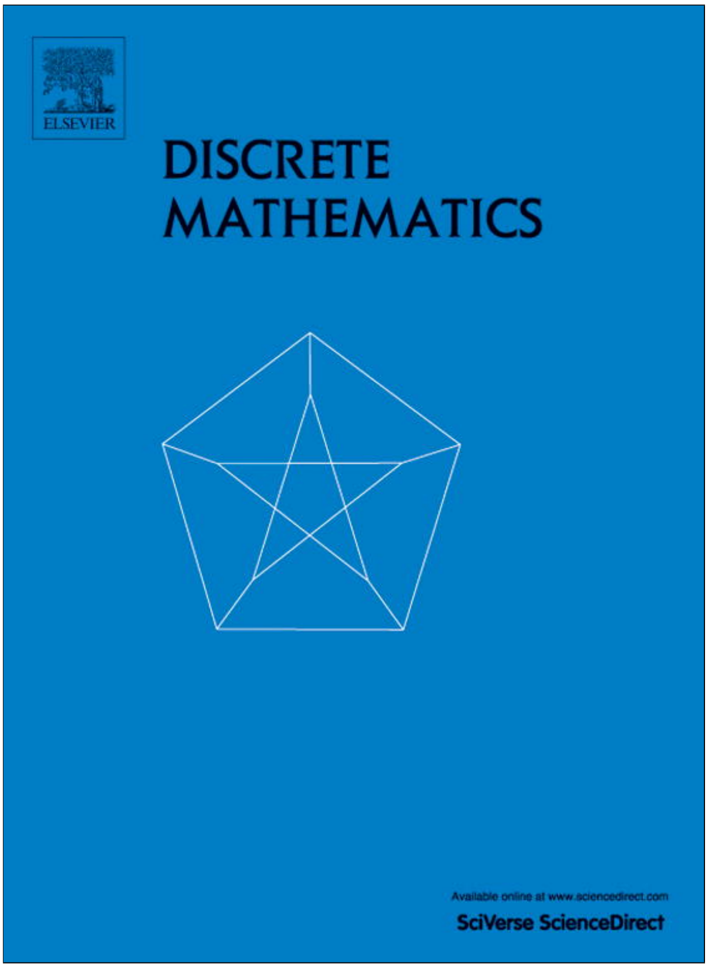
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Hamilton cycles in 3-connected claw-free and net-free graphs[☆]

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ABSTRACT

For an integer $s_1, s_2, s_3 > 0$, let N_{s_1,s_2,s_3} denote the graph obtained by identifying each vertex of a K_3 with an end vertex of three disjoint paths P_{s_1+1}, P_{s_2+1}, and P_{s_3+1} of length s_1, s_2, and s_3, respectively. We determine a family $\mathcal{F}$ of graphs such that, every 3-connected $(K_{1,3}, N_{s_1,s_2,1})$-free graph Γ with $s_1 + s_2 + 1 \le 10$ is hamiltonian if and only if the closure of Γ is $L(G)$ for some graph $G \notin \mathcal{F}$. We also obtain the following results.

(i) Every 3-connected $(K_{1,3}, N_{s_1,s_2,s_3})$-free graph with $s_1 + s_2 + s_3 \le 9$ is hamiltonian.
(ii) If G is a 3-connected $(K_{1,3}, N_{s_1,s_2,0})$-free graph with $s_1 + s_2 \le 9$, then G is hamiltonian if and only if the closure of G is not the line graph of a member in $\mathcal{F}$.
(iii) Every 3-connected $(K_{1,3}, N_{s_1,s_2,0})$-free graph with $s_1 + s_2 \le 8$ is hamiltonian.

1. Introduction

We consider finite loopless graphs. Undefined terms and notation will follow [2]. For a graph G which contains at least one cycle, the *circumference* of G, denoted by $c(G)$, is the length of a longest cycle contained in G; and the *girth* of G, denoted by $g(G)$, is the length of a shortest cycle contained in G. By $H \subseteq G$ we mean that H is a subgraph of G. If $H \subseteq G$, then the set of *vertices of attachments* of H in G is defined as

$$A_G(H) = \{v \in V(H) : N_G(v) - V(H) \neq \emptyset\}.$$

For an integer $i \ge 0$ and $v \in V(G)$, define

$$D_i(G) = \{v \in V(G) : d_G(v) = i\}, \quad \text{and} \quad E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}.$$

For a vertex $v \in V(G)$, define $N_G(v) = \{u \in V(G) \mid vu \in E(G)\}$. The subscript G in the notations above might be omitted if G is understood from the context.

Let G be a graph and $X \subseteq E(G)$ be an edge subset. The *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. We define $G/\emptyset = G$. If $H \subseteq G$, then we write G/H for $G/E(H)$. If H is a connected subgraph of G, and if v_H is the vertex in G/H onto which H is contracted, then H is the *preimage* of v_H, and is denoted by $PI_G(v_H)$. If H is the preimage of v_H in G/H, then we also say that v_H is lifted to H in G. When the graph G is understood from the context, we often use $PI(v)$ for $PI_G(v)$. A vertex v in a contraction of G is *nontrivial* if $PI(v)$ has at least

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one edge. If P' is a path (or cycle, respectively) in G', then since for any $v \in V(P')$, by the definition of contractions, $PI(v)$ is a connected subgraph of G, then P' can be extended to a path (or a cycle, respectively) of G by adding (possibly empty) paths in each $PI(v)$ to P', viewed as the induced subgraph $G[E(P')]$. In this case, we say that P' *is lifted to* P, or P is a lifting of P'.

For integer $s_1, s_2, s_3, k \geq 0$, let P_k denote a path of k vertices and N_{s_1,s_2,s_3} be the graph obtained by identifying each vertex of a K_3 with an end vertex of three disjoint paths $P_{s_1+1}, P_{s_2+1}, P_{s_3+1}$, respectively. The graph $N_{0,0,k}$ is also known as Z_k. For graphs $H_1, H_2, \dots, H_s$, a graph G is $\{H_1, H_2, \cdots H_s\}$-free if it contains no induced subgraph isomorphic to a copy of H_i for any i. A graph G is called *claw-free* if it is $K_{1,3}$-free.

The *line graph* of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common. Beineke [1] and Robertson [9] showed that line graphs are $K_{1,3}$-free graphs.

Two fascinating conjectures on hamiltonian line graphs and hamiltonian claw-free graphs have attracted the attention of many researchers.

Conjecture 1.1. (i) *(Thomassen, [17]) Every 4-connected line graph is hamiltonian.*
(ii) *(Matthews and Sumner [14]) Every 4-connected $K_{1,3}$-free graph is hamiltonian.*

Ryjáček [15] introduced the line graph *closure* $cl(G)$ of a claw-free graph G and used it to show that Conjecture 1.1(i) and (ii) are equivalent. Motivated by Conjecture 1.1, many have investigated forbidden induced subgraph conditions for hamiltonicity. In 1999, Brousek, Ryjáček and Favaron proved the following theorem.

Theorem 1.2 (*Brousek, Ryjáček and Favaron [3]*)**.** *Every 3-connected $\{K_{1,3}, N_{0,0,4}\}$-free graph is hamiltonian.*

In 2010, Lai et al. extended Theorem 1.2 by showing a best possible result stated below.

Theorem 1.3 (*Lai, Xiong, Yan, and Yan [11]*)**.** *Every 3-connected $\{K_{1,3}, N_{0,0,8}\}$-free graph is hamiltonian.*

A recent research by Ma et al. [13] determined two well characterized families of graphs $\mathcal{F}_1$ and $\mathcal{F}_2$ such that both conclusions in the following hold.

Theorem 1.4 (*Ma et al. [13]*)**.** (i) *A 3-connected $\{K_{1,3}, N_{0,0,9}\}$-free graph G is hamiltonian if and only if the closure of G is not the line graph of a graph in $\mathcal{F}_1$.*
(ii) *A 3-connected $\{K_{1,3}, P_{12}\}$-free graph G is hamiltonian if and only if the closure of G is not the line graph of a graph in $\mathcal{F}_2$.*

The main purpose of this paper is to extend the theorems above.

Throughout this paper, we use $P(10)$ to denote the Petersen graph. Let $s \geq 1$ be an integer. When $s > 1$, the vertex of degree s in $K_{1,s}$ is the *center* of $K_{1,s}$. When $s = 1$, any vertex of $K_{1,1}$ is a center of it. A graph is a *star* if it is isomorphic to a $K_{1,s}$. Let $\mathcal{F}$ denote the family of graphs such that $L \in \mathcal{F}$ if and only if F is obtained from $P(10)$ by identifying every vertex $v \in V(P(10))$ with the center of a star $K_{1,s(v)}$, where $s(v) \geq 1$.

Theorem 1.5. *Let $s_1, s_2, s_3 > 0$ be integers such that $s_1 + s_2 + s_3 \leq 10$.*
(i) *If $s_1 + s_2 + 1 \leq 10$, every 3-connected $\{K_{1,3}, N_{s_1,s_2,1}\}$-free graph Γ is hamiltonian if and only if the closure of Γ is the line graph $L(G)$ for some graph $G \notin \mathcal{F}$.*
(ii) *If $s_1 + s_2 + s_3 \leq 9$, every 3-connected $\{K_{1,3}, N_{s_1,s_2,s_3}\}$-free graph is hamiltonian.*
(iii) *If $s_1 + s_2 \leq 9$, every 3-connected $\{K_{1,3}, N_{s_1,s_2,0}\}$-free graph Γ is hamiltonian if and only the closure of Γ is the line graph $L(G)$ for some graph $G \notin \mathcal{F}$.*
(iv) *If $s_1 + s_2 \leq 8$, every 3-connected $\{K_{1,3}, N_{s_1,s_2,0}\}$-free graph is hamiltonian.*

This result motivates the following conjecture. If $s_1 + s_2 + s_3 \leq 10$, every 3-connected $\{K_{1,3}, N_{s_1,s_2,s_3}\}$-free graph Γ is hamiltonian if and only if the closure of Γ is the line graph $L(G)$ for some graph $G \notin \mathcal{F}$. Our strategy in this paper is to apply Ryjáček's line graph closure to convert the problem to a line graph problem. Therefore, we want to prove that if a 3-connected line graph $L(G)$ does not have the indicated N_{s_1,s_2,s_3} as an induced subgraph, then either $L(G)$ has a Hamilton cycle or $G \in \mathcal{F}$. Using a recent theorem of Ma et al. in [13], we approach the problem via two routes: when G can be contracted to the Petersen graph, we show that $G \in \mathcal{F}$; and when G cannot be contracted to the Petersen graph, we show that $L(G)$ will have an induced N_{s_1,s_2,s_3} to obtain a contradiction. Our arguments will apply Catlin's reduction method. In Section 2, we display the basics of Catlin's reduction method and other related tools we have developed to be used in the arguments. The proof of the main result is in the last section.

2. Catlin's reduction and Ryjáček's line graph closure

Following [2], $\kappa(G)$ and $\kappa'(G)$ denote connectivity and edge connectivity of G, respectively. Given vertices $u, v \in V(G)$, a path P in G from u to v is referred to as a (u, v)-path, and is often denoted by $P(u, v)$ to emphasize the end vertices. A *subpath* of a path P is defined to be a path that is a subgraph of P. For convenience of discussion, cycles are often given with an orientation. For a cycle $C = u_1u_2 \cdots u_lu_1$, $C[u_i, u_j]$ denotes the consecutive vertices on C from u_i to u_j in the chosen direction of C, and $C(u_i, u_j] = C[u_i, u_j] - \{u_i\}$, $C[u_i, u_j) = C[u_i, u_j] - \{u_j\}$ and $C(u_i, u_j) = C[u_i, u_j] - \{u_i, u_j\}$.

2.1. Catlin's reduction method

We shall apply Catlin's reduction using collapsible graphs. For a graph G, let $O(G)$ denote the set of odd degree vertices in G. In [4], Catlin discovered collapsible graphs. A graph G is *collapsible* if for any $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph T_R with $O(T_R) = R$. Catlin showed in [4] that for any graph G, every vertex of G lies in a unique maximal collapsible subgraph of G. The *reduction* of G, denoted by G', is obtained from G by contracting all maximal collapsible subgraphs of G. A graph is *reduced* if it is the reduction of some graph. The next theorem summarizes the most frequently applied properties.

Theorem 2.1 (*Catlin, [4]*). *Let G be a connected graph, H be a collapsible subgraph of G, v_H the vertex in G/H with $PI_G(v_H) = H$, and G' the reduction graph of G. Then each of the following holds.*
(i) *(Theorem 3 of [4]) G is collapsible if and only if G/H is collapsible. In particular, G is collapsible if and only if the reduction $G' = K_1$.*
(ii) *(Theorem 5 of [4]) G is reduced if and only if G has no nontrivial collapsible subgraphs.*
(iii) *(Theorem 8 of [4]) G' is simple, $g(G') \geq 4$ and $\delta(G') \leq 3$.*
(iv) *(Theorem 8 of [4]) G is supereulerian if and only if G' is supereulerian.*
(v) *(Theorem 8 of [4]) If L' is an open (or closed, respectively) trail of G/H such that $v_H \in V(L')$, then G has an open (or closed, respectively) trail L with $E(L') \subseteq E(L)$ and $V(H) \subseteq V(L)$.*
(vi) *(Lemma 1 of [5]) $K_{3,3} - e$ is collapsible.*
(vii) *(Theorem 1.3 of [6]) If a connected graph G is at most two edges short of having two edge-disjoint spanning trees, then the reduction of G must be in $\{K_1, K_2\} \cup \{K_{2,t} : t \geq 1\}$.*

Chen [7] showed that every 3-edge-connected graph with at most 11 vertices is either supereulerian or contractible to the Petersen graph. It has also been observed [8] that Petersen graph is the smallest obstacle in searching for spanning eulerian subgraphs. Ma et al. proved something more general.

Theorem 2.2 (*Ma et al., [13]*). *Let G be a 3-edge-connected simple graph. If $c(G) \leq 11$, then G is supereulerian if and only if G is not contractible to the Petersen graph.*

We need a few more lemmas in the arguments of our proofs. Lemma 2.3 can be routinely verified.

Lemma 2.3. *Let $P(10)$ denote the Petersen graph. For any vertices $u_1, u_2 \in V(P(10))$ such that $u_1u_2 \in E(P(10))$, both $P(10)-u_1$ and $P(10) - \{u_1, u_2\}$ are hamiltonian.*

Lemma 2.4. *Let G be a graph with $\kappa'(G) \geq 3$, $H \subset G$ be an induced connected subgraph of G, and let v_H be the vertex in G/H with $PI(v_H) = H$. If v_H has degree at most 3 in G/H, each of the following holds.*
(i) *If $|V(H)| \leq 5$, then H is collapsible unless $H \cong K_{2,3}$ with $A_G(H) = D_2(H)$.*
(ii) *If H is not collapsible, then for any $u \in A_G(H)$, H has a path of length at least 4 with u being an end vertex.*

Proof. We argue by induction on $|V(H)|$. Since $\kappa'(G) \geq 3$ and $d_{G/H}(v_H) \leq 3$, $\kappa'(H) \geq 2$. By Theorem 2.1(iii), any 2-edge-connected graph with at most 3 vertices must be collapsible. Hence the lemma holds when $|V(H)| \leq 3$. Assume that $|V(H)| \geq 4$ and the lemma holds for smaller values of $|V(H)|$. Suppose that H has a nontrivial collapsible subgraph L. Let $G' = G/L$ and $H' = H/L$. By the definition of contraction, $\kappa(G') \geq \kappa'(G) \geq 3$, $H' \subseteq G'$ is an induced subgraph, and $G/H \cong G'/H'$ such that v_H has degree 3 in both $G/H \cong G'/H'$. Since L is nontrivial, $|V(H')| < |V(H)|$. By induction, either H' is collapsible, whence by Theorem 2.1(i), H is collapsible; or $H' \cong K_{2,3}$, whence $5 = |V(H')| < |V(H)| \leq 5$, a contradiction. If H has a cut vertex z, then H has two connected subgraphs H_1 and H_2 with $\min\{|V(H_1)|, |V(H_2)|\} \geq 2$ such that $V(H_1) \cap V(H_2) = \{z\}$ and $H = H_1 \cup H_2$. By induction, either both H_i's are collapsible, whence by Theorem 2.1(i), H is collapsible; or one of the H_i' is isomorphic to $K_{2,3}$, whence $5 = |V(H')| < |V(H)| \leq 5$, a contradiction. Thus, we may assume that

$$H \text{ is reduced and } \kappa(H) \geq 2. \tag{2.1}$$

Let C be a longest cycle of H that contains u. If $|V(H)| = 4$, then $|V(C)| = 4$ and so $V(C) = V(H)$. Since $|A_G(H)| \leq 3$, and since $\delta(G) \geq \kappa'(G) \geq 3$, H must have a cycle of length at most 3, whence H must be collapsible, contrary to (2.1). Hence we assume that $|V(H)| \geq 5$.

If $|V(H)| = |V(C)| = 5$, then by $\delta(G) \geq \kappa'(G) \geq 3$, C has two chords, whence H must be collapsible, contrary to (2.1). Hence $|V(C)| = 4$. As $\kappa'(G) \geq 3$ and $|V(H)| = 5$, $H \cong K_{2,3}$ and $A_G(H) = D_2(H)$, and so (i) holds.

Now assume that H is not collapsible. If $H \cong K_{2,3}$ or if $|V(C)| \geq 5$, then (ii) holds trivially. Hence we assume that $|V(H)| \geq 6$ and $|V(C)| = 4$. Choose a maximum $t \geq 2$ such that $K_{2,t} \subset H$ and $u \in V(K_{2,t})$. If $V(K_{2,t}) = V(H)$, then since $|V(H)| \geq 6$, $t \geq 6 - 2 = 4$. Since $|A_G(H)| \leq 3$, there must be a vertex $u' \in D_2(H) - A_G(H)$. By $\kappa'(G) \geq 3$, $|E_H(u')| \geq 3$, implying that H contains a cycle of length at most 3, contrary to (2.1) and Theorem 2.1(iii).

Thus there must be a vertex $u'' \in V(H) - V(K_{2,t})$. Since $\kappa(H) \geq 2$, H has two paths P_1', P_2' from u'' to two distinct vertices of $K_{2,t}$. It follows that H has a cycle of length at least 5, contrary to the assumption that C is a longest cycle of H containing u. This proves Lemma 2.4. $\square$

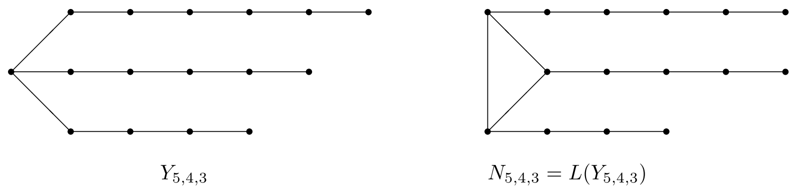

Fig. 1. Examples of Y_{s_1,s_2,s_3} and N_{s_1,s_2,s_3}.

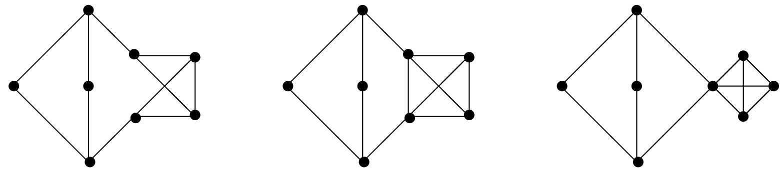

Fig. 2. Non-collapsible graphs in Lemma 2.7.

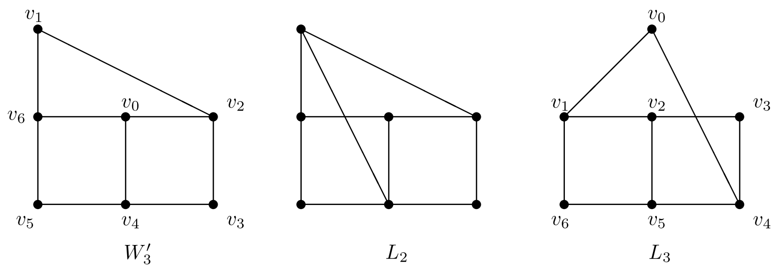

Fig. 3. Some collapsible graphs: W_3', L_2 and L_3.

Lemma 2.5. *Let G be a 2-connected graph, then every 3 vertices are on a path.*

Proof. Let $u_1, u_2, u_3 \in V(G)$. Since $\kappa(G) \geq 2$, u_1 and u_2 are in a cycle C. Since G is connected, G has a path from u_3 to $V(C)$, and so G has a path containing u_1, u_2 and u_3. $\square$

For integers $s_1 \geq s_2 \geq s_3 \geq 1$, let Y_{s_1,s_2,s_3} be the graph obtained from disjoint paths P_{s_1+2}, P_{s_2+2} and P_{s_3+2} by identifying an end vertex of each of these three paths. (An example is depicted in Fig. 1.) For integers $h, k > 0$, let $L(h, k)$ be the graph obtained by identifying an end vertex of a path P_{k+1} and a vertex in a cycle C_h which was disjoint from P_{k+1}. The following lemma follows from straightforward observations and definitions.

Lemma 2.6. *Let $s_1, s_2, s_3 \geq 1$ and h, k be integers, and let G be a graph.*

(i) *If $k \geq s_3 + 1$ and $h \geq s_1 + s_2 + 3$, then $Y_{s_1,s_2,s_3} \subseteq L(h, k)$.*

(ii) *If $Y_{s_1,s_2,s_3} \subseteq G$, then for any $s_1' \leq s_1$, $s_2' \leq s_2$, $s_3' \leq s_3$, $Y_{s_1',s_2',s_3'} \subseteq G$.*

(iii) *If $Y_{8,1,1}, Y_{7,2,1}, Y_{6,3,1}, Y_{6,2,2}, Y_{5,4,1}, Y_{5,3,2}, Y_{4,4,2}, Y_{4,3,3} \subseteq G$, then for any $s_1, s_2, s_3 \geq 1$ with $s_1 + s_2 + s_3 \leq 10$, $Y_{s_1,s_2,s_3} \subseteq G$.*

(iv) *If $Y_{5,2,2}, Y_{4,3,2}, Y_{3,3,3} \subseteq G$, then for any $s_1, s_2, s_3 \geq 2$ with $s_1 + s_2 + s_3 \leq 9$, $Y_{s_1,s_2,s_3} \subseteq G$.*

(v) *If $Y_{8,1,0}, Y_{7,2,0}, Y_{6,3,0}, Y_{5,4,0} \subseteq G$, then for any $s_1, s_2 \geq 1$ with $s_1 + s_2 \leq 9$, $Y_{s_1,s_2,0} \subseteq G$. In particular, if G has an $L(h, 1)$ as a subgraph with $h \geq 12$, then for any $s_1, s_2 \geq 1$ with $s_1 + s_2 \leq 9$, $Y_{s_1,s_2,0} \subseteq G$.*

Lemma 2.7 (*Li, Lai, and Zhan, Lemma 2.1 of [12]*). *Let G be a connected simple graph with $n \leq 8$ vertices and with $D_1(G) = \emptyset$, $|D_2(G)| \leq 2$. Then either G is one of three graphs depicted in Fig. 2, or the reduction of G is K_1 or K_2.*

Let $C_6 = v_1v_2v_3v_4v_5v_6v_1$ denote a 6-cycle, and u_0, v_0 be vertices not in $V(C_6)$. Define $W_3' \cong C_6 + \{v_0v_1, v_0v_3, v_0v_5\}$, $L_1 \cong C_6 + \{u_0v_1, u_0v_3, u_0v_5, v_0v_1, v_0v_3, v_0v_5\}$, $L_2 \cong W_3' + v_1v_4$, and $L_3 \cong C_6 + \{v_0v_1, v_0v_4, v_2v_5\}$ (see Fig. 3).

An edge cut X of a graph G is an *essential edge cut* if both sides of $G - X$ are nontrivial. A graph G is essentially k-edge-connected if G does not have an essential edge cut of size less than k.

Lemma 2.8. *Each of the following holds.*

(i) *L_1, L_2, and L_3 are collapsible.*

(ii) *Let G be an essentially 3-edge-connected graph with $\kappa'(G) \geq 2$, $|D_2(G)| \leq 3$ and $c(G) \leq 6$. Then either G is collapsible or the reduction of G is in $\{K_{2,3}, W_3'\}$.*

Proof. Part (i) can be proved using the same method in the proof of Lemma 1 in [5]. It suffices to prove (ii). By contradiction, assume that

G is a counterexample to Lemma 2.8(ii) with $|V(G)|$ the smallest. (2.2)

Since contraction does not decrease edge-connectivity and essential edge-connectivity, and does not increase circumference, by Theorem 2.1(iii) and (2.2), G must be reduced with $g(G) \geq 4$. If G has a cut vertex, then each block of G satisfies the hypothesis of the lemma, and so by (2.2), and by the fact that G is reduced, every block of G is in $\{K_{2,3}, W_3'\}$. If G has at least two end blocks, then $|D_2(G)| \geq 4$, contrary to the assumption that $|D_2(G)| \leq 3$. Hence $\kappa(G) \geq 2$. Let $c = c(G) \leq 6$ and $C = v_1v_2v_3v_4 \dots v_cv_1$ be a longest cycle of G.

If C has a chord, then since G is reduced, we may assume that $v_2v_5 \in E(G)$. Since $|D_2(G)| \leq 3$, we may assume $V(G)-V(C)$ has a vertex z with $zv_1 \in E(G)$. By $\kappa(G) \geq 2$, G has a path Q' with $zv_1 \in E(Q')$ and $|V(Q') \cap V(C)| = 2$. Since $c(G) \leq 6$ and G has no 3-cycle, $Q' = v_1zv_5$ or $Q' = v_1zv_3$. By $|D_2(G)| \leq 3$ and $\kappa(G) \geq 2$ again, we may assume that z (or by symmetry, v_6) is adjacent to a vertex in $V(G) - (V(C) \cup \{z\})$, and so a (z, v_i)-path Q'' for some $v_i \in V(C)$ such that $V(Q'') \cap V(C) = \{v_i\}$. Hence G has a cycle of length at least 7, contrary to $c(G) \leq 6$. Therefore, we conclude that C does not have a chord.

If $V(G) = V(C)$, then as G is essentially 3-edge-connected with $\kappa'(G) \geq 2$, $|D_2(G)| \leq 2$, and by Lemma 2.7, G must be collapsible, contrary to (2.2). Hence $V(G) - V(C) \neq \emptyset$.

Since $\kappa(G) \geq 2$, $V(G) - V(C)$ has a vertex u such that, without loss of generality, $uv_1 \in E(G)$. As $\kappa(G) \geq 2$, uv_1, v_1v_2 must be contained in a cycle of G, and so G has a (v_1, v_i)-path Q_1, such that $V(Q_1) \cap V(C) = \{v_1, v_i\}$ with $i \neq 1$. Since C is longest in G, $i \notin \{2, c\}$ and $i \in \{3, c-1\}$ only if $Q_1 = v_1uv_i$. A path Q of G satisfying $|V(Q) \cap V(C)| = 2$ and $|E(Q)| \geq 2$ will be referred to as a *long chord* of C. As $c(G) \leq 6$ and as G is reduced, every long chord of C has length at most 3.

Case 1 $c = 6$.

Since $V(G) - V(C) \neq \emptyset$, C must have a long chord with length 2 or 3.

Case 1.1 C has a long chord of length 3. We may assume that $Q_1 = v_1uu'v_4$ is a long chord of C.

Subcase 1.1.1 $v_2, v_3 \notin D_2(G)$ or $v_5, v_6 \notin D_2(G)$.

We may assume that $v_2, v_3 \notin D_2(G)$. Then for some $w_2, w_3 \in V(C) \cup \{u, u'\}$, G has a (v_2, w_2)-path Q_2 and a (v_3, w_3)-path Q_3 such that $V(Q_2) \cap (V(C) \cup \{u, u'\}) = \{v_2, w_2\}$ and $V(Q_3) \cap (V(C) \cup \{u, u'\}) = \{v_3, w_3\}$, and such that $v_2 \neq w_2$ and $v_3 \neq w_3$. If $\{w_2, w_3\} \cap \{u, u', v_5, v_6\} \neq \emptyset$, or if $w_2 \in \{v_1, v_3\}$, or $w_3 \in \{v_2, v_4\}$, then G has a cycle of length at least 7, contrary to the assumption that $c(G) = 6$. Hence we must have $w_2 = v_4$ and $w_3 = v_1$. As G is reduced and by $c(G) = 6$, $|E(Q_2)| = |E(Q_3)| = 2$. But then, $G[E(Q_1) \cup E(Q_2) \cup E(Q_3)]$ is a cycle of length 8, contrary to $c(G) = 6$.

Subcase 1.1.2 $\{v_2, v_3\} \cap D_2(G) \neq \emptyset$ and $\{v_5, v_6\} \cap D_2(G) \neq \emptyset$.

Since $|D_2(G)| \leq 3$, we assume that $u \notin D_2(G)$. Then by $\kappa(G) \geq 2$, G has a (u, w)-path Q_u with $V(Q_u) \cap (V(C) \cup \{u, u'\}) = \{u, w\}$. Since $c(G) = 6$ and since G is reduced, we must have $w = v_4$ and $|E(Q_u)| = 2$. By symmetry, if $u' \notin D_2(G)$, then G has a (u', v_1)-path $Q_{u'}$ with $V(Q_{u'}) \cap (V(C) \cup \{u'\}) = \{u, v_1\}$ and with $|E(Q_{u'})| = 2$. It follows that $G[E(Q_u) \cup E(Q_{u'}) \cup \{uu', v_1v_2, v_2v_3, v_3v_4\}]$ is a cycle of length at least 7, contrary to the assumption of $c(G) = 6$. Therefore, we must have $u' \in D_2(G)$, and for some $w' \in V(G) - V(C)$. $Q_u = uw'v_4$. By the assumption of Subcase 1.1.2, $w' \notin D_2(G)$, and so by $\kappa(G) \geq 2$, G has a (w', w'')-path $Q_{w'}$ such that $V(Q_{w'}) \cap (V(C) \cup \{u\}) = \{w', w''\}$ with $w' \neq w''$. But then, G always has a cycle of length at least 7, contrary to the assumption of $c(G) = 6$.

Case 1.2 C does not have a long chord of length 3.

We may assume that $Q_1 = v_1uv_3$. If $u \notin D_2(G)$, then by the assumption of Case 1.2, and by the fact that G is reduced, we must have $uv_5 \in E(G)$, and so $W_3' \subseteq G$. If W_3' spans G, then $G = W_3'$ as adding any edge to W_3' will create a cycle of length at most 3, or a collapsible L_2 (Lemma 2.8(i)). Hence we conclude that G does not contain a W_3' and $u \in D_2(G)$.

If $v_2 \notin D_2(G)$, then by $\kappa(G) \geq 2$, G has a (v_2, z_2)-path Z_2 with $V(Z_2) \cap (V(C) \cup \{u\}) = \{v_2, z_2\}$ and with $v_2 \neq z_2$. If $z_2 = v_5$, then either G has a cycle of length at least 7, or $W_3' \subseteq G$, contrary to the assumption that $c = 6$ and $W_3' \not\subseteq G$. If $z_2 \in \{v_4, v_6\}$, then G has a cycle of length at least 7, contrary to the assumption of $c(G) = 6$. Hence $v_2 \in D_2(G)$.

Since $u, v_2 \in D_2(G)$ and since $|D_2(G)| \leq 3$, $\{v_4, v_5, v_6\} - D_2(G) \neq \emptyset$. If $v_5 \notin D_2(G)$, then G has a (v_5, z_5)-path Z_5 with $V(Z_5) \cap V(C) = \{v_5, z_5\}$ and with $v_5 \neq z_5$. Since $v_2, u \in D_2(G)$, and since $c = 6$, $z_5 \in \{v_1, v_3\}$. By symmetry and by the fact that G is reduced, assume that $z_5 = v_1$, and $Z_5 = v_1u'v_5$. Since $|D_2(G)| \leq 3$, we may assume that $v_6 \notin D_2(G)$ and so G has a (v_6, z_6)-path Z_6 with $V(Z_6) \cap (V(C) \cup \{u'\}) = \{v_6, z_6\}$ and with $v_6 \neq z_6$. But then, $z_6 \in \{v_1, v_3, v_4, v_5, u'\}$, and in any case, G has a cycle of length at least 7, contrary to $c = 6$.

Hence we assume that $z_5 \in D_2(G)$ and so $v_4, v_6 \notin D_2(G)$. As $\kappa(G) \geq 2$, for $i \in \{4, 6\}$, G has a (v_i, z_i)-path Z_i with $V(Z_i) \cap V(C) = \{v_i, z_i\}$ and with $v_i \neq z_i$. If $z_4 = v_6$ (or $v_4 = z_6$), then by $c = 6$ and by the fact that G is reduced, we assume that $Z_4 = v_4u''z_6$. But then, by symmetry, since $v_5 \in D_2(G)$, we must have $u'' \in D_2(G)$, and so $v_2, u.v{-}5, u'' \in D_2(G)$, contrary to $|D_2(G)| \leq 3$. Hence $z_6 = v_3$. By symmetry, $z_4 = v_1$. Thus $v_1v_4, v_3v_6 \in E(G)$, and so $G[V(C_6) \cup \{u\}]$ is a 2-connected graph with 7 vertices and 3 vertices of degree 2. By Lemma 2.7, this is a collapsible graph, contrary to the assumption that G is reduced. This proves Case 1.

Case 2 $c = 5$.

Since $V(G) - V(C) \neq \emptyset$ and since $c(G) = 5$, C must have a long chord with length 2. By symmetry, suppose that $Q_1 = v_1uv_3$ is a long chord of C. Since $|D_2(G)| \leq 3$, $\{u, v_2, v_4, v_5\} - D_2(G) \neq \emptyset$. Suppose first that $u \notin D_2(G)$ (or by symmetry, $v_2 \notin D_2(G)$), then by $\kappa(G) \geq 2$, G has a (u, w_2)-path Q_2 with $u \neq w_2$ and $V(Q_2) \cap (V(C) \cup \{u\}) = \{u, w_2\}$. But then, in any case, G has a cycle of length at least 6, contrary to the assumption of $c(G) = 5$.

Hence we may assume that $u, v_2 \in D_2(G)$ and so, by symmetry, $v_4 \notin D_2(G)$. By $\kappa(G) \geq 2$, G has a (v_4, w_4)-path Q_4 with $v_4 \neq w_4$ and $V(Q_4) \cap V(C) = \{v_4, w_4\}$. By the assumptions that $c = 5$ and G is reduced, we must have $Q_4 = v_4 u' v_1$. By $|D_2(G)| \leq 3$, we may assume that $v_5 \notin D_2(G)$ and so G has a (v_5, w_5)-path Q_5 with $v_5 \neq w_5$ and $V(Q_5) \cap (V(C) \cup \{u'\}) = \{v_5, w_5\}$. As G is reduced and as $c = 5$, $w_5 \in \{u', v_1, v_3, v_4\}$. In any case, G has a cycle of length at least 6, contrary to $c = 5$.

Case 3 $c = 4$.

Again by $V(G) - V(C) \neq \emptyset$ and $c(G) = 4$, C must have a long chord with length 2. By symmetry, suppose that $Q_1 = v_1 u v_3$ is a long chord of C. Then G contains subgraph isomorphic to $K_{2,3}$. Let $H \cong K_{2,t}$ be a subgraph of G with $t \geq 3$ maximized.

If $V(G) = V(H)$, then since adding any edge to join two vertices of $K_{2,t}$ will result in a collapsible graph (by Theorem 2.1(vii)), it follows by $D_2(G) \leq 3$ that $G = K_{2,3}$. If $V(G) - V(H)$ has a vertex u, then by $\kappa(G) \geq 2$, G has a (w, w')-path Q with $w \neq w'$ such that $V(Q) \cap V(H) = \{w, w'\}$. If $\{w, w'\} - D_t(H) \neq \emptyset$, then G has a cycle of length at least 5, contrary to the assumption that $c(G) = 4$. If $w, w' \in D_t(H)$, then H is contained in a subgraph isomorphic to $K_{2,t+1}$ of G, contrary to the choice of H. This proves Case 3, as well as the lemma. □

Lemma 2.9. *Let G be a graph with $\kappa'(G) \geq 3$ such that G is contracted to $P(10)$. If for some vertex $u \in V(P(10))$, $PI(u)$ is not collapsible, then both of the following hold.*

(i) *for any integer $s_1, s_2, s_3 \geq 1$ with $s_1 + s_2 + s_3 \leq 10$, $Y_{s_1,s_2,s_3} \subseteq G$.*

(ii) *for any integer $s_1, s_2 \geq 1$ with $s_1 + s_2 \leq 9$, $Y_{s_1,s_2,0} \subseteq G$.*

Proof. It suffices to prove (i). We argue by contradiction and assume that G is a counterexample with $|V(G)|$ minimized. Since $P(10)$ is reduced, if L is a collapsible subgraph of G, then G/L is also contractible to $P(10)$. Thus by the minimality of G, we assume that

$$G \text{ is reduced.} \tag{2.3}$$

Let $H = PI(u)$. Suppose that $c(H) \geq 7$. Let C' be a longest cycle of H. By Lemma 2.3, $P(10) - u$ has a cycle of length 9, which can be left to a cycle C of length $c \geq 9$ in G. Let $e \in E_{P(10)}(u)$. Lift e to a path in G joining a vertex in C' and a vertex in C. Then $C' \cup C \cup P$ contains an $L(c', 9)$ and an $L(c, 7)$ with $c' \geq 7$ and $c \geq 9$. By Lemma 2.6(i), $Y_{8,1,1}, Y_{7,2,1}, Y_{6,3,1}, Y_{6,2,2}, Y_{5,4,1}, Y_{5,3,2}, Y_{4,4,2}, Y_{4,3,3} \subseteq G$, and so the lemma follows from Lemma 2.6(iii).

Therefore, we assume that $c(H) \leq 6$. By Lemma 2.8(ii), $H \in \{K_{2,3}, W_3'\}$ with $A_G(H) = D_2(H)$. In this case, for any integer $s_1, s_2, s_3 \geq 1$ with $s_1 + s_2 + s_3 \leq 10$, $Y_{s_1,s_2,s_3} \subseteq G$. (Detailed verifications can be found in Tables 1 and 2 of the Appendix.) □

The core of a graph is formally introduced by Shao [16]. Let G be a graph such that $\kappa(L(G)) \geq 3$ and such that $L(G)$ is not complete. For each $v \in D_2(G)$, let $E_G(v) = \{e_1^v, e_2^v\}$ and define

$$X_1(G) = \cup_{v \in D_1(G)} E_G(v), \quad \text{and} \quad X_2(G) = \{e_2^v : v \in D_2(G)\}. \tag{2.4}$$

Since $\kappa(L(G)) \geq 3$, $D_2(G)$ is an independent set of G and for any $v \in D_2(G)$, $|X_2(G) \cap E_G(v)| = 1$. Define the *core* of the graph G as

$$G_0 = G/(X_1(G) \cup X_2(G)) = (G - D_1(G))/X_2(G). \tag{2.5}$$

Edges in $\cup_{v \in D_2(G)} E_G(v) - X_2(G)$ are referred to as *nontrivial edges* in G_0. Vertices of G adjacent to a vertex in $D_1(G)$ are viewed as the contraction image of edges in $\cup_{v \in D_1(G)} E_G(v)$. An eulerian graph $H \subseteq G$ is *dominating* in G if $E(G - V(H)) = \emptyset$. Harary and Nash-Williams found a close relationship between dominating eulerian subgraphs and hamiltonian line graphs.

Theorem 2.10 (*Harary and Nash-Williams, [10]*). *Let G be a connected graph with at least 3 edges. The line graph $L(G)$ is hamiltonian if and only if G has a dominating eulerian graph.*

Utilizing Theorem 2.10 and Catlin's collapsible graphs [4], Shao proves the following useful theorem. A justification for Theorem 2.11(iii) can be found in [13].

Theorem 2.11 (*Shao, Section 1.4 of [16]*). *Let G_0 be the core of graph G, then each of the following holds.*

(i) *G_0 is nontrivial and $\delta(G_0) \geq \kappa'(G_0) \geq 3$.*

(ii) *G_0 is well defined.*

(iii) *$L(G)$ is hamiltonian if and only if G_0 has a dominating eulerian subgraph containing all nontrivial vertices and both end vertices of each nontrivial edge.*

By (2.4), the edge set

$$E_1'(G) = \bigcup_{v \in D_2(G)} E_G(v) - X_2(G) \tag{2.6}$$

is the set of nontrivial edges in G_0. Let G_0' be the reduction of G_0. Then G_0' is a contraction of both G_0 and G, and so we can view $E(G_0') \subseteq E(G_0) \subseteq E(G)$. Define

$$\Lambda(G_0) = \{v \in V(G_0) : PI_{G_0}(v) \text{ is nontrivial or } v \text{ is an end of a nontrivial edge of } G_0\}, \text{ and} \tag{2.7}$$

$$\Lambda'(G_0) = \{v \in V(G'_0) : PI_G(v) \text{ is nontrivial or contains an end of a nontrivial edge of } G_0\}. \tag{2.8}$$

Lemma 2.12. *Let G be a connected simple graph satisfying $\kappa(L(G)) \geq 3$, G_0 be the core of G and G'_0 be the reduction of G_0. Suppose that $G'_0 = P(10)$, and $L(G)$ is not hamiltonian. Then each of the following holds.*

(i) *$V(G'_0) = \Lambda'(G_0)$.*

(ii) *If G'_0 contains at least one nontrivial edge, then for any integers $s_1 \geq s_2 > 0$ with $s_1 + s_2 + 1 \leq 10$, $Y_{s_1,s_2,1} \subseteq G$.*

(iii) *If G'_0 contains at least one nontrivial edge, then for any integers $s_1 \geq s_2 > 0$, $s_3 \geq 0$ with $s_1 + s_2 + s_3 \leq 9$, $Y_{s_1,s_2,s_3} \subseteq G$.*

(iv) *If G'_0 has a nontrivial vertex v such that $PI_G(v)$ is not a star, then the conclusions of Lemma 2.12(ii) and (iii) must hold.*

(v) *Either $G \in \mathcal{F}$ or the conclusions of Lemma 2.12(ii) and (iii) must hold.*

Proof. (i) If for some $v' \in V(G'_0) - \Lambda'(G_0)$, then as $G'_0 = P(10)$, G'_0 has a cycle C' containing $\Lambda'(G_0)$. Hence C' can be lifted to an eulerian subgraph H' of G_0, containing all vertices in $\Lambda(G_0)$. By Theorem 2.11, $L(G)$ is hamiltonian, contrary to the assumption that $L(G)$ is not hamiltonian.

(ii) Let e' be a nontrivial edge of G'_0, and if G'_0 has at least two nontrivial edges, then let e'' denote another. Since $G'_0 = P(10)$, G'_0 has vertex w such that $G'_0 - w$ has a spanning cycle C' with $e', e'' \in E(C')$. Since $w \notin V(C')$, C' has a vertex $w' \in V(C')$ such that $ww' \in E(P(10))$. (In fact, as $P(10)$ is a 3-regular graph, there are three choices for such w' in C'.) By Lemma 2.12(i), either ww' is a nontrivial edge or both w and w' are nontrivial vertices. It follows that the edge ww' in G'_0 can be lifted to a path Q of length 2 in G. Since C' contains at least one nontrivial edge, C' can be lifted to a cycle C of length at least 10 in G. It follows that $C \cup Q$ is an $L(h, 2)$ with $h \geq 10$. If $h \geq 12$, then by Lemma 2.6, for any integers $s_1 \geq s_2 > 0$ with $s_1+s_2+1 \leq 10$, $Y_{s_1,s_2,1} \subseteq G$.

Hence we may assume that $h \in \{10, 11\}$. By Lemma 2.6, for any integers $s_1 \geq s_2 > 0$ with $s_1 + s_2 + 1 \leq 10$, $Y_{l_1,l_2,1} \subseteq G$, where $(l_1, l_2) \in \{(s_1, s_2 - 1), (s_1 - 1, s_2)\}$ if $h = 11$, or $(l_1, l_2) = (s_1 - 1, s_2 - 1)$ if $h = 10$. Let z_0 be the only vertex of degree 3 in $Y_{l_1,l_2,1}$, and let Q'_1, Q_2, Q_3 be the three internally disjoint paths from z_0 in $Y_{l_1,l_2,1}$, of length $l_1 + 1$, $l_2 + 1$, 2, respectively. Let z'_i denote the other end vertex of Q'_i, $1 \leq i \leq 2$. By Lemma 2.12(i), z'_i is either a nontrivial vertex, or an end of a nontrivial edge not in C'. Since w' has more than one choices, we can choose w' so that z'_i's are independent in G'_0. If follows that Q'_i can be lifted to a path of length $s_1 + 1$, and so $Y_{s_1,s_2,1} \subseteq G$.

(iii) The proof is similar to that for (ii). We outline the idea here. Since $s_1 + s_2 + s_3 \leq 9$ and $s_1 \geq s_2 \geq s_3$, $s_3 \leq 3$. If $s_3 = 3$, then by inspection, $G'_0 = P(10)$ contains a $Y_{2,2,2}$ such that $D_1(Y_{2,2,2}) = \{w_1, w_2, w_3\}$ is independent in G'_0. By Lemma 2.12(i), each w_i is either nontrivial or an end of a nontrivial edge. Thus this $Y_{2,2,2}$ of G'_0 can be lifted to a $Y_{3,3,3}$ in G.

If $s_3 = 2$, then $P(10)$ has an $L(8, 2)$ and so by Lemma 2.6, for any integers $s_1 \geq s_2 > 0$ with $s_1+s_2+2 \leq 9$, $Y_{s_1-1,s_2-1,1} \subseteq G'_0$ such that $D_1(Y_{2,2,2})$ is independent in G'_0. By Lemma 2.12(i), This $Y_{s_1-1,s_2-1,1}$ of G'_0 can be lifted to a $Y_{s_1,s_2,2}$ in G.

If $s_3 = 1$, then by Lemma 2.12(ii), for any integers $s_1 \geq s_2 > 0$ with $s_1 + s_2 + 1 \leq 9$, $Y_{s_1,s_2,1}$ in G.

(iv) Suppose G'_0 has a vertex z_0 such that $PI_G(z_0)$ is not a star. Since $PI_G(z_0)$ is connected, for any $z' \in A_G(PI_G(z_0))$, $PI_G(z_0)$ has a path of length at least 2 from z'. Since $G'_0 = P(10)$, for any $v \in V(G'_0)$, G'_0 has an $L(9, 1)$ and an $L(8, 2)$ such that v is the only vertex of degree 3 in $L \in \{L(9, 1), L(8, 2)\}$. Using this property, it follows that for any $s_1 \geq s_2 > 0$ and $s_3 \geq 0$ with either $s_1 + s_2 + 1 \leq 10$ or $s_1 + s_2 + s_3 \leq 9$, G'_0 has a $Y = Y_{s_1-2,s_2-1,0}$ (if $s_1 + s_2 + 1 \leq 10$) or a $Y = Y_{s_1-2,s_2-1,l_3}$, (where $l_3 = \max\{s_3 - 1, 0\}$ if $s_1 + s_2 + s_3 \leq 9$), such that the path Q of length $s_1 - 1$ in Y ends at z_0. By Lemma 2.12(i) and by the choice of z_0, $Y_{s_1,s_2,1}$ (if $s_1 + s_2 + 1 \leq 10$) and Y_{s_1,s_2,s_3} (if $s_1 + s_2 + s_3 \leq 9$) are subgraphs of G, whence the conclusions in Lemma 2.12(ii) and (iii) must hold.

(v) By Lemma 2.12(i)–(iv), we may assume that G'_0 has no nontrivial edges and for every vertex $v \in V(G'_0)$, $PI_G(v)$ is a star. Therefore, $G \in \mathcal{F}$. □

2.2. Closure of claw-free graphs

Ryjáček [15] introduced the line graph *closure* $cl(G)$ of a claw-free graph G, which becomes a useful tool in investigating hamiltonian claw-free graphs. We refer the reader to [15] for the definition of $cl(G)$.

Theorem 2.13 (*Ryjáček, [15]*). *Let G be a claw-free graph. Then*

(i) *$\mathrm{cl}(G)$ is uniquely determined;*
(i) *$\mathrm{cl}(G)$ is the line graph of a triangle-free graph;*
(iii) *G is hamiltonian if and only if $\mathrm{cl}(G)$ is hamiltonian.*

Theorem 2.14 (*Brousek, Ryjáček and Favaron [3]*). *Let G be a claw-free graph, and let $s_1, s_2, s_3 \geq 0$ be integers. If G is N_{s_1,s_2,s_3}-free, then $\mathrm{cl}(G)$ is also N_{s_1,s_2,s_3}-free.*

3. Proof of the main result

We will prove Theorem 1.5 in this section. Let $s_1 \geq s_2 \geq s_3 \geq 0$ be integers and $N = N_{s_1,s_2,s_3}$ such that either $s_3 = 1$ and $s_1 + s_2 + 1 \leq 10$, or $s_3 > 0$ and $s_1 + s_2 + s_3 \leq 9$, or $s_3 = 0$ and $s_1 + s_2 \leq 9$. By Theorems 2.13 and 2.14, it suffices to prove Theorem 1.5 for 3-connected N-free line graphs of simple graphs.

Throughout this section, we assume that G is a connected simple graph such that $L(G)$ is a 3-connected $\{K_{1,3}, N\}$-free graph. Let G_0 be the core of G, and G_0' be the reduction of G_0. Let $\Lambda(G_0)$ and $\Lambda'(G_0)$ be given by (2.7), and (2.8). By Theorem 2.11, $\kappa'(G_0') \geq 3$. By Theorem 2.1, if G_0' has a dominating eulerian subgraph containing all vertices in $\Lambda'(G_0)$, then G_0 has a dominating eulerian subgraph containing all vertices in $\Lambda(G_0)$, and so by Theorem 2.11, $L(G)$ is hamiltonian.

We argue by contradiction to prove Theorem 1.5, and assume that

$$G \text{ is a counterexample to Theorem 1.5 with } |V(G_0)| \text{ minimized.} \tag{3.1}$$

By the discussion above, by (3.1) and by Theorem 2.1, we may assume that

$$\kappa(G_0') \geq 2,\ G_0' \text{ is reduced and does not have an eulerian subgraph containing } \Lambda'(G_0). \tag{3.2}$$

For the given values s_1, s_2, s_3, since $L(G)$ is N_{s_1,s_2,s_3}-free, we conclude that

$$G \text{ does not contain } Y_{s_1,s_2,s_3} \text{ as a subgraph.} \tag{3.3}$$

Lemma 3.1. *If G_0 is contractible to the Petersen graph, then $G \in \mathcal{F}$.*

Proof. If for some $v \in V(P(10))$, $PI_{G_0}(v)$ is not collapsible, then by Lemma 2.9, (3.3) is violated. Hence we may assume that $G_0' = P(10)$, and so Lemma 3.1 follows from Lemma 2.12. □

If $c(G_0') \leq 11$, then by Theorem 2.2, either G_0' is supereulerian, contrary to (3.2), or is contractible to the Petersen graph, whence by Lemma 3.1, $G \in \mathcal{F}$, contrary to (3.1). Therefore, we may assume that $c(G_0') \geq 12$.

Lemma 3.2. *Theorem 1.5(iii) and (iv) must hold.*

Proof. If $c(G_0') \leq 11$, then by Theorem 2.2 and by (3.2), G_0' is contractible to the Petersen graph. By Lemma 3.1, Theorem 1.5(iii) and (iv) must hold. Now assume that G_0' has a cycle C of length $h \geq 12$. By (3.2), C is not spanning, and so G has an $L(h, 1)$ as a subgraph. By Lemma 2.6, for any $s_1 \geq s_2 > 0$ with $s_1 + s_2 \leq 9$, G has a $Y_{s_1,s_2,0}$ as a subgraph, and so Theorem 1.5(iii) and (iv) hold also. □

It remains to prove Theorem 1.5(i) and (ii). By Theorem 2.2 and by Lemma 3.1, we assume that $h = c(G_0') \geq 12$ and $C' = v_1v_2 \ldots v_hv_1$ is a longest cycle of G_0' such that

$$|V(C') \cap \Lambda(G_0')| \text{ is maximized.} \tag{3.4}$$

The cycle C' can be lifted to a cycle C of G with length $|V(C)| \geq h \geq 12$. By (3.2), $\Lambda(G_0) - V(C') \neq \emptyset$. Since G_0' is connected, G_0' has a path P' with $|E(P')| \geq 1$ such that $|V(P') \cap V(C')| = 1$. We call any such path a C'-path of G_0'. Let $l = \max\{|E(P')| : P'$ is a C'-path P' in $G_0'\}$. Note that $h \geq 12$. If $l \geq 4$, then G_0' has an $L(h, 4)$ as a subgraph. By Lemma 2.6, we have a violation to (3.3). Thus $l \leq 3$.

Lemma 3.3. *If $l = 3$, then for any $s_1, s_2, s_3 > 0$ with $s_1 + s_2 + s_3 \leq 10$, $Y_{s_1,s_2,s_3} \subseteq G$.*

Proof. Suppose $l = 3$. Then G has an $L(h, 3)$ as a subgraph. By Lemma 2.6, Y_{s_1,s_2,s_3} is in G for any $s_1 \geq s_2 \geq s_3 \geq 1$ with $s_3 \leq 2$. It remains to show that $Y_{4,3,3} \subseteq G$. Without loss of generality, we may assume that $P' = v_1u_1u_2u_3$. Since $\kappa'(G_0') \geq 3$, u_3 must be adjacent to two different vertices in $V(C')$, and so one such vertex v_i satisfies $i \neq 1$. If $i \in \{2, 3, 4, h-2, h-1, h\}$, then G_0' would have a cycle longer than C', contrary to $c(G_0') = |V(C')|$. If $6 \leq i \leq h-4$, then $C' \cup P' - v_iv_{i+1}$ or $C' \cup P' - v_{i-1}v_i$ can be lifted to an $L(h', k')$ in G with $h' \geq 9$ and $k' \geq 5$, and so $Y_{4,3,3} \subseteq G$. Hence $i \in \{5, h-3\}$. By symmetry, we assume that $u_3v_5 \in E(G_0')$, and either u_3v_1 or $u_3v_{h-3} \in E(G_0')$.

Since $\kappa'(G_0') \geq 3$, u_2 is adjacent to a vertex $u' \notin V(P')$. If $u' \notin V(C')$, then the discussion on the neighbors of u_3 indicates that $N_{G_0'}(u') \subseteq \{u_2, v_1, v_5, v_{h-3}\}$. If $u'v_{h-3} \in E(G_0')$, then the union of the paths $u_2u'v_{h-3}v_{h-2}v_{h-1}v_h$, $u_2u_1v_1v_2v_3$ and $u_2u_3v_5v_6v_7$ is a $Y_{4,3,3}$ of G_0', whence $Y_{4,3,3} \subseteq G$. Hence we must have $u'v_1, u'v_5, u_3v_1, u_3v_5 \in E(G_0')$. But then $G_0'[\{u'v_1, u'v_5, u_3v_1, u_3v_5, u_2u'\} \cup E(P')] \cong K_{3,3} - e$. By Theorem 2.1(vi), it is collapsible, contrary to (3.2).

Therefore, $u' \in V(C')$. Arguing similarly to the discussion on the neighbors of u_3 above, we conclude that $u' = v_j \in V(C')$ with $4 \leq j \leq 6$ or $h-4 \leq j \leq h-2$. Since $u_3v_5 \in E(G_0')$, if $4 \leq j \leq 6$, then G_0' has a cycle longer than C', contrary to the choice of c. Thus $h-4 \leq j \leq h-2$, and so as $c(G_0') = |V(C_0')|$ and as G_0' has no 3 cycles, $u_3v_{h-3} \notin E(G_0')$. Hence $v_1u_3 \in E(G_0')$.

If $j = h-2$, then $C_0' \cup \{u_3v_5, u_2v_{h-2}, u_2u_3\} - v_{h-3}v_{h-2}$ can be lifted to an $L(h'', 4)$ with $h'' \geq 10$. By Lemma 2.6, G_0' has a $Y_{4,3,3}$ whence $Y_{4,3,3} \subseteq G$. If $j = h-3$, then the union of the paths $u_3u_2v_{h-3}v_{h-2}v_{h-1}v_h$, $u_3v_1v_2v_3v_4$ and $u_3v_5v_6v_7v_8$ is a $Y_{4,3,3}$ in G_0'. If $j = h-4$, then $C_0' \cup \{u_3v_1, u_2v_{h-4}, u_2u_3\} - v_{h-3}v_{h-4}$ can be lifted to an $L(h'', 4)$ with $h'' \geq 10$, and so by Lemma 2.6, G_0' has a $Y_{4,3,3}$. Therefore in any case, $Y_{4,3,3} \subseteq G$, and so the lemma follows. □

Lemma 3.4. *If $l = 2$, then for any $s_1, s_2, s_3 > 0$, $Y_{s_1,s_2,1} \subseteq G$ if $s_1 + s_2 + 1 \le 10$ and $Y_{s_1,s_2,s_3} \subseteq G$ if $s_1 + s_2 + s_3 \le 9$.*

Proof. Suppose $l = 2$. Then $L(h, 2) \subseteq G$ with $h \ge 12$. By Lemma 2.6, $Y_{s_1,s_2,1} \subseteq G$ for any $s_1 \ge s_2 \ge 1$ with $s_1 + s_2 + 1 \le 10$. Thus by Lemma 2.6(iv), it suffices to show $Y_{5,2,2}, Y_{4,3,2}, Y_{3,3,3} \subseteq G$.

Without loss of generality, we may assume that $P' = v_1u_1u_2$. Since $\kappa'(G_0') \ge 3$ and since G_0' has no 3-cycles, there exist i and j with $2 < i < j < h$ such that $u_2v_i, u_2v_j \in E(G_0')$. By symmetry, we may assume that $h + 1 - j \ge i - 1$. Since C' is longest in G_0', $4 \le i < j \le h - 2$. If $\{i, j\} \cap \{6, h - 4\} \ne \emptyset$, then, assuming $i = 6$, $C_0' \cup P' \cup \{u_2v_6\} - v_1v_2$ is a $L(h', k')$ with $h' \ge 10$ and $k' \ge 4$, whence by Lemma 2.6, $Y_{5,2,2}, Y_{4,3,2}, Y_{3,3,3} \subseteq G$. The same conclusion can be made if $h \ge 13$, and $v_7, v_{h-5} \in N_{G_0'}(u_2)$. Thus we assume that if $h = 12$, then $v_6, v_8 \notin N_{G_0'}(u_2)$, and if $h \ge 13$, then $v_6, v_7, \ldots, v_{h-4} \notin N_{G_0'}(u_2)$.

Suppose that $h \ge 13$. Then as C_0' is longest and G_0' is reduced, we must have $i \in \{4, 5\}$ and $j \in \{h - 3, h - 2\}$. If $(i, j) = (4, h-2)$, then $C_0' \cup \{u_2v_i, u_2v_j\} - v_3v_4$ is an $L(h', 5)$ with $h' \ge 9$; and if $(i, j) = (5, h-3)$, then $C_0' \cup \{u_2v_i, u_2v_j\} - v_5v_6$ contains an $L(h'', 4)$ with $h'' \ge 10$. Thus by Lemma 2.6, in either case, $Y_{5,2,2}, Y_{4,3,2}, Y_{3,3,3} \subseteq G$. If $(i, j) = (4, h - 3)$ (or by symmetry, $(i, j) = (5, h - 2)$), then $C_0' \cup P' \cup \{u_2v_{h-3}\} - v_1v_h$ is an $L(h''', 3)$ with $h''' \ge 12$, whence by Lemma 2.6, $Y_{5,2,2}, Y_{4,3,2} \subseteq G$; and the union of $u_2u_1v_1v_2v_3$, $u_2v_{h-3}v_{h-2}v_{h-1}v_h$, and $u_2v_4v_5 \cdots v_8v_9$ contains a $Y_{3,3,3}$. Therefore, if $h \ge 13$, then Lemma 3.4 holds.

Hence we assume that $h = 12$. By symmetry, we assume that $i \le 6$ and $12 - j + 1 \ge i - 1$. As G_0' is reduced, $4 \le i \le 6$. It is shown above that if v_6 or v_8 is in $N_{G_0'}(u_2)$, then $Y_{5,2,2}, Y_{4,3,2}, Y_{3,3,3} \subseteq G$. Hence we assume that $v_6, v_8 \notin N_{G_0'}(u_2)$, and so $i \in \{4, 5\}$.

If $i = 5$, then $j \in \{7, 9\}$, and $C_0' \cup P' \cup \{u_2v_5\} - v_1v_2$ is an $L(11, 3)$. By Lemma 2.6, $Y_{5,2,2}, Y_{4,3,2}, \subseteq G$. If $j = 9$, then the union of $u_2u_1v_1v_2v_3$, $u_2v_5v_6v_7v_8$ and $u_2v_9v_{10}v_{11}v_{12}$ is a $Y_{3,3,3}$. If $j = 7$, then the union of $v_7v_6v_5v_4v_3$, $v_7u_2u_1v_1v_2$ and $v_7v_8v_9v_{10}v_{11}$ is a $Y_{3,3,3}$. Hence we assume that $v_5 \notin N_{G_0'}(u_2)$.

Suppose that $i = 4$. Then $j \in \{7, 9, 10\}$. If $j = 7$, then $(C' - v_1v_2) \cup P' \cup \{u_2v_7\}$ is an $L(9, 5)$ and so by Lemma 2.6, $Y_{4,3,2}, Y_{3,3,3} \subseteq G$. The union of $u_2v_4v_5v_6$, $u_2u_1v_1v_2$ and $u_2v_7v_8v_9v_{10}v_{11}v_{12}$ is a $Y_{5,2,2}$, and so the lemma holds if $j = 7$. If $j = 9$, $(C' - v_4v_5) \cup P' \cup \{u_2v_4, u_2v_9\}$ is an $L(9, 4)$. By Lemma 2.6, $Y_{3,3,3}, Y_{4,3,2}, \subseteq G$. The union of $v_9v_8v_7v_6v_5v_4v_3$, $v_9u_2u_1v_1$ and $v_9v_{10}v_{11}v_{12}$ is a $Y_{5,2,2}$, and so the lemma holds if $j = 9$.

Assume that $j = 10$. Then $(C' - v_3v_4) \cup P' \cup \{u_2v_4, u_2v_{10}\}$ is an $L(8, 5)$ and so by Lemma 2.6, $Y_{4,3,2} \subseteq G$. The union of $u_2v_4v_3v_2$, $u_2u_1v_1v_{12}$ and $u_2v_{10}v_9v_8v_7v_6v_5$ is a $Y_{5,2,2}$. It remains to show that $Y_{3,3,3} \subseteq G$.

By $\kappa'(G_0') \ge 3$ and by $\kappa(G_0') \ge 2$, $N_{G_0'}(u_1) - \{v_1, u_2\}$ has a vertex u_1' and G_0' has a (u_1, v)-path Q such that $(V(C') \cup V(P')) \cap V(Q) = \{u_1, v\}$ (with $u_1' = v$ possible). If $u_1' \ne v$, then replacing u_2 by u_1' in the arguments above, we conclude that $v_4, v_{10} \in N_{G_0'}(u_1')$, and so $(C_0' - v_3v_4) \cup \{u_2v_{10}, u_1u_2, u_1u_1', u_1'v_4\}$ is an $L(10, 5)$, and so by Lemma 2.6, $Y_{3,3,3} \subseteq G$. Hence we assume that $u_1' = v \in V(C_0')$. Since c is longest and since G_0' is reduced, we must have $u_1' = v_7$. In this case, $v_{10}v_9v_8v_7u_1$, $v_{10}u_2v_4v_5v_6$ and $v_{10}v_{11}v_{12}v_1v_2$ form a $Y_{3,3,3}$. This completes the proof of the lemma. $\square$

Lemma 3.5. *If $l = 1$, then for any $s_1, s_2, s_3 > 0$, $Y_{s_1,s_2,1} \subseteq G$ if $s_1 + s_2 + 1 \le 10$, and $Y_{s_1,s_2,s_3} \subseteq G$ if $s_1 + s_2 + s_3 \le 9$.*

Proof. By (3.2), $\Lambda(G_0') - V(C') \ne \emptyset$. Since $l = 1$, every vertex $u \in \Lambda(G_0') - V(C')$ is adjacent to a vertex in C'. Choose $u \in \Lambda(G_0') - V(C')$ such that

$$|V(PI_{G_0}(u))| \text{ is maximized.} \tag{3.5}$$

If u is an end of a nontrivial edge of G_0, then we view that $PI(u)$ is the contraction image of an edge incident with a vertex of degree 2 in G. With this convention, $|V(PI_{G_0}(u))| \ge 2$. We assume that $uv_1 \in E(G_0')$.

Claim 1. *Each of the following holds.*

(i) $Y_{8,1,1}, Y_{7,2,1}, Y_{6,3,1}, Y_{5,4,1} \subseteq G$.
(ii) *If $|V(PI_{G_0}(u))| \ge 3$, then $Y_{6,2,2}, Y_{5,3,2}, Y_{4,4,2}, Y_{4,3,3} \subseteq G$.*
(iii) *For any $u \in V(G_0') - V(C')$, $PI_{G_0}(u)$ is not a nontrivial collapsible subgraph of G_0.*

Proof of Claim 1. Since $u \in \Lambda(G_0')$, $|V(PI_{G_0}(u))| \ge 2$, and so $G[V(C) \cup V(PI_G(u))]$ contains a $L(h_0, 2)$ for some $h_0 = |V(C)| \ge h \ge 12$. By Lemma 2.6, Claim 1(i) follows.

To prove (ii), we assume that $|V(PI_{G_0}(u))| \ge 3$. Since $PI_{G_0}(u)$ is collapsible, $G_0[V(PI_{G_0}(u)) \cup \{v_1\}]$ contains a $L(h', 3)$ with $h' \ge h \ge 12$. By Lemma 2.6, $Y_{6,2,2}, Y_{5,3,2}, Y_{4,4,2} \subseteq G$. If $|V(PI_{G_0}(u))| \ge 4$, then a similar argument implies $Y_{4,3,3} \subseteq G$. Hence we assume that $|V(PI_{G_0}(u))| = 3$, and so $PI_{G_0}(u)$ is spanned by a K_3. Since $d_{G_0'}(u) \ge \kappa'(G_0') \ge 3$, and since G_0' is reduced, $|N_{G_0'}(u)| \ge 3$. We proceed with the proof by examining the distribution of the vertices of $N_{G_0'}(u)$ in C_0'.

Case 1. Suppose that $N_{G_0'}(u)$ has distinct vertices x, y with distance d on C', such that $5 \le d \le \frac{h}{2}$. Since $PI_{G_0}(u)$ is spanned by a K_3, $G_0[PI_{G_0}(u) \cup \{x, y\}]$ has an (x, y)-path Q_1 of length 4. The two paths from x in $C' - y$ can be lifted to two paths Q_2 and Q_3 from x in G_0 of length at least 4 and 5, respectively. Hence $G_0[E(Q_1) \cup E(Q_2) \cup E(Q_3)]$ contains a $Y_{4,3,3}$.

Case 2. Suppose that $N_{G_0'}(u)$ has distinct vertices $v_{i_1}, v_{i_2}, v_{i_3}$ with $1 \le i_1 < i_2 < i_3 \le h$ and with $i_2 \equiv i_1 + 2 \pmod h$ such that

$$\text{either } i_3 \equiv i_2 + 2 \pmod h \quad \text{or} \quad i_3 \equiv i_2 + 3 \pmod h.$$

Since $i_2 \equiv i_1 + 2 \pmod h$, if $i_3 \equiv i_2 + 3 \pmod h$, then the distance between v_{i_1} and v_{i_3} on C' is 5. Thus by Case 1, we only consider that case when $i_3 \equiv i_2 + 2 \pmod h$.

Relabeling if needed, we assume that $i_1 = 1$, and so $v_1, v_3 \in N_{G_0'}(u)$, and $v_5 \in N_{G_0'}(u) \cup N_{G_0'}(u)$. If $v_5 \in N_{G_0'}(u)$, then by (3.4), $|PI_G(v_4)| \geq 2$ (if v_4 is an end of a nontrivial edge, then view that $PI_G(v_4)$ is the contraction image of an edge incident with a vertex of degree 2 in G), as otherwise, $G_0'[E(C' - v_4) \cup \{uv_3, uv_5\}]$ is a cycle violating (3.4). Denote the vertex in $PI_G(v_4)$ incident with the edge v_3v_4 in G_0 by v_4'. Then $PI_G(v_4)$ contains an edge $v_4'v_4''$. Since $PI_{G_0}(u)$ is spanned by a K_3, $G_0[PI_{G_0}(u) \cup \{v_1, v_5\}]$ has an (v_1, v_5)-path which can be lifted to a path Q_1 in G of length at least 4. Furthermore, $C' - v_5$ has a path from v_1 to v_4 which can be lifted to a path Q_2 in G of length at least 4 from v_1 to v_4'', using the edge $v_4'v_4''$; and a path from v_1 to v_6 which can be lifted to a path Q_3 in G of length at least $k - 5 \geq 7$. It follows that $G[E(Q_1) \cup E(Q_2) \cup E(Q_3)]$ contains a $Y_{4,3,3}$.

Case 3. Case 1 and Case 2 do not occur.

Without loss of generality, we assume that $v_1, v_i, v_j \in N_{G_0'}(u)$ with $1 < i < j$ and with $i \leq \frac{h}{2}$. Since G_0' has no 3-cycles, and since Case A and Case B do not occur, $i \in \{3, 4, 5\}$ and $j \in \{h-3, h-2, h-1\}$. By Case 1, the distance between any two of the these three vertices v_1, v_i, v_j must be at most 4. It follows by Case 2 that $h = 12$, $i = 5$ and $j = 9$. Since $PI_{G_0}(u)$ is spanned by a K_3, denote $V(PI_{G_0}(u)) = \{u_1, u_2, u_3\}$. By $\kappa(L(G)) \geq 3$, we may assume that $u_1v_5, u_2v_9 \in E(G)$. It follows that the cycle C lifted from $u_1v_5v_6v_7v_8v_9u_2u_1u_3$ and the path lifted from $v_{10}v_{11}v_{12}v_1v_2v_3v_4v_5$ will form an $L(h, k)$ in G with $h \geq 8$ and $k \geq 7$. It follows by Lemma 2.6, $Y_{6,2,2}$, $Y_{5,3,2}$ and $Y_{4,2,2}$ are subgraphs of G. This proves (ii).

If for some $u \in V(G_0') - V(C')$, $PI_{G_0}(u)$ is a nontrivial collapsible subgraph, then as G_0 is simple, $|PI_{G_0}(u)| \geq 3$. By Claim 1(i) and (ii), and by Lemma 2.6, G has Y_{s_1,s_2,s_3} as a subgraph, contrary to (3.3). This proves (iii), and completes the proof for Claim 1. $\square$

By Claim 1(i) and by Lemma 2.6, it remains to prove

$$Y_{5,2,2}, Y_{4,3,2}, Y_{3,3,3} \subseteq G. \tag{3.6}$$

In the rest of the proof, we always assume that u is a vertex that satisfies $u \in \Lambda(G_0) - V(C')$. By Claim 1(iii), either $PI_G(u)$ consists of an edge incident with a vertex in $D_2(G)$, or for some $u' \in V(G)$, every edge in $PI_G(u)$ is in $E_G(u')$ and is incident with a vertex in $D_1(G)$. To simplify notations, throughout the rest of the proof of this lemma, we assume that u, u' are vertices in $PI_G(u)$ such that $u'u \in E(PI_G(u))$ and u is the vertex in G incident with the edge uv_1 in G_0'.

As $|N_{G_0'}(u) \cap V(C')| \geq \kappa'(G_0') \geq 3$, relabeling if needed, we may assume that $v_1, v_i, v_j \in N_{G_0'}(u)$, such that if $n_1 = i - 1$, $n_2 = j - i$, and $n_3 = h - j + 1$, then $n_3 \geq n_2 \geq n_1$. Note that $n_1 + n_2 + n_3 = h$, and so

$$n_3 \geq \frac{h}{3} \geq n_1, \quad \text{and so } G_0' \text{ has an } L(n_2 + n_3 + 2, n_1 - 1). \tag{3.7}$$

Since G_0' is reduced, $n_1 \geq 2$. Suppose $n_1 \geq 4$. By (3.7), $n_2 + n_3 + 2 \geq 10$ and $n_1 - 1 \geq 4$, by (3.7) and by Lemma 2.6, (3.6) holds. Hence $2 \leq n_1 \leq 3$.

Claim 2. *If $n_1 = 3$, then (3.6) holds.*

Proof of Claim 2. Suppose that $n_1 = 3$. If $n_2 \geq 5$, then G_0' has an $L(n_1 + n_3 + 2, n_2 - 1)$. Since $n_2 - 1 \geq 4$ and $n_1 + n_3 + 2 \geq 3 + 5 + 2 = 10$, G_0' has an $L(10, 4)$. By Lemma 2.6, (3.6) holds.

Assume that $n_2 = 4$. Then G_0' has an $L(n_1 + n_3 + 2, 3)$. As $n_1 + n_3 + 2 \geq k - 4 + 2 = 10$, G_0' has an $L(10, 3)$. By Lemma 2.6, $Y_{4,3,2}, Y_{5,2,2} \subseteq G$. Since $n_1 = 3$ and $n_2 = 4$, $uv_8 \in E(G_0')$. The union of $v_1v_2v_3v_4v_5$, $v_1uv_8v_7v_6$ and $v_1v_hv_{h-1}v_{h-2}v_{h-3}$ is a $Y_{3,3,3}$, and so (3.6) holds.

Hence $n_2 = n_1 = 3$ and so $v_4, v_7 \in N_{G_0'}(u)$. It follows that $(C' - v_1v_h) \cup \{uv_1, uv_7\}$ contains an $L(8, 5)$, and so by Lemma 2.6, $Y_{4,3,2} \subseteq G$. It remains to show that $Y_{5,2,2}, Y_{3,3,3} \subseteq G$.

Since $\kappa'(G_0') \geq 3$ and $\kappa(G_0') \geq 2$, for $l \in \{2, 3\}$, G_0' has a path Q_l such that $V(Q_l) \cap V(C') = \{v_l, v_{i_l}\}$ with $i_l \neq l$. As C' is longest and as G_0' is reduced, $i_2, i_3 \notin \{1, 2, 3, 4\}$ unless $Q_2 = v_2w_2v_4$ and $Q_3 = v_1w_3v_3$ for some $w_2 \neq w_3$ and $w_2, w_3 \in V(G_0') - V(C')$. But if $Q_2 = v_2w_2v_4$ and $Q_3 = v_1w_3v_3$, then $(C' \cup Q_2 \cup Q_3) - \{v_1v_2, v_3v_4\}$ is a cycle of length at least $h + 2$, contrary to the assumption that C' is longest. Hence we assume that $i_2, i_3 \notin \{1, 2, 3, 4\}$.

Since C' is longest, $u \notin V(Q_2) \cup V(Q_3)$ and $i_3 \notin \{5, 6, 12\}$. (If $i_3 = 5$, then as G_0' is reduced, $|E(Q_3)| \geq 2$, and $C'[v_7, v_3]Q_3(v_3, v_5]v_4uv_7$ is longer than C'. If $i_3 = 6$, then $C'[v_7, v_3]Q_3(v_3, v_6]v_5v_4uv_7$ is longer than C'. If $i_3 = 12$, then $C'[v_4, v_{12}]Q_3(v_{12}, v_3]v_2v_1uv_4$ is longer than C'.)

If $i_3 = 7$, then $(C' - v_3v_4) \cup Q_3 \cup \{uv_4\}$ is an $L(h - 4 + |E(Q_3)|, 4)$. As $h - 4 + |E(Q_3)| \geq 9$, by Lemma 2.6, $Y_{3,3,3} \subset G$. The union of v_3v_4uu', $Q_3[v_3, v_7]v_6v_5$ and $v_3v_2v_1v_hv_{h-1}v_{h-2}v_{h-3}$ contains a $Y_{5,2,2}$. Hence (3.6) holds.

If $i_3 = 8$, then $(C' - v_3v_4) \cup \{uu', uv_4\}$ is an $L(h - 5, 6)$. As $h - 5 \geq 7$, by Lemma 2.6, $Y_{5,2,2} \subset G$. The union of $Q_3[v_8, v_3]v_2v_1u$, $v_8v_7v_6v_5v_4$ and $v_8v_9v_{10}v_{11}v_{12}$ contains a $Y_{3,3,3}$. Hence (3.6) holds.

If $i_3 = 9$, then $(C' - \{v_1v_h, v_3v_4\}) \cup E(Q_3) \cup \{uv_1, uv_4\}$ is an $L(9 + |E(Q_3)|, h - 9)$. As $9 + |E(Q_3)| \geq 10$ and $h - 9 \geq 3$, by Lemma 2.6, $Y_{5,2,2} \subset G$. The union of $v_9v_{10}v_{11}v_{12}v_1$, $Q_3[v_9, v_3]v_4uu'$ and $v_9v_8v_7v_6v_5$ contains a $Y_{3,3,3}$. Hence (3.6) holds.

If $i_3 = 10$, then the union of $v_{10}v_{11}v_{12}v_1v_2$, $Q_2[v_{10}, v_3]v_4uu'$ and $v_{10}v_9v_8v_7v_6$ contains a $Y_{3,3,3}$; and the union of $Q_3[v_3, v_{10}]v_{11}v_{12}$, $v_3v_2v_1u$ and $v_3v_4v_5v_6v_7v_8v_9$ has a $Y_{5,2,2}$. Hence (3.6) holds.

Table 1
Existence of Y_{s_1,s_2,s_3} when $PI(v) = K_{2,3}$ in the proof of Lemma 2.9.

Cases			Y_{s_1,s_2,s_3}		
s_1	s_2	s_3	P_{s_1+1}	P_{s_2+1}	P_{s_3+1}
8	1	1	$vv_1v_4v_2v_9v_8v_7v_6v_5v_3$	vw_1u_2	vw_2u_1
7	2	1	$vv_1v_4v_2v_5v_3v_8v_7v_6$	$vw_1u_2v_9$	vw_2u_1
6	3	1	$vv_1v_3v_5v_2v_4v_7v_6$	$vw_1u_2v_9v_8$	vw_2u_1
6	2	2	$vv_1v_4v_2v_5v_3v_8v_7$	$vw_2u_1v_6$	$vw_1u_2v_9$
5	4	1	$vv_1v_4v_2v_5v_6v_7$	$vw_1u_2v_9v_8v_3$	vw_2u_1
5	3	2	$vv_1v_3v_5v_2v_4v_7$	$vw_1u_2v_9v_8v_7$	$vw_2u_1v_6$
4	4	2	$vv_1v_3v_5v_2v_4$	$vw_1u_2v_9v_8v_7$	$vw_2u_1v_6$
4	3	3	$vv_1v_3v_5v_2v_4$	$vw_1u_2v_9v_8$	$vw_2u_1v_6v_7$

Table 2
Existence of Y_{s_1,s_2,s_3} when $PI(v) = K'_{1,3}$ in the proof of Lemma 2.9.

Cases			Y_{s_1,s_2,s_3}		
s_1	s_2	s_3	P_{s_1+1}	P_{s_2+1}	P_{s_3+1}
8	1	1	$uw_5w_4v_1v_4v_2v_5v_3v_8v_7$	uw_1w_6	uw_3w_2
7	2	1	$uw_5w_4v_1v_4v_2v_5v_3v_8$	$uw_3w_2v_9$	uw_1w_6
6	3	1	$uw_5w_4v_1v_4v_2v_5v_3$	$uw_3w_2v_9v_8$	uw_1w_6
6	2	2	$uw_5w_4v_1v_4v_2v_5v_3$	$uw_3w_2v_9$	$uw_1w_6v_6$
5	4	1	$uw_5w_4v_1v_4v_2v_5$	$uw_3w_2v_9v_8v_7$	uw_1w_6
5	3	2	$uw_5w_4v_1v_4v_2v_5$	$uw_3w_2v_9v_8$	$uw_1w_6v_6$
4	4	2	$uw_5w_4v_1v_4v_2$	$uw_3w_2v_9v_8v_7$	$uw_1w_6v_6$
4	3	3	$uw_5w_4v_1v_4v_2$	$uw_3w_2v_9v_8$	$uw_1w_6v_6v_7$

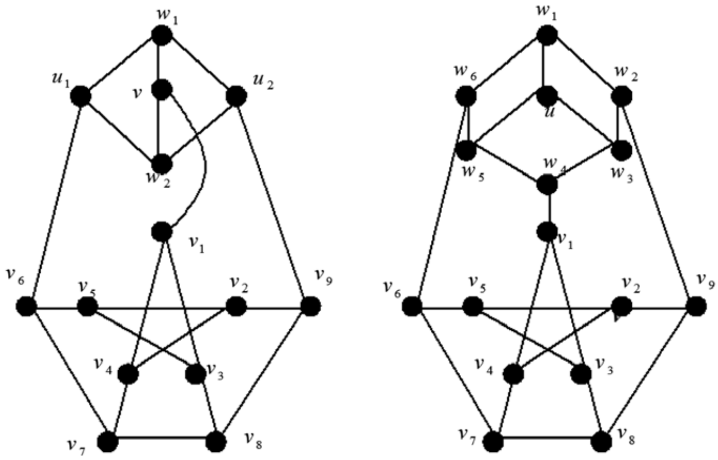

Fig. 4. Graphs in Tables 1 and 2.

If $i_3 = 11$, then $(C' - \{v_1v_2, v_2v_3\}) \cup E(Q_3) \cup \{uv_1, uu'\}$ has an $L(8 + |E(Q_3)|, h - 10 + 2)$. As $8 + |E(Q_3)| \geq 9$ and $h - 10 + 2 \geq 4$, by Lemma 2.6, $Y_{3,3,3} \subset G$. The union of v_3v_4uu', $v_3v_2v_1v_{12}$, and $Q_3[v_{v_3}, v_{11}]v_{10}v_9v_8v_7v_6$ and has a $Y_{5,2,2}$. Hence (3.6) holds. This proves Claim 2.

By Claim 2, $n_1 = 2$, and so $v_3 \in N_{G'_0}(u)$. By (3.4), $v_2 \in \Lambda(G'_0)$. For notational convenience, let v_2, v'_2 be vertices in $PI_G(v_2)$ such that $v_2v'_2 \in E(G)$ and such that v_2 is the vertex in $PI_G(v_2)$ incident with the edge uv_2 in G'_0. If $n_2 \geq 6$, or $n_2 = 5$ and $n_3 \geq 6$, then $h = n_3 + n_2 + n_1 \geq 13$. Thus G'_0 has an $L(h - n_2 + 2, n_2 - 1)$ as a subgraph. Since $h - n_2 + 2 = n_1 + n_3 + 2 \geq 10$ and $n_2 - 1 \geq 4$, by Lemma 2.6, (3.6) holds. Hence either $n_3 = n_2 = 5$, or $n_2 \leq 4$.

Case 1. $n_3 = n_2 = 5$ and $n_1 = 2$.

Therefore, $h = 12$, and $v_1, v_3, v_8 \in N_{G'_0}(u)$. Hence $(C' - v_8v_9) \cup \{uv_1, uv_8\}$ is an $L(9, 4)$, and so by Lemma 2.6, $Y_{4,3,2}, Y_{3,3,3} \subseteq G$. It suffices to show that $Y_{5,2,2} \subseteq G$. By $\kappa'(G'_0) \geq 3$, G'_0 has a path P such that $V(C') \cap V(P) = \{v_2, v_i\}$ for some $i \neq 2$. Since C' is longest, $i \notin \{1, 3\}$. By symmetry, we may only examine the cases when $i \in \{4, 5, 6, 7, 8\}$. Table 3 in the Appendix shows that $Y_{5,2,2} \subseteq G$ in any of these cases.

Case 2. $2 \leq n_2 \leq 3$ and $n_1 = 2$.

If $n_2 = 2$, then $v_5 \in N_{G'_0}(u)$, and so $(C' - v_1v_2) \cup \{v_2v'_2, uv_1, uv_5\}$ is an $L(h - 4 + 2, 4)$. As $h - 2 \geq 10$, by Lemma 2.6, (3.6) holds.

If $n_2 = 3$, then $v_6 \in N_{G'_0}(u)$, and so $(C' - v_1v_2) \cup \{uv_1, uv_6, v_2v'_2\}$ is an $L(h - 5 + 2, 5)$. As $h - 3 \geq 9$, by Lemma 2.6, $Y_{4,3,2}, Y_{3,3,3} \subset G$. The union of $uv_1v_2v'_2$, $uv_3v_4v_5$ and $uv_6v_7v_8v_9v_{10}v_{11}$ is a $Y_{5,2,2}$. Hence (3.6) holds. This proves Case 2.

Table 3
Existence of $Y_{5,2,2}$ in Lemma 3.5 when $n_1 = 2$ and $n_2 = n_3 = 5$.

v_i	$Y_{5,2,2}$
v_4	The union of v_2v_3uu', $P[v_2, v_4]v_5v_6$ and $v_2v_1v_{12}v_{11}v_{10}v_9v_8$
v_5	The union of $v_5v_4v_3u$, $P[v_5, v_2]v_1v_{12}$ and $v_5v_6v_7v_8v_9v_{10}v_{11}$
v_6	The union of $v_6v_5v_4v_3$, $P[v_6, v_2]v_1u$ and $v_6v_7v_8v_9v_{10}v_{11}v_{12}$
v_7	The union of $v_7v_8v_9v_{10}$, $P[v_7, v_2]v_1v_{12}$ and $v_7v_6v_5v_4v_3uu'$
v_8	The union of $v_8v_9v_{10}v_{11}$, $v_8uv_1v_{12}$ and $P[v_8, u]v_8v_7v_6v_5v_4v_3v_2$

Case 3. $n_2 = 4$ and $n_1 = 2$.

Thus $v_3, v_7 \in N_{G'_0}(u)$ and so $(C' - v_3v_4) \cup \{uv_3, uv_7\}$ is an $L(h-4+2, 3)$. As $h-2 \geq 10$, by Lemma 2.6, $Y_{4,3,2}, Y_{5,2,2} \subseteq G$. It remains to show that $Y_{3,3,3} \subseteq G$.

Recall that $PI_G(u)$ has an edge uu'. By $\kappa'(G'_0) \geq 3$ and $\kappa(G'_0) \geq 2$, G'_0 has a (v_2, v_i)-path Q_2 such that $V(C') \cap V(Q_2) = \{v_2, v_i\}$ for some $i \neq 2$. Since G'_0 is reduced and since C' is longest, $i \notin \{h, 1, 2, 3, 4\}$.

If $i = 5$, then the union of paths $v_5v_4v_3uu'$, $Q_2[v_5, v_2]v_1v_hv_{h-1}$ and $v_5v_6v_7v_8v_9$ contains a $Y_{3,3,3}$. If $i = 6$, then the union of paths $v_6v_5v_4v_3u$, $Q_2[v_6, v_2]v_1v_hv_{h-1}$ and $v_6v_7v_8v_9v_{10}$ contains a $Y_{3,3,3}$. Therefore, by symmetry and since uv_7 is not used in the proof for $i \in \{5, 6\}$, we may assume that $i \notin \{5, 6, h-1, h-2\}$.

If $i = 7$, then the union of $v_7v_8v_9v_{10}v_{11}$, $v_7v_6v_5v_4v_3$ and $v_7v_2v_3uu'$ is a $Y_{3,3,3}$.

If $i = 8$, then $(C' \cup Q_2 \cup \{uv_1, uv_3\}) - \{v_1v_2, v_7v_8\}$ is an $L(h-6+|E(Q_2)|+2, 4)$. Since $h \geq 12$, $h-4+|E(Q_2)| \geq 9$, and so by Lemma 2.6, $Y_{3,3,3} \subseteq G$.

If $i = 9$, then the union of $v_9v_{10}v_{11}\ldots v_hv_1$, $v_9v_8v_7v_6v_5$ and $v_9v_2v_3uu'$ contains a $Y_{3,3,3}$.

If $10 \leq i \leq h-3$, then the union of the cycle $v_2v_3v_4\ldots v_iv_2$ and the path $v_iv_{i+1}\ldots v_hv_1uu'$ is an $L(h', k')$ with $h' \geq 10$ and $k' \geq 5$. By Lemma 2.6, $Y_{3,3,3} \subseteq G$. This proves Case 3 and completes the proof of the lemma. □

Continuation of the proof of Theorem 1.5(i) and (ii). Suppose that (3.1) holds. If $c(G'_0) \geq 12$, then by Lemmas 3.3–3.5, either $Y_{s_1,s_2,1} \subset G$ for any $s_1, s_2 > 0$ with $s_1 + s_2 + 1 \leq 10$, or $Y_{s_1,s_2,s_3} \subset G$ for any $s_1, s_2, s_3 > 0$ with $s_1 + s_2 + s_3 \leq 9$, contrary to (3.3). Hence we assume that $c(G'_0) \leq 11$. By Theorem 2.2, either G is supereulerian, whence by Theorem 2.11, $L(G)$ is hamiltonian, contrary to (3.1); or G is contractible to $P(10)$, whence by Lemma 3.1, $L(G)$ is not hamiltonian and $G \in \mathcal{F}$, contrary to (3.1). This completes the proof of Theorem 1.5. □

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Appendix

See Tables 1–3 and Fig. 4.

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