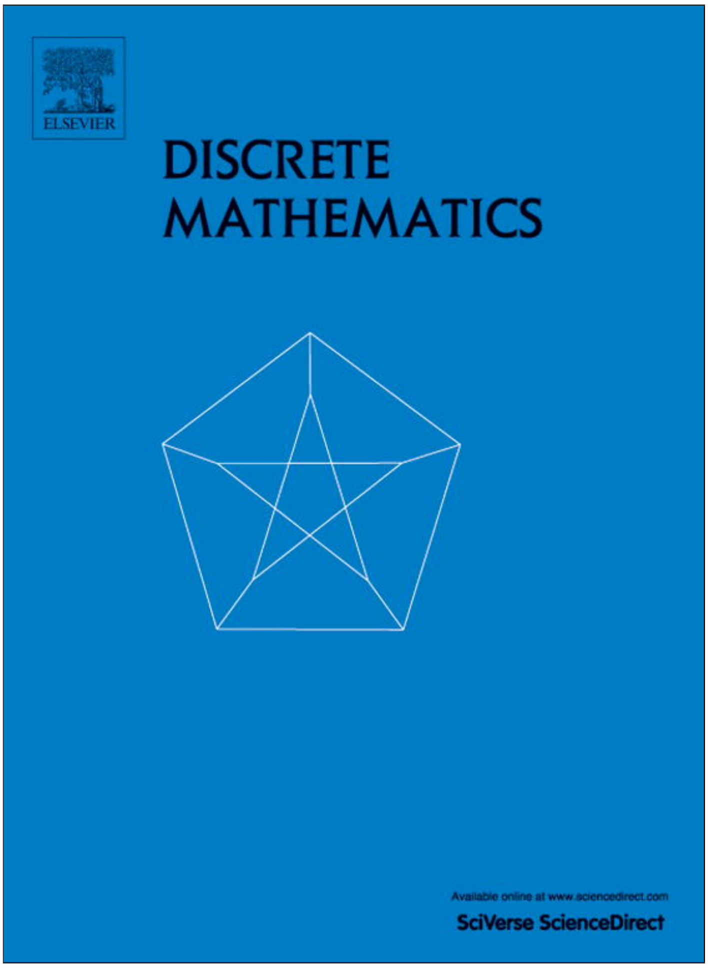
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Non-separating subgraphs☆

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ABSTRACT

Lovász conjectured that there is a smallest integer $f(l)$ such that for every $f(l)$-connected graph G and every two vertices s, t of G there is a path P connecting s and t such that $G - V(P)$ is l-connected. This conjecture is still open for $l \geq 3$. In this paper, we generalize this conjecture to a k-vertex version: is there a smallest integer $f(k, l)$ such that for every $f(k, l)$-connected graph and every subset X with k vertices, there is a tree T connecting X such that $G - V(T)$ is l-connected? We prove that $f(k, 1) = k + 1$ and $f(k, 2) \leq 2k + 1$.

1. Introduction

A well-known conjecture due to Lovász [7] says the following.

Conjecture 1.1 (*Lovász [7]*)**.** *There exists a smallest integer $f(l)$ such that for every $f(l)$-connected graph G and two vertices s and t in G, there exists a path P connecting s and t such that $G - V(P)$ is k-connected.*

This conjecture is still open except for the cases $l \leq 2$. A theorem of Tutte [8] shows that $f(1) = 3$. For $l = 2, f(2) = 5$ is proved by Chen, Could, and Yu [2] and Kriesell [6]independently. Actually, they proved that the path to be deleted is in fact an induced path. Later, Kawarabayashi, Lee and Yu [5] further proved that every 4-connected graph G other than the double wheel has the property that for any given pair of vertices x and y, G has a path joining x and y such that $G - V(P)$ is 2-connected. Although a weaker version of Conjecture 1.1 is proved in [4], the prospect is still not clear.

We follow [1] for general notations and terminology. For a graph G and an integer $i > 0$, let $D_i(G)$ denote the set of vertices of degree i in G. As in [1], $\kappa(G)$ and $\delta(G)$ denote the connectivity and the minimum degree of G, respectively. For a subset $X \subseteq V(G)$, a subtree T of G is said to be *connecting* X if $X \subseteq V(T)$ and if $D_1(T) \subseteq X$. Thus a path connecting two vertices s and t can be viewed as a tree connecting $\{s, t\}$. With this viewpoint, we generalize the path connecting two vertices to a tree connecting X for some specific vertex set X. Lovász's conjecture can be extended to a more general version.

Conjecture 1.2. *There exists a smallest integer $f(k, l)$ such that for any vertex set X with order k of an $f(k, l)$-connected graph G, there is a tree T connecting X such that $G - V(T)$ is l-connected.*

When $|X| = 1$, the tree connecting X is trivial, and so $f(1, l) = l+1$. When $k = 2$, Conjecture 1.2 becomes Conjecture 1.1. Hence $f(2, 1) = 3$ and $f(2, 2) = 5$. The purpose of this paper is to extend these former results as follows.

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Theorem 1.3. *Let $k \geq 1$ be an integer. Then each of the following holds.*

(i) $f(k, 1) = k + 1$.
(ii) $f(k, 2) \leq 2k + 1$.

In the next section, we give some terminologies and some previous results used in this paper.

2. Preliminaries

Let G be a graph and X be a vertex subset of G. As in [1], $G[X]$ denotes the subgraph induced by X and $G-X$ is the subgraph induced by $V(G) - X$. The *neighborhood* $N_G(X)$ of X is the set of vertices in $V(G) - X$ which are adjacent to some vertex in X. If $X = \{x\}$, we also use $G - x$ and $N_G(x)$ for $G - \{x\}$ and $N_G(\{x\})$, respectively. If H is a subgraph of G, we often use $N_G(H)$ for $N_G(V(H))$ and let $|H| = |V(H)|$. If $e \in E(G)$, then $V(e)$ denotes the set of vertices incident with e, and G/e denotes the graph obtained from G by contracting e.

In [3], Yanmei Hong, Liying Kang and Xingxing Yu define an X-tree as follows.

Definition 2.1 (*X-Tree [3]*)**.** Let X be a vertex subset of a graph G. An X-tree is a minimal connected induced subgraph of G containing X.

When $|X| = 1$, $G[X]$ is the unique X-tree. When $|X| = 2$, an X-tree is simply an induced path in G between the two vertices in X. When $|X| \geq 3$, an X-tree need not be a tree. The following lemma shows the relation between an X-tree and a tree connecting X.

Lemma 2.2. *Let X be a vertex subset of a graph G and T be an X-tree of G. Then every spanning tree of T is connecting X.*

Proof. Let T_0 be a spanning tree of T. Then $X \subseteq V(T_0)$. It suffices to show that every leaf of T_0 lies in X. Assume that $v \in V(T) - X$ is a leaf of T_0. Then $T_0 - v$ is connected and thus $T - v$ is also connected and contains X, contradicting the minimality of T. □

Lemma 2.2 shows that an X-tree is somewhat like an "induced" tree connecting X. Hence to find a tree connecting X, it suffices to find an X-tree. In fact, in Section 3, we prove the existence of an X-tree instead of a tree connecting X.

In [3], Yanmei Hong, Liying Kang and Xingxing Yu studied some properties of X-trees and defined a partition, called an H-partition, of an X-tree T according to a subgraph H in $G - V(T)$. Since we emphasize how to partition $V(T)$, we only mention the properties of an H-partition as follows.

Lemma 2.3 (*[3]*)**.** *Let $X \subseteq V(G)$ be a subset with $|X| = k$ and T be an X-tree of G. For any connected subgraph H of $G - V(T)$, there exists a partition (V_1, V_2, V_3) (called an H-partition) of $V(T)$ corresponding to H such that*

(a) $N_G(V_1) \cap (V(H) \cup V_3) = \emptyset$,
(b) $X \subseteq V_1 \cup V_2$,
(c) *for any $u \in V_3$, each component of $T - u$ contains a neighbor of H (so $G[V(T) \cup V(H)] - u$ is connected),*
(d) *$|V_2| \leq k$, where $|V_2| = |X|$ only if every component of $G[V_1 \cup V_2] - E(G[V_2])$ is a path between X and V_2 with all internal vertices (if any) in V_1.*

3. Main result

In this section, we first prove Theorem 3.1, which implies $f(k, 1) \leq k + 1$. We will then present an example to show that $f(k, 1) \geq k + 1$ which will establish Theorem 1.3(i).

Theorem 3.1. *Let G be a $(k+1)$-connected graph. For any vertex subset $X \subseteq V(G)$ with $|X| = k$ and for a vertex $v \in V(G)-X$, G has an X-tree T such that $v \notin V(T)$ and $G - V(T)$ is connected.*

Proof. If $k = 1$, then $G[X]$ is the unique X-tree. Since G is 2-connected, $G - X$ is connected. Hence we assume that $k \geq 2$ in the rest of the proof. Thus $G - v$ is connected.

For each X-tree T in $G - v$, there is a component of $G - V(T)$ containing v, say C_0. Let $C_1, \ldots, C_q$ denote the other components of $G - V(T)$ such that $|C_1| \geq \cdots \geq |C_q|$, and let $\mathcal{S}(T) = (|C_0|, |C_1|, \ldots, |C_q|)$. Choose an X-tree T in $G - v$ such that $\mathcal{S}(T)$ is maximized with respect to the lexicographic ordering.

If $q = 0$, then $G-V(T) = C_0$ is connected and the theorem holds in this case. Assume that $q > 0$. Then by Lemma 2.3, $V(T)$ has a C_q-partition (V_1, V_2, V_3). By Lemma 2.3(a), $N_G(C_q) \subseteq V_2 \cup V_3$. By Lemma 2.3(d), $|V_2| \leq k$. Since $|N_G(C_q)| \geq \kappa(G) \geq k+1$, and since $|V_2| \leq k$, we conclude that $N_G(C_q) \cap V_3 \neq \emptyset$.

For any vertex $u \in V_3$, by Lemma 2.3(b) and (c), $X \subseteq V(T) \cup V(C_q) - u$ and $G[V(T) \cup V(C_q)] - u$ is connected. It follows that G has another X-tree T' as a subgraph of $G[V(T) \cup V(C_q)] - u$. If u has a neighbor in C_i for some $0 \leq i \leq q-1$, one component of $G - V(T')$ contains $V(C_i) \cup \{u\}$, and so $\mathcal{S}(T')$ would be bigger than $\mathcal{S}(T)$ in the lexicographic order, contradicting the choice of T. Therefore, $N_G(u) \cap V(C_i) = \emptyset$ for $0 \leq i \leq q - 1$. Furthermore, as u in the argument above can be any vertex in V_3, it follows that $N_G(V_3) \cap V(C_i) = \emptyset$ for $0 \leq i \leq q - 1$. Thus by Lemma 2.3(d), $|N_G(V_3 \cup V(C_q))| \leq |V_2| \leq k$ contrary to the assumption that $\kappa(G) \geq k + 1$. □

Let G be a graph obtained from a K_k, whose vertex set is denoted by X, by adding $m \geq k+1$ isolated vertices, denoted by $v_1, \dots, v_m$, and all possible edges from these m vertices to X. It is routine to verify that $\kappa(G) = k$. If there is a tree T connecting X such that $G - V(T)$ is connected, then $G - V(T)$ is just an isolated vertex, say v_m. So $V(T) = V(G) - \{v_m\} = X \cup \{v_1, \dots, v_{m-1}\}$. Since T is a tree connecting X, for each $i = 1, \dots, m-1$, v_i has degree at least 2 in T. Hence, $|E(T)| \geq 2(m-1)$. On the other hand, $|E(T)| = |V(T)| - 1 = k + m - 2$. It follows that $2(m-1) \leq k + m - 2$, contradicting $m \geq k+1$. So $g(k, 1) > k$. Together with Theorem 3.1, $f(k, 1) = k + 1$.

The main idea of the proof of Theorem 3.2 is similar to that of Theorem 3.1, with much more complicated and different details. As in [1], a *block* of a graph G is a maximal subgraph without a cut vertex. Thus every block with more than 2 vertices is 2-connected.

Theorem 3.2. *For any set X with k vertices in a $(2k+1)$-connected graph G, there is an X-tree T such that $G - V(T)$ is 2-connected.*

Proof. When $k = 1$, $G[X]$ is the unique X-tree, and so $\kappa(G - X) \geq 2$. Arguing by contradiction, we assume that $k \geq 2$, and X is a vertex subset of G with $|X| = k$ such that

$$G \text{ does not have an } X\text{-tree } L \text{ such that } \kappa(G - V(L)) \geq 2. \tag{3.1}$$

For each X-tree T of G, let B be a block of $G - V(T)$ with maximum order. Denote by $C_1, \dots, C_q$ the components of $G - (V(T) \cup V(B))$ such that $|C_1| \geq \cdots \geq |C_q|$. Let $\mathcal{S}(T) = (|C_1|, \dots, |C_q|)$. Choose an X-tree T in G such that

$$|B| \text{ is maximum,} \tag{3.2}$$

and, subject to (3.2),

$$\mathcal{S}(T) \text{ is maximum with respect to the lexicographic ordering.} \tag{3.3}$$

Claim 1. $q > 0$.

By contradiction, assume that $q = 0$. If $|B| \geq 3$, then $G - V(T) = B$ is 2-connected, contrary to (3.1). Thus $|B| \leq 2$. If $V(T) = X$, then $|G| = |X| + |B| \leq k + 2$, contrary to the assumption that $\kappa(G) \geq 2k + 1$ with $k \geq 2$. Hence there exists a vertex $u \in V(T) - X$. By the definition of an X-tree, the minimality of T implies that $V(B) \cup \{u\}$ is a vertex cut of G, contrary to the assumption that $\kappa(G) \geq 2k + 1 \geq 5$. This proves Claim 1.

By Lemma 2.3 with $H = C_q$, $V(T)$ has a C_q-partition (V_1, V_2, V_3). Similar to the proof of Theorem 3.1, we will show $|N_G(V(C_q) \cup V_3)| \leq 2k$, which forces the order of B is at most k, and leads to a contradiction.

In fact, by Lemma 2.3(a), both C_q and V_3 has no neighbors in V_1. By the definition of B and C_i, C_q has no neighbors in C_i for $1 \leq i \leq q-1$ and, since B is a block of $G - V(T)$,

$$|N_G(C_q) \cap V(B)| \leq 1. \tag{3.4}$$

Hence it suffices to determine the number of neighbors of V_3 in B and in the C_i's. Next, we construct a subset U_3 of V_3 by a sequence of edge contractions, aiming at determining the number of neighbors of U_3 in B and C_i.

To this end, we start with $G_0 = G$ and $T_0 = T$, and construct two sequences $T_0, T_1, \dots$ and $G_0, G_1, \dots$ as follows. Suppose G_i and T_i have been obtained. An edge $e = uv$ in T_i is *contractible* if both $u, v \notin V_1 \cup V_2$ and $G_i[V(T_i) \cup V(C_q)] - \{u, v\}$ is connected. If T_i has a contractible edge e, then define $T_{i+1} = T_i/e$ and $G_{i+1} = G_i/e$ (we also view V_1 and V_2 as vertex subsets of T_{i+1} and G_{i+1}). Otherwise, we stop. Assume that we stop at $i = r$ and let $U_3 = V(T_r) - V_1 \cup V_2$. Since all contractions are taken in $G[V_3]$, for notational convenience, vertices and subgraphs in $G - V_3$ will be viewed as vertices and subgraphs of G_i, for any i with $0 \leq i \leq r$.

Claim 2. *For any $i \leq r$, and for any vertex $u \in V(T_i) - V_1 \cup V_2$, $T_i - u$ is disconnected and $G_i[V(T_i) \cup V(C_q)] - u$ is connected.*

In fact, any vertex $u \in U_3$ corresponds to a vertex subset, disjoint with X, of T. By the minimality of an X-tree, $T_r - u$ is disconnected.

It suffices to verify that $G_i[V(T_i) \cup V(C_q)] - u$ is connected. We argue by induction on i. When $i = 0$, it holds by Lemma 2.3(c). Suppose $i > 0$ and $G_j[V(T_j) \cup V(C_q)] - u$ is connected for any value of $0 \leq j < i$.

Assume zw is the contractible edge of T_{i-1} such that $T_i = T_{i-1}/zw$ and z_0 is the vertex of T_i onto which the edge zw is contracted. Then since zw is a contractible edge, $G_{i-1}[V(T_{i-1}) \cup V(C_q)] - \{z, w\}$ is connected. Note that u is a vertex of $T_i - V_1 \cup V_2$. If $u = z_0$ then $G_i[V(T_i) \cup V(C_q)] - u = G_{i-1}[V(T_{i-1}) \cup V(C_q)] - \{z, w\}$ is connected, by the definition of a contractible edge. If $u \neq z_0$, then $G_i[V(T_i) \cup V(C_q)] - u = (G_{i-1}[V(T_{i-1}) \cup V(C_q)] - u)/zw$ is also connected by induction. This proves Claim 2.

By Claim 2, and from the fact that T_r has no contractible edges, we conclude that

$$\text{for each } u \in U_3,\ T_r - u \text{ is disconnected, and } G_r[V(T_r) \cup V(C_q)] - u \text{ is connected,} \tag{3.5}$$

and that

$$\text{for any edge } e \text{ of } G[U_3],\ G[V(T') \cup V(C_q)] - V(e) \text{ is disconnected.} \tag{3.6}$$

Based on (3.5) and (3.6), we make the following observations.

Claim 3. *$N_{G_r}(U_3) \cap V(C_i) = \emptyset$ for $1 \le i \le q-1$.*

Suppose that for some i with $1 \le i \le q-1$, U_3 has a vertex u such that $N_{G_r}(u) \cap V(C_i) \neq \emptyset$. Without loss of generality, we may assume i is as small as possible. Let V_u be the set of vertices in T contracted to u. By definition of contraction, $G[V_u]$ is connected and $T_r - u$ can be obtained from $T - V_u$ by contraction. By (3.5), $G_r[V(T_r) \cup V(C_q)] - u$ (and so $G[V(T) \cup V(C_q)] - V_u$) is connected and so G has an X-tree T' contained in $G[V(T) \cup V(C_q)] - V_u$. As $B, C_1, C_2, \ldots, C_{i-1}$ remain unchanged in $G - V(T')$ and as $C_i \cup V_u$ is in a component of $G - V(T')$, $\mathcal{S}(T')$ is bigger than $\mathcal{S}(T)$ in the lexicographic ordering, contrary to (3.3). This proves Claim 3.

Claim 4. *$|N_{G_r}(U_3) \cap V(B)| \le k-1$.*

We shall show that for each $u \in U_3$, $|N_{G_r}(u) \cap V(B)| \le 1$ and $|U_3| \le k-1$, which leads to the validity of Claim 4.

By contradiction, suppose that for some $u \in U_3$, $|N_{G_r}(u) \cap V(B)| \ge 2$. Let $u_1, u_2 \in N_{G_r}(u) \cap V(B)$ and let V_u be the set of vertices in T contracted onto u. Then $G[V_u]$ is connected and $u_1, u_2 \in N_G(V_u)$. Therefore, $G[V_u \cup \{u_1, u_2\}]$ has a path P joining u_1 and u_2 with internal vertices in V_u. By (3.6), $G_r[V(T_r) \cup V(C_q)] - u$ is connected, and so $G[V(T) \cup V(C_q)] - V_u$ is also connected. It follows that G has an X-tree T' contained in $G[V(T) \cup V(C_q)] - V_u$. As B is a block in $G - V(T)$, and as P is disjoint from T', $B \cup P$ is in a block of $G - V(T')$, which implies the maximal block of $G - V(T')$ is bigger than B, contrary to (3.2). This contradiction proves that $|N_{G_r}(u) \cap V(B)| \le 1$ for any $u \in U_3$.

We shall show that $|U_3| \le k-1$ by a few steps. For each edge $e = zw \in E(G_r[U_3])$, we will define a subgraph F_e, as follows. Let $Z_1, \ldots, Z_a, F_1, \ldots, F_b, W_1, \ldots, W_c$ be the components of $T_r - \{w, z\}$ such that $w \notin N_{T_r}(Z_i)$ for $i = 1, \ldots, a$, $z \notin N_{T_r}(W_j)$ for $j = 1, \ldots, c$, and $z, w \in N_{T_r}(F_p)$ for $1 \le p \le b$. Since $z \in U_3$, $G_r[V(T_r) \cup V(C_q)] - z$ is connected by (3.5), and so $N_{G_r}(C_q) \cap V(Z_i) \neq \emptyset$ for $i = 1, \ldots, a$. Similarly, $N_{G_r}(C_q) \cap V(W_j) \neq \emptyset$ for $j = 1, \ldots, c$. By (3.6), $G_r[V(T_r) \cup V(C_q)] - \{z, w\}$ is disconnected, and so $N_{G_r}(C_q) \cap V(F_p) = \emptyset$ for some $1 \le p \le b$. Fix one such value p and define $F_e = F_p$. Hence

$$N_{G_r}(C_q) \cap V(F_e) = \emptyset. \tag{3.7}$$

Subclaim 4.1. *For any edge $e = zw$ of $G_r[U_3]$, the component F_e satisfies $N_{T_r}(F_e) = \{z, w\}$, $V(F_e) \cap U_3 = \emptyset$, and $V(F_e) \cap V_2 \neq \emptyset$.*

By the definition of F_e, $N_{T_r}(F_e) = \{z, w\}$. If $V(F_e) \cap U_3$ has a vertex x, then by (3.5), $T_r - x$ is disconnected, and so $T_r - x$ has a component C'_x such that $V(C'_x) \subseteq V(F_e)$. By (3.5), C'_x contains some neighbor of C_q, contrary to (3.7). Hence $V(F_e) \cap U_3 = \emptyset$. Since F_e is a component of $T_r - \{z, w\}$, $V(F_e) \cap U_3 = \emptyset$ and $N_{T_r}(V_1) \cap U_3 = \emptyset$, it follows that $V(F_e) \cap V_2 \neq \emptyset$. This proves Subclaim 4.1.

Subclaim 4.2. *For any two edges $e, f \in E(G_r[U_3])$, $V(F_e) \cap V(F_f) = \emptyset$.*

Denote $e = u_1v_1, f = u_2v_2$. Then $u_1, v_1, u_2, v_2 \in U_3$. By Subclaim 4.1, $V(F_e) \cap U_3 = \emptyset$, and so $u_2, v_2 \notin V(F_e)$. Thus F_e is still connected in $T_r - \{u_2, v_2\}$. As F_f is a component of $T_r - \{u_2, v_2\}$, if $V(F_e) \cap V(F_f) \neq \emptyset$, then $V(F_e) \subseteq V(F_f)$. Similarly, $V(F_f) \subseteq V(F_e)$. It follows that $V(F_e) = V(F_f)$, and so $\{u_1, v_1\} = N_{G_r}(F_e) = N_{G_r}(F_f) = \{u_2, v_2\}$, a contradiction. This proves Subclaim 4.2.

Let $D_1, \ldots, D_t$ be the components of $G_r[U_3]$ and $H_1, \ldots, H_{k'}$ be the components of $G_r[V_1 \cup V_2]$. Since T_r is connected, each H_i contains some vertices in V_2, and so by Lemma 2.3(d), $k' \le |V_2| \le k$. Let $\mathcal{H}$ be the graph obtained from T_r by contracting $D_1, \ldots, D_t$ to $d_1, \ldots, d_t$ and contracting $H_1, \ldots, H_{k'}$ to $h_1, \ldots, h_{k'}$. Then $\mathcal{H}$ is bipartite and connected.

Suppose that $E(G_r[U_3]) = \{e_1, \ldots, e_m\}$. By Subclaim 4.1, F_{e_i} is a component of $G_r[V_1 \cup V_2]$. By Subclaim 4.2 and without loss of generality, we may assume $F_{e_i} = H_i$ for $i = 1, \ldots, m$. Let $\mathcal{F} = \{h_1, \ldots, h_m\}$. Then by Subclaim 4.1, each vertex in $\mathcal{F}$ has degree 1 in $\mathcal{H}$. Hence $\mathcal{H} - \mathcal{F}$ is still connected.

Subclaim 4.3. *Every d_i is a cut vertex of $\mathcal{H} - \mathcal{F}$.*

Suppose that this is not the case. Without loss of generality, we may assume $\mathcal{H} - (\mathcal{F} \cup \{d_1\})$ is still connected. As T is a minimal connected induced subgraph containing X, $T_r - V(D_1)$ (and hence $\mathcal{H} - d_1$) is disconnected. Therefore there must be some $h_i \in \mathcal{F}$ only adjacent to d_1. Since $h_i \in \mathcal{F}$ is only adjacent to d_1 and since $H_i = F_{e_i}$, it follows that $e_i \in E(D_1)$ and so $|D_1| \ge 2$. Pick an arbitrary $u \in V(D_1)$. Then $T_r - u$ has a component, say C, with $V(C) \cap V(D_1) \neq \emptyset$. For any $v_1 \in V(C) \cap V(D_1)$, $T_r - v_1$ has a component C'_1 contained in C. Choose a $v_1 \in V(C) \cap V(D_1)$ and such a component C'_1 so that $|C'_1|$ is minimized. Then $V(C'_1) \cap V(D_1) = \emptyset$ and $N_{T_r}(C'_1) = \{v_1\}$. Since $u \in V(D_1)$ is arbitrary, we may let $u = v_1$. Then there must be another vertex $v_2 \neq v_1$ and C'_2 such that $V(C'_2) \cap V(D_1) = \emptyset$ and $N_{T_r}(C'_2) = \{v_2\}$. Similar to the proof of Subclaim 4.2, we conclude that $V(C'_1) \cap V(C'_2) = \emptyset$.

For $i \in \{1, 2\}$, let $\mathcal{F}_i = \left\{h_j : V(C'_i) \cap V(H_j) \neq \emptyset\right\} \cup \left\{d_j : V(C'_i) \cap V(D_j) \neq \emptyset\right\} - \mathcal{F}$. We shall show that $\mathcal{F}_1$ and $\mathcal{F}_2$ induce two components of $\mathcal{H} - \mathcal{F} - d_1$, which implies d_1 is a cut vertex, whence a contradiction is obtained.

By symmetry, we only need to show $\mathcal{F}_1$ induces a component of $\mathcal{H} - (\mathcal{F} \cup \{d_1\})$. Let F be a component of $G_r[U_3]$ or $G_r[V_1 \cup V_2]$. If $V(F) \cap V(C'_1) \neq \emptyset$, then since $V(C'_1) \cap V(D_1) = \emptyset$, both $F \neq D_1$ and $V(F) \cap V(D_1) = \emptyset$ hold. Hence F is still connected in $T_r - V(D_1)$. As C'_1 is a component of $T_r - V(D_1)$, $V(F) \subseteq V(C'_1)$. Hence, every component corresponding to a vertex of $\mathcal{F}_1$ is in fact contained in C'_1. Thus in order to show $\mathcal{F}_1$ induces a component of $\mathcal{H} - \mathcal{F}$, it suffices to show $\mathcal{F}_1 \neq \emptyset$. In fact, let w_1 be a neighbor of v_1 in C_1'. Then for some j_1, $w_1 \in V(H_{j_1})$, and so $V(H_{j_1}) \subseteq V(C_1)$ since $V(H_{j_1}) \cap V(C_1') \neq \emptyset$. Hence, v_1 is the only neighbor of H_{j_1} in D_1. By Subclaim 4.1, $h_{j_1} \notin \mathcal{F}$, which implies $\mathcal{F}_1 \neq \emptyset$. Therefore, $\mathcal{F}_1$ induces a component of $\mathcal{H} - (\mathcal{F} \cup \{d_1\})$. Similarly, $\mathcal{F}_2$ induces a component of $\mathcal{H} - (\mathcal{F} \cup \{d_1\})$. Then d_1 is a cut vertex of $\mathcal{H} - \mathcal{F}$, a contradiction which implies Subclaim 4.3.

By the definition of the d_i's, $\{d_1, \ldots, d_t\}$ is an independent set of $\mathcal{H}$. By Subclaim 4.3, each d_i is a cut vertex of $\mathcal{H} - \mathcal{F}$. It follows that $\mathcal{H} - (\mathcal{F} \cup \{d_1, \ldots, d_t\})$ has at least $t + 1$ component, and so $k' - m \geq t + 1$. This implies $|U_3| \leq m + t \leq k' - 1 \leq k - 1$. This proves Claim 4.

By (3.4), by Lemma 2.3(d) and by Claim 4,

$$\begin{aligned} |N_G(V_3 \cup V(C_q))| &\leq |N_G(C_q) \cap V(B)| + |N_G(V_3) \cap V(B)| + |V_2| \\ &= |N_G(C_q) \cap V(B)| + |N_{G_r}(U_3) \cap V(B)| + |V_2| \leq 2k. \end{aligned} \tag{3.8}$$

By $\kappa(G) \geq 2k+1$ and by (3.8), $N_G(V_3 \cup V(C_q))$ is not a vertex cut of G, which implies $V(G) = (V_3 \cup V(C_q)) \cup N_G(V_3 \cup V(C_q))$. By (3.2), by (3.4) and by Claim 4, we conclude that for every X-tree T, the maximum block B of $G - V(T)$ satisfies

$$|B| \leq |N_{G_r}(U_3) \cap B| + |N_G(C_q) \cap B| \leq k. \tag{3.9}$$

Next, we will find another X-tree T' in G such that $G - V(T')$ has a block with order at least $k + 1$, leading to a contradiction to (3.9).

Choose an X-tree T' such that

(a) $|V(T')|$ is minimized, and
(b) subject to (a), $|E(G - V(T'))|$ is maximized.

Let $\delta = \delta(G - V(T'))$ and $x \in V(G - V(T'))$ be a vertex with degree δ. By Lemma 2.3, $V(T')$ has an $\{x\}$-partition (V_1', V_2', V_3'). We shall show that

$$\delta(G - V(T')) \geq k, \tag{3.10}$$

and so $G - V(T')$ has a block of order at least $k + 1$. This will be justified by the next few claims.

Claim 5. *For any vertex $u \in V_3'$, $|N_G(u) - V(T')| \leq \delta + 1$.*

Suppose, for the sake of contradiction, that there is a vertex $u \in V_3'$ such that $|N_G(u) - V(T')| \geq \delta + 2$. By Lemma 2.3(c), $G[V(T') \cup \{x\}] - u$ is connected, and so $G[V(T') \cup \{x\}] - u$ has an X-tree T''. By (a), $T'' = G[V(T') \cup \{x\}] - u$ is also an X-tree with minimum order. However, $|E(G - V(T''))| \geq |E(G - V(T'))| - \delta + (\delta + 2) - 1 > |E(G - V(T'))|$, contradicting the choice (b) of T'. This proves Claim 5.

Let δ' be the minimum degree of $G[V_3']$ and u a vertex of $G[V_3']$ with degree δ'. Denote $A_1 := N_G(u) \cap V_2'$ and $A_2 = V_2' - A_1$. Then by Claim 5, $|A_1| \geq 2k + 1 - (\delta + 1) - \delta' = 2k - \delta - \delta'$, and

$$|A_2| = |V_2'| - |A_1| \leq k - (2k - \delta - \delta') = \delta + \delta' - k. \tag{3.11}$$

With a similar idea in the proof of Subclaim 4.1, we also have the next claim.

Claim 6. *For any edge $e = zw$ of $G[V_3']$ not incident with u, there is a component F_{zw} of $T' - \{z, w\}$ such that $N(F_{zw}) \cap V_3' = \{z, w\}$, $V(F_{zw}) \cap V_3' = \emptyset$, $V(F_{zw}) \cap A_1 = \emptyset$ and $V(F_{zw}) \cap A_2 \neq \emptyset$.*

Let $e = zw$ be an edge of $G[V_3']$ not incident with u. Let $Z_1, \ldots, Z_a, F_1, \ldots, F_b, W_1, \ldots, W_c$ be the components of $T' - \{w, z\}$ such that $w \notin N_G(Z_i)$ for $i = 1, \ldots, a$, $z \notin N_G(W_j)$ for $j = 1, \ldots, c$, and $z, w \in N_G(F_p)$ for $1 \leq p \leq b$. By Lemma 2.3(c), each of $Z_1, \ldots, Z_a, W_1, \ldots, W_c$ contains some neighbor of x. By (a), $G[V(T') \cup \{x\}] - \{z, w\}$ is not connected, and so $x \notin N_G(F_p)$ for some $1 \leq p \leq b$. Define F_{zw} to be this F_p. By the definition of F_{zw}, $N(F_{zw}) \cap V_3' = \{z, w\}$. If there exists $z' \in V(F_{zw}) \cap V_3'$, then $T' - z'$ has a component, say F, contained in F_{zw}. By Lemma 2.3(c), $G[V(T') \cup \{x\}] - z'$ is connected, and so $x \in N_G(F)$, contrary to the choice of p. Hence $V(F_{zw}) \cap V_3' = \emptyset$, and so $u \notin N_G(F_{zw})$, which implies $V(F_{zw}) \cap A_1 = \emptyset$. Thus $V(F_{zw}) \cap A_2 \neq \emptyset$, completing the proof of Claim 6.

Claim 7. *If $|V_3'| \geq 2$ then there exists $v \in V_3' - \{u\}$ and a component F_v of $T' - \{v\}$ such that $N_G(F_v) \cap V_3' = \{v\}$, $V(F_v) \cap A_1 = \emptyset$ and $V(F_v) \cap A_2 \neq \emptyset$.*

Suppose $|V_3'| \geq 2$. Then $T' - u$ has a component, say F, with $V(F) \cap V_3' \neq \emptyset$. For any $v \in V(F) \cap V_3'$, at least one component of $T' - v$, say F_v, is contained in F. Choose v and F_v so that $|F_v|$ is minimum. Then $V(F_v) \cap V_3' = \emptyset$ and $N_G(F_v) \cap V_3' = \{v\}$. Thus $u \notin N_G(F_v)$, and so $V(F_v) \cap A_1 = \emptyset$. Hence, $V(F_v) \cap A_2 \neq \emptyset$, completing the proof of Claim 7.

Let $|V_3'| = t'$ and $|E(G[V_3'])| = m'$. If $t' = 1$, then $\delta' = 0$. By (3.11) and as $|A_2| \geq 0$, $\delta \geq k$, and so (3.10) holds. Thus we assume $t' \geq 2$. Let $\mathcal{S} := \{zw : zw \text{ is an edge of } G[V_3'] - u\} \cup \{v\}$, where v is the vertex found in Claim 7. By an argument similar to the proof of Subclaim 4.2, we have the following observation:

$$\text{For any } e, f \in \mathcal{S},\ V(F_e) \cap V(F_f) = \emptyset. \tag{3.12}$$

By Claims 6 and 7 and by (3.12), $|A_2| \geq |\mathcal{S}| = m' - \delta' + 1 > \delta'(t'/2 - 1)$. This, together with (3.11), implies $\delta'(t'/2 - 1) < \delta + \delta' - k$. It follows that $\delta - k > \delta'(t'/2 - 2)$. Suppose that $\delta < k$. Then $t' < 4$ and so $t' = 2$ or 3. By $\delta' \leq t' - 1$, $\delta - k > (t' - 1)(t'/2 - 2) = -1$, contradicting our assumption $\delta < k$. Hence $\delta \geq k$ and so (3.10) must hold.

Let P be a longest path of $G - V(T')$ and y an end of P. Since P is longest, $N_G(y) \subseteq V(P)$. Let z be the neighbor of y with maximum distance to y on P. Then the (y, z)-segment of P and the edge yz form a cycle of order at least $k + 1$ by the fact $\delta \geq k$, which implies there is a block of $G - V(T')$ with order at least $k + 1$, contrary to (3.9), which completes the proof. □

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