Linear Algebra and its Applications 438 (2013) 1393–1397

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Note on the spectral characterization of some cubic graphs with maximum number of triangles ☆

Fenjin Liu [a,b,*], Qiongxiang Huang [b], Hong-Jian Lai [b,c]

[a] College of Science, Chang'An University, Xi'an, Shaanxi 710064, PR China
[b] College of Mathematics and System Science, Xinjiang University, Urumqi 830046, PR China
[c] Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

ARTICLE INFO

Article history:
Received 10 May 2012
Accepted 17 September 2012
Available online 1 November 2012

Submitted by R.A. Brualdi

AMS classification:
05C50

Keywords:
Cubic graphs
Eigenvalues
Graph spectrum
Determined by adjacency spectrum

ABSTRACT

Cubic graphs of given order with maximum number of triangles are characterized. Consequently, it is proved that certain cubic graphs with maximum number of triangles are determined by their adjacency spectrum.

1. Introduction

All the graphs considered in this paper are finite, simple and undirected. Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. Let $A(G)$ be the adjacency matrix of G, then the *eigenvalues of* G are the eigenvalues of $A(G)$. Since $A(G)$ is real and symmetric, its eigenvalues are all real numbers, which will be ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The *spectrum* of G is the multiset of eigenvalues of $A(G)$. Two non-isomorphic graphs G and H are called *cospectral* if they share the same spectrum. We say a graph G is *determined by its adjacency spectrum* (DAS for short) if there is no nonisomorphic graph H with the same spectrum as G. The question whether or not a given graph is DAS, has been investigated by some authors, see for instance [2,7–9,14]. For a recent survey of results on this subject we refer the reader to [4,5]. Generally speaking, not all regular graphs are DAS

☆ This work is supported by NSFC Grant No. 10961023 and NSFC Grant No. 11261059.
* Corresponding author at: College of Mathematics and System Science, Xinjiang University, Urumqi 830046, PR China.
E-mail addresses: fenjinliu@yahoo.cn (F. Liu), huangqiongxiang@sohu.com (Q. Huang), hjlai@math.wvu.edu (H.-J. Lai).

0024-3795/$ - see front matter
http://dx.doi.org/10.1016/j.laa.2012.09.015

(e.g., [3, p. 157]) and distance regular graphs have attracted more attention, see [1,6] for examples. It is also proved that the Cartesian product of an odd cycle with the complete graph K_2 is DAS [12]. In this note, we investigate the structure of cubic graphs of given order that have maximum number of triangles, and characterize the extremal cubic graphs. Four families of cubic graphs with maximum number of triangles are determined to be DAS.

Recall that K_n and C_n denote the complete graph and cycle on n vertices, respectively. Let K_4^- denote the graph obtained from K_4 by deleting an edge. Denote by G_a (G_b) the graph obtained from C_3 (C_4) by appending a pendant edge, G_c the bowtie graph (two triangles share a common vertex). The *theta graph* $\theta(a, b, c)$ is the graph consisting of three internally disjoint paths with common endpoints and lengths $a+1, b+1, c+1$ and with $a \geq b \geq c \geq 0$. The *dumbbell graph* $D(d, e, f)$ which is formed by joining two disconnected cycles C_{d+1} and C_{e+1} by a path P_{f+2} (two endpoints of P_{f+2} are coalescing to one vertex of C_{d+1} and one vertex of C_{e+1}, respectively.) where $d \geq e \geq 1, f \geq 0$. For a vertex $v \in V(G)$, $N(v)$ is the neighbors of v, i.e., $N(v) = \{u \in V(G) | uv \in E(G)\}$, and $N[v] = N(v) \cup \{v\}$. Let x_k be the number of vertices of degree k of a graph G, then we write the degree sequence of G as $d_G = (0^{x_0}, 1^{x_1}, \ldots, k^{x_k}, \ldots, \Delta^{x_\Delta})$. The notion and symbols not defined here are standard, see [3] for any undefined terms.

2. Cubic graphs with maximum number of triangles

In this section we characterize cubic graphs with maximum number of triangles. Moreover, we determine four families DAS extremal graphs.

Let G ($G \ncong K_4$) be a connected cubic graph with vertex set $V(G)$ and edge set $E(G)$. Clearly, each vertex of a cubic graph G belongs to at most two triangles. We call a vertex v_i is an i-type vertex of G if v_i is exactly contained in i triangles ($i = 0, 1, 2$). Denote by $V_i(G)$ the set of i-type vertices in G and $n_i = |V_i(G)|$, then the vertex set $V(G)$ can be partitioned into $V(G) = V_0(G) \cup V_1(G) \cup V_2(G)$. By definition, for $v \in V_2(G)$, $N[v]$ induces K_4^- in G in which v and v' belong to exactly two triangles and u and u' belong respectively to one triangles (see Fig. 1), such an induced subgraph is said to be a K_4^-*-part* of G. Thus we have the claim bellow.

Claim 1. Let G ($G \ncong K_4$) be a connected cubic graph, then n_2 is even, G contains exactly $\frac{n_2}{2}$ numbers of K_4^--part and $n_1 \geq n_2$.

If $x \in V_1(G)$ then $N[x] = \{x, y, z, w\}$ induces a G_a where x, y and z are all contained in a common triangle (see Fig. 1), which we call a C_3*-part* of G, and w lies in at most on triangle. Thus we have the claim bellow.

Claim 2. $n_1 - n_2 \equiv 0 \pmod 3$.

Since a cubic graph G (except for K_4) with n vertices consists of some K_4^--parts, C_3-parts and some vertices belonging to $V_0(G)$. If the K_4^- parts and C_3 parts are respectively contracted as a vertex then we get a cycle or a (2, 3)-bidegree graph $\tilde{G}$ from G. For example, in Fig. 1 (a)-(g) we illustrate all cubic graphs consisting of four numbers of K_4^--parts and two vertices contained in no triangles, and (a')-(e') the cubic graphs consisting of three numbers of K_4^--parts and two numbers of C_3-parts. In general we denote by $\mathcal{G}_n(k, \ell, (n-4k-3\ell))$ the cubic graphs with n vertices consisting of k numbers of K_4^--parts, ℓ numbers of C_3-parts and $(n-4k-3\ell)$ vertices lying in no triangles. For instance, the graphs shown in Fig. 1 (a)-(g) are the graphs in $\mathcal{G}_{18}(4, 0, 2)$, and ($a'$)-($e'$) the graphs in $\mathcal{G}_{18}(3, 2, 0)$, where the small thick dots are the vertices belonging to $V_0(G)$, big thick dots denote the subgraphs K_4^- and big hollow dots denote the subgraphs K_3, such graphs shown in Fig. 1 are called the *part-contracted graphs* of cubic graphs. (Note that these graphs may have cycles of length two). It is worth mentioning that $\mathcal{G}_n(k, 0, 0)$ contains exactly one part-contracted graph (i.e., a cycle C_k), and the part-contracted graph of a cubic graph will be itself if it contains no triangles. It is easy to verify the following two claims.

Claim 3. If $G \in \mathcal{G}_n(k, \ell, n-4k-3\ell)$ then G contains exactly $2k+\ell$ triangles.

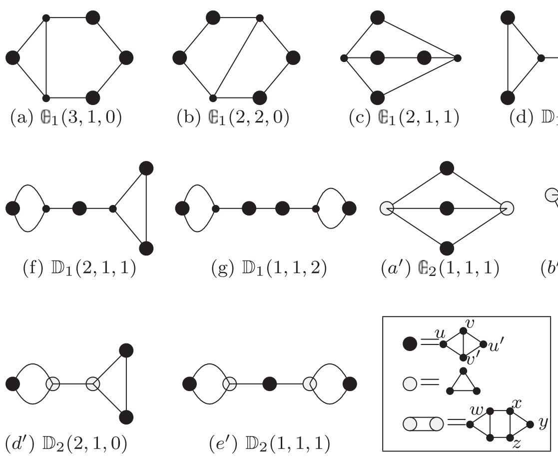

Fig. 1. $\mathcal{G}_{18}(4,0,2)$ and $\mathcal{G}_{18}(3,2,0)$.

Claim 4. Let G ($G \ncong K_4$) be a connected cubic graph, then the number of triangles of G is $t(G) = \frac{1}{3}\sum_{i=0}^{2} in_i$.

The following theorem characterizes all connected cubic graphs with maximum number of triangles.

Theorem 2.1. *Let G be a connected cubic graphs with $n = 2r$ vertices, $m = 3r$ edges ($r \geq 3$) and maximum number of triangles. If r is even then $G \cong \mathcal{G}_n\left(\frac{r}{2},0,0\right)$; if r is odd then $G \in \mathcal{G}_n\left(\frac{r-1}{2},0,2\right)$ or $\mathcal{G}_n\left(\frac{r-3}{2},2,0\right)$.*

Proof. To determine such an extremal cubic graph G with maximum number of triangles is equivalent to solving the following optimization problem:

$$\max\ t(G) = \frac{1}{3}(n_1 + 2n_2), \qquad \text{s.t.} \begin{cases} n_0 + n_1 + n_2 = |V(G)| = 2r \\ n_i = |V_i(G)| \geq 0 \quad (i = 0, 1, 2.) \end{cases} \tag{1}$$

From (1) we see that $t(G)$ is a decreasing function in n_0 and an increasing function in $n_2 - n_1$. A natural idea is when $t(G)$ achieves its maximum then n_0 is smallest and n_2 is largest or nearly largest (by nearly largest we mean the value is one less than the maximum). Hence we need to consider the largest n_2. Suppose that G is an extremal cubic graph satisfying (1). From Claim 1 we get the following inequalities:

$$\begin{cases} n_2 \equiv 0 \pmod 2 \\ n_1 \geq n_2 \\ n_0 \geq 0 \end{cases} \tag{2}$$

Combing (1) and (2), we obtain $2r - n_0 - n_1 = n_2 \leq n_1 \leq n_1 + n_0 = 2r - n_2$, thus

$$n_2 \leq r. \tag{3}$$

If r is even then $n_2 = r$ achieves maximum value. By Claim 1, $r = n_2 \leq n_1 \leq n - n_2 = r$, which implies $n_0 = 0$, and so G is a graph in $\mathcal{G}_n(r,0,0)$. Hence $t(G) = r$ achieves its maximum value.

If r is odd then $n_2 = r - 1$ achieves maximum value. By Claim 2, $n_1 = r - 1$ and so $n_0 = 2$, we obtain a solution $t(G) = r - 1$ for (1) and find $G \in \mathcal{G}_n\left(\frac{r-1}{2}, 0, 2\right)$ (see Fig. 1 for example). Next, if $n_2 = r - 3$ takes nearly largest value then G contains $\frac{r-3}{2}$ numbers of K_4^--parts. By Claim 2 and Claim 3 we know $G \in \mathcal{G}\left(\frac{r-3}{2}, 2, 0\right)$ has maximum number of triangles: $t(G) = r - 3 + 2 = r - 1$. Finally, if $n_2 = r - (2s + 1)$ $(s \geq 2)$ then G has exactly $\frac{r-(2s+1)}{2}$ numbers of K_4^--parts. By Claim 2 and Claim 3, G has at most $r - (2s + 1) + \frac{2(2s+1)}{3}$ triangles, but $r - (2s + 1) + \frac{2(2s+1)}{3} < r - 1$, and so $t(G)$ cannot achieves its maximum value if $n_2 = r - (2s + 1)$ $(s \geq 2)$. $\square$

Theorem 2.2. *The order of the set $\mathcal{G}_n(k, 0, 2)$ and $\mathcal{G}_n(k-1, 2, 0)$ $(k \geq 2)$ are $\left\lfloor \frac{k(2k+3)}{6} \right\rfloor$ and $\left\lfloor \frac{(k-1)(2k+1)}{6} \right\rfloor + 1$, respectively.*

Proof. Denote by $p_i(k)$ the number of the unordered partitions of k into i parts. We have $p_1(k) = 1$ and $p_2(k) = \left\lfloor \frac{k}{2} \right\rfloor$. It is known in [13, p. 152] that $p_3(k) = \left\lfloor \frac{k^2+6}{12} \right\rfloor$. Let G be a graph in $\mathcal{G}_n(k, 0, 2)$ and $\tilde{G}$ be its part contracted graph. Since $d_{\tilde{G}} = (3^2, 2^k)$, $\tilde{G} \in S_1 = \{\theta(a, b, c) | a + b + c = k\}$ or $S_2 = \{D(d, e, f) | d + e + f = k\}$. For the order of S_1, we obtain if $c = 0$, then $|S_1| = p_2(k)$; if $c > 0$, then $|S_1| = p_3(k)$. For the cardinality of S_2, we conclude if $f = 0$, then $|S_2| = p_2(k)$; if $f > 0$, then $|S_2| = \sum_{j=1}^{k-2} p_2(k - j)$. Hence direct calculation shows that

$$|\mathcal{G}_n(k, 0, 2)| = |S_1| + |S_2| = (p_2(k) + p_3(k)) + \left(p_2(k) + \sum_{j=1}^{k-2} p_2(k - j)\right) = \left\lfloor \frac{k(2k + 3)}{6} \right\rfloor. \quad (4)$$

Let $\tilde{H}$ be the part contracted graph of the graph H in $\mathcal{G}_n(k - 1, 2, 0)$. If $\tilde{H} \in \{\theta(a, b, c) | a + b + c = k - 1\}$, unlike the graph in $\mathcal{G}_n(k, 0, 2)$, the unique different case $a = k - 1, b = c = 0$ may occur. For the other cases, the argument is exactly the same as the above, it suffices to substitute $k - 1$ into (4) and which gives $\left\lfloor \frac{(k-1)(2k+1)}{6} \right\rfloor$. Hence we get $|\mathcal{G}_n(k - 1, 2, 0)| = \left\lfloor \frac{(k-1)(2k+1)}{6} \right\rfloor + 1$. $\square$

Denote by $\Theta_1(a, b, c)$, $\Theta_2(a', b', c')$ the cubic graphs in $\mathcal{G}_n(k, 0, 2)$ and $\mathcal{G}_n(k - 1, 2, 0)$ whose part contracted graphs are $\theta_1(a, b, c)$ and $\theta_2(a', b', c')$ $(a + b + c = k, a' + b' + c' = k - 1)$, respectively. Denote by $\mathbb{D}_1(d, e, f)$, $\mathbb{D}_2(d, e, f)$ the cubic graphs in $\mathcal{G}_n(k, 0, 2)$ and $\mathcal{G}_n(k - 1, 2, 0)$ whose part contracted graphs are $D(d, e, f)$ and $D(d', e', f')$ $(d + e + f = k, d' + e' + f' = k - 1)$, respectively. Now we turn to the spectral characterization of graphs $\mathcal{G}_n(k, 0, 0)$ $(k \geq 2)$ and $\mathbb{D}_1(1, 1, k - 2)$, $\Theta_1(k - 1, 1, 0)$, $\Theta_2(k - 1, 0, 0)$.

Let $G \in \mathcal{G}_n(k, 0, 2) \cup \mathcal{G}_n(k - 1, 2, 0)$ and H be cospectral with G. We obtain H is also a connected cubic graph with $2k$ numbers of triangles [4]. Hence $H \in \mathcal{G}_n(k, 0, 2) \cup \mathcal{G}_n(k-1, 2, 0)$. If $G \cong \mathcal{G}_n(k, 0, 0)$, Theorem 2.1 implies $\mathcal{G}_n(k, 0, 0)$ is the unique cubic graph with $4k$ vertices and $2k$ triangles, it follows that $\mathcal{G}_n(k, 0, 0)$ is DAS.

Theorem 2.3. *$\mathcal{G}_n(k, 0, 0)$ $(k \geq 2)$ is determined by its adjacency spectrum.*

In order to show that the graphs $\mathbb{D}_1(1, 1, k - 2)$, $\Theta_1(k - 1, 1, 0)$, $\Theta_2(k - 1, 0, 0)$ are also DAS, we enumerate the number of closed walks of length 4, 5, 6 of graphs in $\mathcal{G}_n(k, 0, 2)$ and $\mathcal{G}_n(k - 1, 2, 0)$ to rule out their possible cospectral mate. Let $N_G(H)$ be the number of subgraphs of a graph G which are isomorphic to H and let $N_G(i)$ be the number of closed walks of length i in G. The following lemma provides some formulae for calculating the number of closed walks of length 4, 5, 6 for any graph.

Lemma 2.4 [10,11]. *The number of closed walks of length 4, 5, 6 of a graph G are giving in the following:*

(i) $N_G(4) = 2|E(G)| + 4N_G(P_3) + 8N_G(C_4)$.
(ii) $N_G(5) = 30N_G(K_3) + 10N_G(C_5) + 10N_G(G_a)$,

(iii) $N_G(6) = 2|E(G)| + 12N_G(P_3) + 6N_G(P_4) + 12N_G(K_{1,3}) + 24N_G(K_3) + 48N_G(C_4) + 36N_G(K_4^-) + 12N_G(G_b) + 24N_G(G_c) + 12N_G(C_6)$,

Theorem 2.5. *$\mathbb{D}_1(1,1,k-2)$, $\theta_1(k-1,1,0)$ and $\theta_2(k-1,0,0)$ $(d \geq 1, k \geq 2)$ are determined by their adjacency spectrum.*

Proof. It suffices to show $\mathbb{D}_1(1,1,k-2)$, $\theta_1(k-1,1,0)$ and $\theta_2(k-1,0,0)$ have no cospectral mate in $\mathcal{G}_n(k,0,2) \cup \mathcal{G}_n(k-1,2,0)$.

We consider the number of closed walks of length 4 of a graph G in $\mathcal{G}_n(k,0,2) \cup \mathcal{G}_n(k-1,2,0)$. Since $|E(G)| = 3(2k+1)$ and $N_G(P_3) = \sum_{v\in V(G)} \binom{d(v)}{2} = 6(2k+1)$, by Lemma 2.4 (*i*), $N_G(4) = 60k + 30 + 8N_G(C_4)$. Thus $N_G(4)$ is determined by $N_G(C_4)$. From the graph structure we see that $N_{\mathbb{D}_1(1,1,k-2)}(C_4) = k+4$, $N_{\mathbb{D}_1(d,1,k-d-1)}(C_4) = k+2$ $(2 \leq d \leq k-1)$, $N_{\theta_1(a,b,c)}(C_4) = N_{\mathbb{D}_1(d,e,f)}(C_4) = k$ $(2 \leq e \leq d)$, $N_{\theta_2(k-1,0,0)}(C_4) = k$. For any graph $G' \in \mathcal{G}_n(k-1,2,0) \setminus \{\theta_2(k-1,0,0)\}$, $N_{G'}(C_4) = k-1$. Thus $\mathbb{D}_1(1,1,k-2)$ is DAS. For $\theta_1(k-1,1,0)$ and $\theta_2(k-1,0,0)$, we show that they have no cospectral mate in the set $S_3 = \{\theta_1(a,b,c), \theta_2(k-1,0,0), \mathbb{D}_1(d,e,f)(2 \leq e \leq d)\}$. We need to count the number of closed walks of length 5 and 6.

For a graph G in S_3, Since $N_G(K_3) = 2k$ and $N_G(G_a) = 6k$, by Lemma 2.4 (*ii*) $N_G(5) = 120k + 10N_G(C_5)$ is determined by $N_G(C_5)$. Clearly, $N_{\theta_1(k-1,1,0)}(C_5) = N_{\theta_2(k-1,0,0)}(C_5) = 2$. For $G' \in S_3 \setminus \{\theta_1(k-1,1,0), \theta_2(k-1,0,0)\}$, it is easy to verify that $N_{G'}(C_5) = 0$. Thus $\theta_1(k-1,1,0)$ and $\theta_2(k-1,0,0)$ are not cospectral with the graphs in $S_3 \setminus \{\theta_1(k-1,1,0), \theta_2(k-1,0,0)\}$.

Finally we enumerate the number of closed walks of length 6 of graphs $\theta_1(k-1,1,0)$ and $\theta_2(k-1,0,0)$. Since $\theta_1(k-1,1,0)$ and $\theta_2(k-1,0,0)$ have the same number of edges, P_3, K_3, $K_{1,3}$, P_4 $(N_{\theta_1(k-1,1,0)}(P_4) = 4|E(\theta_1(k-1,1,0))| - 3N_{\theta_1(k-1,1,0)}(K_3) = 4|E(\theta_2(k-1,0,0))| - 3N_{\theta_2(k-1,0,0)}(K_3) = N_{\theta_2(k-1,0,0)}(P_4))$, C_4 and G_c, moreover $N_{\theta_1(k-1,1,0)}(K_4^-) = k$, $N_{\theta_2(k-1,0,0)}(K_4^-) = k-1$; $N_{\theta_1(k-1,1,0)}(G_b) = 2k$, $N_{\theta_2(k-1,0,0)}(G_b) = 2k+2$; $N_{\theta_1(k-1,1,0)}(C_6) = 2$, $N_{\theta_2(k-1,0,0)}(C_6) = 1$. By Lemma 2.4 (*iii*), $N_{\theta_1(k-1,1,0)}(6) - N_{\theta_2(k-1,0,0)}(6) = 24$. Therefore $\theta_1(k-1,1,0)$ and $\theta_2(k-1,0,0)$ are not cospectral, and they are determined by their adjacency spectrum. □

Acknowledgment

The authors would like to thank the anonymous referees for their valuable suggestions used in improving this paper.

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