \ldots, y_r\}$. Let H' be a hypergraph obtained from H by deleting edges E_0 and F_0, and adding edges $\{v, x_2, x_3, \ldots, x_r\}$ and $\{u, y_2, y_3, \ldots, y_r\}$. Let Z be a nonempty proper subset of $V(H)$. If $d_{H'}(Z) < d_H(Z)$, then one of the following must hold.*

(i) *$u, y_2, y_3, \ldots, y_r \in Z$, $v \in \overline{Z}$ and at least one of $x_2, x_3, \ldots, x_r$ is in $\overline{Z}$;*
(ii) *$u, y_2, y_3, \ldots, y_r \in \overline{Z}$, $v \in Z$ and at least one of $x_2, x_3, \ldots, x_r$ is in Z;*
(iii) *$v, x_2, x_3, \ldots, x_r \in Z$, $u \in \overline{Z}$ and at least one of $y_2, y_3, \ldots, y_r$ is in $\overline{Z}$;*
(iv) *$v, x_2, x_3, \ldots, x_r \in \overline{Z}$, $u \in Z$ and at least one of $y_2, y_3, \ldots, y_r$ is in Z.*

Proof. By symmetry, it suffices to show one of the cases. Since $d_{H'}(Z) < d_H(Z)$, at least one of the two new edges of H' is not $(Z, \overline{Z})$-crossing. Without loss of generality, we may assume that $u, y_2, y_3, \ldots, y_r \in Z$. Then $v \in \overline{Z}$, otherwise, F_0 is not $(Z, \overline{Z})$-crossing in H, and thus removing F_0 will not decrease the number of $(Z, \overline{Z})$-crossing edges, contrary to $d_{H'}(Z) < d_H(Z)$. Similarly, if $x_2, x_3, \ldots, x_r \in Z$, then E_0 is not $(Z, \overline{Z})$-crossing in H and thus removing E_0 will not decrease the number of $(Z, \overline{Z})$-crossing edges, contrary to $d_{H'}(Z) < d_H(Z)$. Thus at least one of $x_2, x_3, \ldots, x_r$ is in $\overline{Z}$, completing the proof of (i). □

Let h be a positive integer, an *h-minimal set* of a hypergraph H is a nonempty proper subset X of $V(H)$ with $d_H(X) = h$ such that for any nonempty proper subset X' of X, $d_H(X') > h$. By definition, if H is h-edge-connected, then any subset $S \subseteq V(H)$ with $d_H(S) = h$ contains an h-minimal set of H.

Lemma 2.5. *Suppose that X is an h-minimal set of an r-uniform hypergraph H. Let X_1 and X_2 be nonempty proper subsets of X with $X_1 \cup X_2 = X$. Then each of the following statements holds.*



(a) H. (b) G.

Fig. 1. The construction of G from H.

(i) $|\mathcal{E}^H_{X_1X_2}| \geq |\mathcal{E}^H_{X_1\overline{X}}| + 1$ *and* $|\mathcal{E}^H_{X_1X_2}| \geq |\mathcal{E}^H_{X_2\overline{X}}| + 1$.

(ii) $|\mathcal{E}^H_{X_1X_2}| \geq \frac{h}{2} - \frac{|\mathcal{E}^H_{X_1X_2\overline{X}}|}{2} + 1$.

Proof. (i) Since X is an h-minimal set of H, $d_H(X) = |\mathcal{E}^H_{X_1\overline{X}}| + |\mathcal{E}^H_{X_2\overline{X}}| + |\mathcal{E}^H_{X_1X_2\overline{X}}| = h$ and $d_H(X_1) = |\mathcal{E}^H_{X_1\overline{X}}| + |\mathcal{E}^H_{X_1X_2}| + |\mathcal{E}^H_{X_1X_2\overline{X}}| \geq h + 1$. Thus $|\mathcal{E}^H_{X_1X_2}| \geq |\mathcal{E}^H_{X_2\overline{X}}| + 1$. By symmetry, $|\mathcal{E}^H_{X_1X_2}| \geq |\mathcal{E}^H_{X_1\overline{X}}| + 1$.

(ii) By (i), $2|\mathcal{E}^H_{X_1X_2}| + |\mathcal{E}^H_{X_1X_2\overline{X}}| \geq |\mathcal{E}^H_{X_1\overline{X}}| + 1 + |\mathcal{E}^H_{X_2\overline{X}}| + 1 + |\mathcal{E}^H_{X_1X_2\overline{X}}| = h + 2$. Thus $|\mathcal{E}^H_{X_1X_2}| \geq \frac{h}{2} - \frac{|\mathcal{E}^H_{X_1X_2\overline{X}}|}{2} + 1$. $\square$

Suppose that $[Z, \overline{Z}]$ is an edge-cut of a hypergraph H. Let $X_1, Y_1 \subseteq Z$ with $X_1 \cap Y_1 = \emptyset$ and $X_2, Y_2 \subseteq \overline{Z}$ with $X_2 \cap Y_2 = \emptyset$. Let $\mathcal{E}^H_O$ be the set of all other edges of $[Z, \overline{Z}]$ which are not in $\mathcal{E}^H_{X_1X_2}$ and $\mathcal{E}^{H'}_{Y_1Y_2}$. Then

$$d_H(Z) = |\mathcal{E}^H_{X_1X_2}| + |\mathcal{E}^H_{Y_1Y_2}| + |\mathcal{E}^H_O|. \tag{1}$$

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Suppose that d has a k-edge-connected r-uniform realization H. For any vertex $v \in V(H)$ whose degree is d_i, $d_i = |[\{v\}, V - \{v\}]| \geq k$, for $i = 1, 2, \ldots, n$. When $k = 1$, by Lemma 2.1, $|\mathcal{E}(H)| \geq \lceil \frac{n-1}{r-1} \rceil$, and so $\sum_{i=1}^n d_i \geq \frac{r(n-1)}{r-1}$.

To prove the sufficiency, let h be the maximum edge connectivity among all d-realizations. By contradiction, we assume that

$$h < k. \tag{2}$$

First we prove that $h \geq 1$ by showing that d has a simple connected r-uniform realization. Let H be a simple r-uniform d-realization with l components such that

$$l \text{ is minimized.} \tag{3}$$

If $l = 1$, then H is connected, and we are done. Hence we may assume that $l \geq 2$ and let $H_1, H_2, \ldots, H_l$ be the components of H.

By Lemma 2.3, we may assume that H_1 has an edge $E = \{u_1, u_2, \ldots, u_r\}$ such that $H_1 - E$ has a component U with $u_1, u_2 \in V(U)$. Let $E' = \{v_1, v_2, \ldots, v_r\} \in \mathcal{E}(H_i)$ for some i with $i > 1$. Let G be a hypergraph obtained from H by deleting edges E and E', and adding edges $\{v_1, u_2, u_3, \ldots, u_r\}$ and $\{u_1, v_2, v_3, \ldots, v_r\}$, as shown in Fig. 1. Then $V(H_i)$ and $V(H_1)$ are in the same component of G, and for each j with $1 \leq j \leq l$, vertices in $V(H_j)$ are in the same component of G. Thus the number of components of G is at most $l - 1$, contrary to (3). Therefore there exists a connected r-uniform d-realization, and so $h \geq 1$.

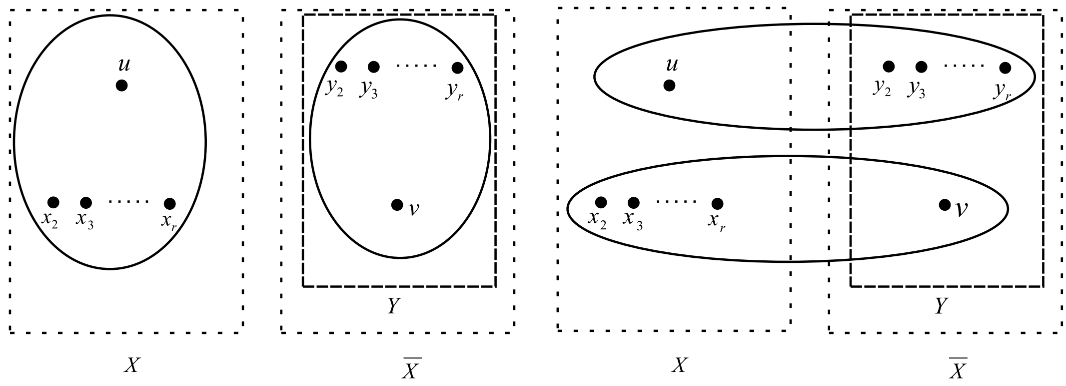
Let H be an r-uniform d-realization with edge connectivity h and

$$\text{with fewest number of edge-cuts of size } h. \tag{4}$$

Let X be an h-minimal set of H. Since $d_H(\overline{X}) = h$, $\overline{X}$ must contain an h-minimal set, denoted by Y. Since H is connected, there exist $u \in X$, $v \in Y$ and a path $P = (u, F_1, w_1, F_2, w_2, \ldots, F_t, v)$ such that

$$F_1 \text{ is } (X, \overline{X})\text{-crossing and } F_t \text{ is } (Y, \overline{Y})\text{-crossing.} \tag{5}$$

By Theorem 1.4(i), $d_H(u) \geq k > h = |[X, \overline{X}]|$. Then by Lemma 2.2, there exist vertices $x_2, x_3, \ldots, x_r \in X$ such that $E_1 = \{u, x_2, x_3, \ldots, x_r\} \in \mathcal{E}(H)$ but $\{v, x_2, x_3, \ldots, x_r\} \notin \mathcal{E}(H)$. Similarly, there exist $y_2, y_3, \ldots, y_r \in Y$ such that $E_2 = \{v, y_2, y_3, \ldots, y_r\} \in \mathcal{E}(H)$ but $\{u, y_2, y_3, \ldots, y_r\} \notin \mathcal{E}(H)$. Let H' be the hypergraph obtained from H by deleting edges E_1



(a) H. (b) H'.

Fig. 2. The construction of H' from H.

and E_2, and by adding edges $E_1' = \{v, x_2, x_3, \ldots, x_r\}$ and $E_2' = \{u, y_2, y_3, \ldots, y_r\}$, as shown in Fig. 2. Then $d_{H'}(X) = h+2$ and $d_{H'}(Y) = h+2$. By the definition of H', $\mathcal{E}(H') = (\mathcal{E}(H) - \{E_1, E_2\}) \cup \{E_1', E_2'\}$. An edge-cut is *new* if it is not an edge-cut of H.

Claim 1. *If H' has a new edge-cut $[Z, \overline{Z}]$ of size at most h, then each of the following holds.*

(i) *H has an $(X \cap Z, X \cap \overline{Z}, Y \cap Z, Y \cap \overline{Z})$-crossing edge.*
(ii) *H has no edges crossing exactly three of $X \cap Z, X \cap \overline{Z}, Y \cap Z$ and $Y \cap \overline{Z}$.*

Proof of Claim 1. Suppose that H' introduces a new edge-cut $[Z, \overline{Z}]$ with size $\le h$. Then $d_{H'}(Z) \le h < d_H(Z)$. By Lemma 2.4 and by symmetry, we may assume that $u, y_2 \in Z$ and $v, x_2 \in \overline{Z}$, as shown in Fig. 3.

Let $X \cap Z = X_1$, $X \cap \overline{Z} = X_2$, $Y \cap Z = Y_1$ and $Y \cap \overline{Z} = Y_2$. By Lemma 2.5, $|\mathcal{E}^H_{X_1X_2}| \ge \frac{h}{2} - \frac{|\mathcal{E}^H_{X_1X_2\overline{X}}|}{2} + 1$ and $|\mathcal{E}^H_{Y_1Y_2}| \ge \frac{h}{2} - \frac{|\mathcal{E}^H_{Y_1Y_2\overline{Y}}|}{2} + 1$. By the construction of H' from H, we have $|\mathcal{E}^{H'}_{X_1X_2}| = |\mathcal{E}^H_{X_1X_2}| - 1$ and $|\mathcal{E}^{H'}_{Y_1Y_2}| = |\mathcal{E}^H_{Y_1Y_2}| - 1$. By (1),

$$\begin{aligned} d_{H'}(Z) &= |\mathcal{E}^{H'}_{X_1X_2}| + |\mathcal{E}^{H'}_{Y_1Y_2}| + |\mathcal{E}^{H'}_O| \\ &= |\mathcal{E}^{H}_{X_1X_2}| + |\mathcal{E}^{H}_{Y_1Y_2}| + |\mathcal{E}^{H'}_O| - 2 \\ &\ge h + |\mathcal{E}^{H'}_O| - \frac{|\mathcal{E}^H_{X_1X_2\overline{X}}|}{2} - \frac{|\mathcal{E}^H_{Y_1Y_2\overline{Y}}|}{2} \\ &= h + \frac{|\mathcal{E}^{H'}_O| - |\mathcal{E}^H_{X_1X_2\overline{X}}|}{2} + \frac{|\mathcal{E}^{H'}_O| - |\mathcal{E}^H_{Y_1Y_2\overline{Y}}|}{2}. \end{aligned}$$

By (5), there must be an edge in $\mathcal{E}^{H'}_O$ contained in the path P and so $\mathcal{E}^{H'}_O \ne \emptyset$. Since $\mathcal{E}^H_{X_1X_2\overline{X}}$ and $\mathcal{E}^H_{Y_1Y_2\overline{Y}}$ are subsets of $\mathcal{E}^{H'}_O$, if one of them is a proper subset of $\mathcal{E}^{H'}_O$, then $d_{H'}(Z) > h$, contrary to $d_{H'}(Z) \le h$. Thus $\mathcal{E}^H_{X_1X_2\overline{X}} = \mathcal{E}^H_{Y_1Y_2\overline{Y}} = \mathcal{E}^{H'}_O \ne \emptyset$. By the definitions of $\mathcal{E}^H_{X_1X_2\overline{X}}$ and $\mathcal{E}^H_{Y_1Y_2\overline{Y}}$, there exists an $(X \cap Z, X \cap \overline{Z}, Y \cap Z, Y \cap \overline{Z})$-crossing edge, and there are no edges crossing exactly three of $X \cap Z, X \cap \overline{Z}, Y \cap Z, Y \cap \overline{Z}$. This completes the proof of Claim 1. □

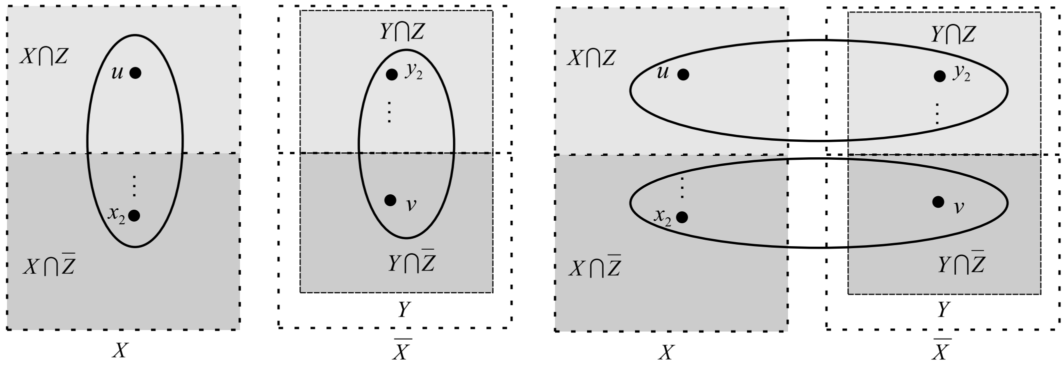
Since $[X, \overline{X}]_{H'}$ is no longer an edge-cut of size h in H', if there is not a new edge-cut with size at most h in H', then the number of edge-cuts with size h of H' is less than that of H, contrary to (4). Thus, we may assume that H' has a new edge-cut $[Z, \overline{Z}]_{H'}$ with size at most h. By Claim 1, there is an edge $E_0 = \{a_1, a_2, \ldots, a_r\} \in \mathcal{E}(H)$ which is $(X \cap Z, X \cap \overline{Z}, Y \cap Z, Y \cap \overline{Z})$-crossing with minimized $|E_0 \cap X|$. (Notice that if $r = 3$, then H can never have such an edge, contrary to Claim 1. Hence we may assume that, in the rest of the proof, $r \ge 4$.)

Denote $E_0 \cap X = \{a_1, a_2, \ldots, a_s\}$, where $2 \le s \le r-2$. As $Y \setminus E_0 \ne \emptyset$, let $b_1 \in Y \setminus E_0$. Since $d_H(b_1) \ge k > h$, by Lemma 2.2, there exist vertices $b_2, b_3, \ldots, b_r \in Y$ such that $F_0 = \{b_1, b_2, \ldots, b_r\} \in \mathcal{E}(H)$ but $E_0' = \{a_1, b_2, b_3, \ldots, b_r\} \notin \mathcal{E}(H)$. See Fig. 4(a).

If $F_0' = \{b_1, a_2, \ldots, a_r\} \in \mathcal{E}(H)$, then F_0' crosses at least three of $X \cap Z, X \cap \overline{Z}, Y \cap Z, Y \cap \overline{Z}$. By Claim 1(ii), F_0' is $(X \cap Z, X \cap \overline{Z}, Y \cap Z, Y \cap \overline{Z})$-crossing, contrary to the minimality of $|E_0 \cap X|$. Thus $F_0' = \{b_1, a_2, \ldots, a_r\} \notin \mathcal{E}(H)$. Let H'' be the hypergraph obtained from H by replacing E_0 and F_0 by E_0' and F_0', as shown in Fig. 4(b).

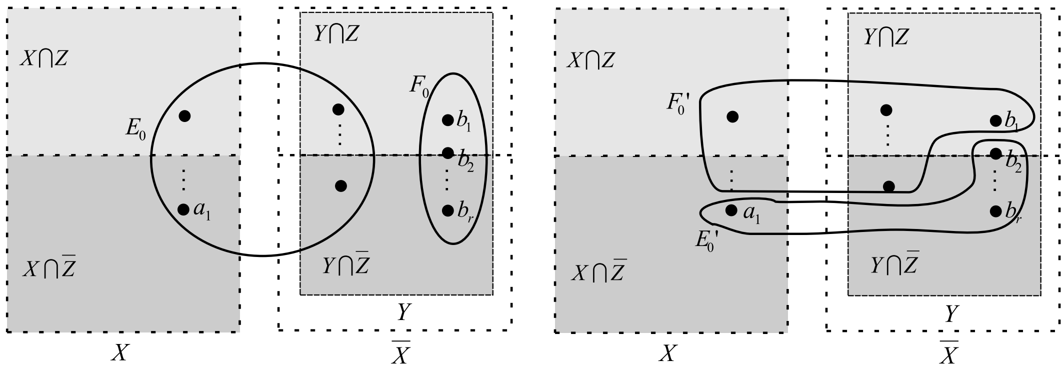
Claim 2. *H'' does not have any new edge-cut of size at most h.*

Proof of Claim 2. Suppose that there is a new edge-cut $[D, \overline{D}]$ of H'' with size at most h. Then $d_{H''}(D) \le h < d_H(D)$. By Lemma 2.4 and by symmetry, we may assume that $a_1 \in D$ and $b_1 \in \overline{D}$, as depicted in Fig. 5.



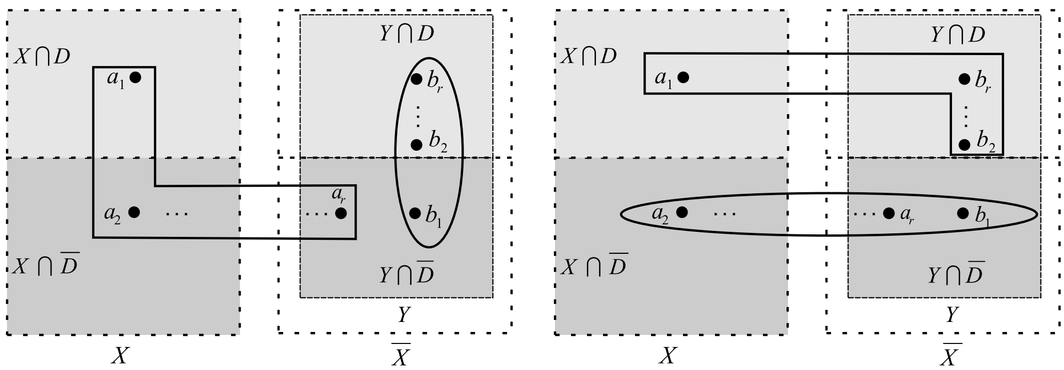
(a) H. (b) H'.

Fig. 3. New edge-cut $[Z, \overline{Z}]$ in H'.



(a) H. (b) H''.

Fig. 4. The construction of H'' from H.



(a) H. (b) H''.

Fig. 5. New edge-cut $[D, \overline{D}]$ in H''.

Let $X \cap D = X_3$, $X \cap \overline{D} = X_4$, $Y \cap D = Y_3$ and $Y \cap \overline{D} = Y_4$. By Lemma 2.5, $|\mathcal{E}^H_{X_3X_4}| \geq \frac{h}{2} - \frac{|\mathcal{E}^H_{X_3X_4\overline{X}}|}{2} + 1$ and $|\mathcal{E}^H_{Y_3Y_4}| \geq \frac{h}{2} - \frac{|\mathcal{E}^H_{Y_3Y_4\overline{Y}}|}{2} + 1$. By the construction of H'' from H, we have $|\mathcal{E}^{H''}_{X_3X_4}| = |\mathcal{E}^H_{X_3X_4}|$ and $|\mathcal{E}^{H''}_{Y_3Y_4}| = |\mathcal{E}^H_{Y_3Y_4}| - 1$. By (1),

$$
\begin{aligned}
d_{H''}(D) &= |\mathcal{E}^{H''}_{X_3X_4}| + |\mathcal{E}^{H''}_{Y_3Y_4}| + |\mathcal{E}^{H''}_{O}| \\
&= |\mathcal{E}^{H}_{X_3X_4}| + |\mathcal{E}^{H}_{Y_3Y_4}| + |\mathcal{E}^{H''}_{O}| - 1 \\
&\geq h + 1 + |\mathcal{E}^{H''}_{O}| - \frac{|\mathcal{E}^H_{X_3X_4\overline{X}}|}{2} - \frac{|\mathcal{E}^H_{Y_3Y_4\overline{Y}}|}{2}
\end{aligned}
$$

$$\geq h + |\mathcal{E}_O^{H''} \cup \{E_0\}| - \frac{|\mathcal{E}^H_{X_3X_4\overline{X}}|}{2} - \frac{|\mathcal{E}^H_{Y_3Y_4\overline{Y}}|}{2}$$

$$= h + \frac{|\mathcal{E}_O^{H''} \cup \{E_0\}| - |\mathcal{E}^H_{X_3X_4\overline{X}}|}{2} + \frac{|\mathcal{E}_O^{H''} \cup \{E_0\}| - |\mathcal{E}^H_{Y_3Y_4\overline{Y}}|}{2}.$$

Since $\mathcal{E}^H_{X_3X_4\overline{X}}$ and $\mathcal{E}^H_{Y_3Y_4\overline{Y}}$ are subsets of $\mathcal{E}_O^{H''} \cup \{E_0\}$, if one of them is a proper subset of $\mathcal{E}_O^{H''} \cup \{E_0\}$, then $d_{H''}(D) > h$, contrary to $d_{H''}(D) \leq h$. Hence $\mathcal{E}^H_{X_3X_4\overline{X}} = \mathcal{E}^H_{Y_3Y_4\overline{Y}} = \mathcal{E}_O^{H''} \cup \{E_0\}$. Then $E_0 \in \mathcal{E}^H_{X_3X_4\overline{X}} \cap \mathcal{E}^H_{Y_3Y_4\overline{Y}}$, which means $E_0 = \{a_1, a_2, \ldots, a_r\}$ must be (X_3, X_4, Y_3, Y_4)-crossing. Thus the new edge $F_0' = \{b_1, a_2, \ldots, a_r\}$ must be in $\mathcal{E}_O^{H''}$. But F_0' is not an edge in H, whence it is not in $\mathcal{E}^H_{X_3X_4\overline{X}}$ and $\mathcal{E}^H_{Y_3Y_4\overline{Y}}$, contrary to $\mathcal{E}^H_{X_3X_4\overline{X}} = \mathcal{E}^H_{Y_3Y_4\overline{Y}} = \mathcal{E}_O^{H''} \cup \{E_0\}$. This completes the proof of Claim 2. □

By Claim 2, the number of edge-cuts of size h of H'' is less than that of H, contrary to (4). Thus a contradiction will always occur if (2) holds, and so we must have $h = k$. □

3. The proof of Theorem 1.5

The necessity of Theorem 1.5 is straightforward. We only need to prove the sufficiency. The argument to prove the sufficiency of Theorem 1.5 is similar to that in the proof of Theorem 1.4. Theorem 1.5 can now be established by combining the two lemmas below.

Lemma 3.1 (*Gale [10], Ryser [11], See also Page 5 of Berge [1]*). *A nonincreasing integer sequence $d = (d_1, d_2, \ldots, d_n)$ is the degree sequence of an r-uniform hypergraph (possibly with multiple edges) if and only if*

(i) $\sum_{i=1}^n d_i$ *is a multiple of* r;
(ii) $\sum_{i=1}^n d_i \geq rd_1$.

Lemma 3.2. *An r-uniform multi-hypergraphic sequence $d = (d_1, d_2, \ldots, d_n)$ has a k-edge-connected realization if and only if*

(i) $d_i \geq k$ *for* $i = 1, 2, \ldots, n$;
(ii) $\sum_{i=1}^n d_i \geq \frac{r(n-1)}{r-1}$ *if* $k = 1$.

Proof. The proof is essentially identical to that of Theorem 1.4 (except that now we do not need to avoid multiple edges), thus, it is omitted here. □

4. Concluding remark

A hypergraph H is linear if for any two distinct edges E and F in H, $|E \cap F| \leq 1$. A sequence d is linear hypergraphic if there is a linear hypergraph with degree sequence d. Usually problems of linear hypergraphic sequences are more difficult than those of hypergraphic sequences. The proof of Theorem 1.4 cannot be applied to linear uniform hypergraphic sequences since the graphs constructed in the proof may not be linear. However, we believe that the following analog of Theorem 1.4 for linear r-uniform hypergraphs holds.

Conjecture 4.1. *A linear r-uniform hypergraphic sequence $d = (d_1, d_2, \ldots, d_n)$ has a k-edge-connected realization if and only if*

(i) $d_i \geq k$ *for* $i = 1, 2, \ldots, n$;
(ii) $\sum_{i=1}^n d_i \geq \frac{r(n-1)}{r-1}$ *if* $k = 1$.

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