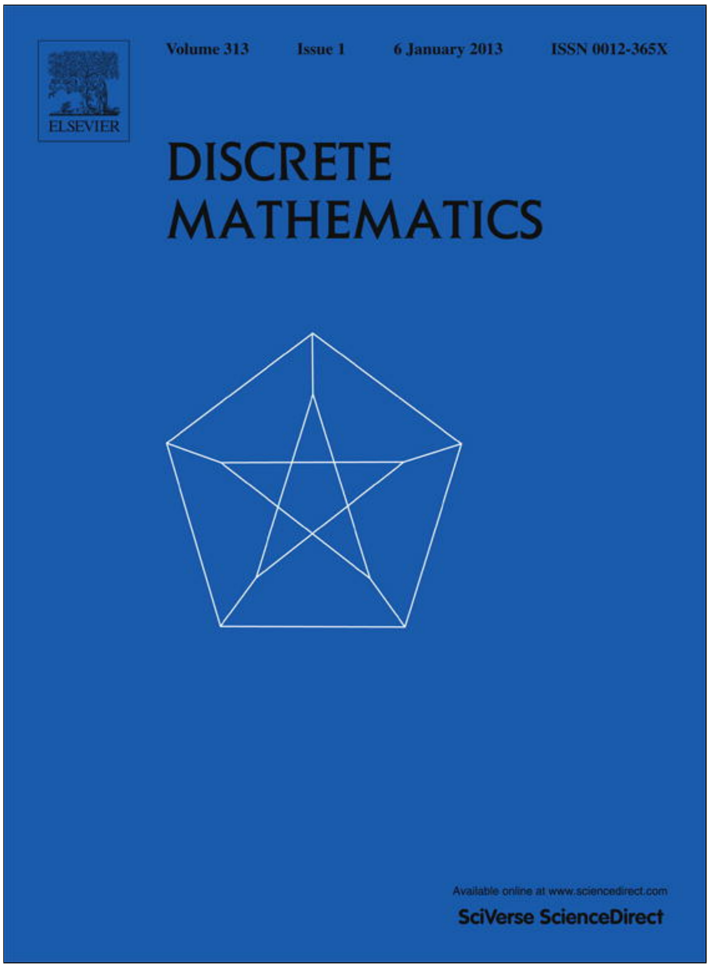
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Note

Group colorability of multigraphs[☆]

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ABSTRACT

Let G be a multigraph with a fixed orientation D and let Γ be a group. Let $F(G, \Gamma)$ denote the set of all functions $f : E(G) \to \Gamma$. A multigraph G is *Γ-colorable* if and only if for every $f \in F(G, \Gamma)$, there exists a Γ-coloring $c : V(G) \to \Gamma$ such that for every $e = uv \in E(G)$ (assumed to be directed from u to v), $c(u)c(v)^{-1} \neq f(e)$. We define the *group chromatic number* $\chi_g(G)$ to be the minimum integer m such that G is Γ-colorable for any group Γ of order $\geq m$ under the orientation D. In this paper, we investigate the properties of χ_g for multigraphs and prove an analogue to Brooks' Theorem.

1. Introduction

In this paper, we use the model of 'graph' that allows multiple edges but not loops; these are also called 'finite loopless multigraphs'. We use terms and notation of [1], unless otherwise stated. A nontrivial 2-regular connected graph is called a *circuit*. For a graph G with vertex set $V(G)$ and edge set $E(G)$, we define an equivalence relation "$\sim$" on $E(G)$ such that $e_1 \sim e_2$ if $e_1 = e_2$ or if e_1 and e_2 form a circuit in G. Edges in the same equivalence class are *parallel edges*, and they have the same endpoints. For $u, v \in V(G)$, let $m(u, v)$ denote the number of parallel edges with endpoints u and v. The *multiplicity* of G, denoted by $M(G)$, is the maximum size of an equivalence class. The *simplification* of G, denoted by G_0, is the simple graph obtained by replacing each equivalence class by a single edge. For a graph H and a positive integer k, we define kH to be the graph obtained by replacing each edge of H by a class of k parallel edges. For $V_1, V_2 \subseteq V(G)$ and $V_1 \cap V_2 = \emptyset$, let $G[V_1]$ denote the induced subgraph of V_1 and let $E[V_1, V_2]$ denote the set of edges joining V_1 and V_2. We abbreviate $E[\{u\}, V_2]$ to $E[u, V_2]$.

Group coloring was first introduced by Jeager et al. [2]. They introduced a concept of group connectivity as a generalization of nowhere-zero flows. They also introduced group coloring as a dual concept to group connectivity.

An *orientation* D of G is a map $h_D : E(G) \to V(G)$ such that $h_D(e)$ is a vertex incident with e in G, and the edge e is oriented from $h_D(e)$ to the other endpoint. The graph G under the orientation D is sometimes denoted by $D(G)$. In $D(G)$, an oriented edge is called an *arc*, and we write $e = uv$ to mean $h_D(e) = u$.

Let Γ be a nontrivial group and $F(G, \Gamma)$ be the set of all functions $f : E(G) \to \Gamma$. For a function $f \in F(G, \Gamma)$, a *(Γ, f)-coloring* of $D(G)$ is a function $c : V(G) \to \Gamma$ such that $c(u)c(v)^{-1} \neq f(e)$ when uv is an arc; $D(G)$ is *Γ-colorable* if and only if for any $f \in F(G, \Gamma)$ there exists a (Γ, f)-coloring. It is known [2] that whether G is Γ-colorable is independent of the choice of the orientation. Therefore, we say simply that G is Γ-colorable when some orientation of G is Γ-colorable. The *group chromatic number* of a graph G, denoted by $\chi_g(G)$, is defined to be the minimum positive integer m for which G is Γ-colorable for every group Γ of order at least m.

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For a subgraph H of G, (G, H) is said to be Γ*-extendible* if for any $f \in F(G, \Gamma)$ and any $(\Gamma, f|_{E(H)})$-coloring c' of H, there is a (Γ, f)-coloring c of G such that $c|_{E(H)} = c'$. The coloring c is then called an *extension* of c'.

Prior studies on group chromatic number were restricted to simple graphs and considered only Abelian groups in the definition of $\chi_g(G)$. The following results were proved under the assumptions that the groups involved are Abelian groups. However, they remain valid without this assumption.

Theorem 1.1 (*Lai and Zhang* [5]). *For any connected simple graph G,*

$$\chi_g(G) \leq \Delta(G) + 1$$

with equality if and only if G is a cycle or G is complete.

Theorem 1.2. *Let G be a simple graph.*

(i) *(Lai and Zhang* [6]*) If G is K_5-minor free, then $\chi_g(G) \leq 5$.*
(ii) *(Lai and Li* [4]*) If G is $K_{3,3}$-minor free, then $\chi_g(G) \leq 5$.*

The bound of Theorem 1.2(ii) is sharp. A 3-colorable simple planar graph with $\chi_g(G) = 5$ is constructed in [3].

Note that although group coloring is a generalization of vertex coloring, they have different behaviors: for any bipartite graph G, $\chi(G) = 2$ while $\chi_g(G)$ can be arbitrary large. Lai and Zhang [5] showed that for any complete bipartite graph $K_{m,n}$ with $n \geq m^m$, $\chi_g(K_{m,n}) = m + 1$. Moreover, group coloring of multigraphs is different from that of simple graphs. Consider K_2, for example. By Brooks' Theorem and Theorem 1.1, $\chi(K_2) = \chi_g(K_2) = 2$ while $\chi_g(mK_2) = m + 1 (m \in Z^+)$.

The main purpose of this paper is to extend Theorem 1.1 to multigraphs.

Theorem 1.3. *For any connected multigraph G,*

$$\chi_g(G) \leq \Delta(G) + 1$$

where equality holds if and only if G is kC_n or kK_n for some positive integer k and n.

In order to prove our theorem, we will first show some properties of group coloring in Section 2. The upper bound of $\chi_g(G)$ for multigraphs will be given in Section 3.

2. Elementary properties

In [5], Lai and Zhang proved the following properties of χ_g. Although they assumed that graphs are simple and groups are Abelian groups, these properties still hold without such assumptions, as their proofs did not utilize the property that the binary operations of the involved groups are commutative.

Proposition 2.1 (*Lai and Zhang* [5]). *If G is a multigraph and Γ is a group, then each of the following holds:*

(i) *If G is Γ-colorable under an orientation D, then G is Γ-colorable under every orientation of G.*
(ii) *G is Γ-colorable if and only if each block of G is Γ-colorable.*

By Proposition 2.1(i), $\chi_g(G)$ is not dependent on the choice of orientations of G. Furthermore, Proposition 2.1(ii) shows that it suffices to consider 2-connected graphs in our proofs.

Proposition 2.2 (*Lai and Zhang* [5]). *If G is a multigraph and Γ is a group, then each of the following holds:*

(i) *Let $H \subseteq G$. If (G, H) is Γ-extendible and H is Γ-colorable, then G is Γ-colorable.*
(ii) *Let $H_2 \subseteq H_1 \subseteq G$. If (G, H_1) and (H_1, H_2) are Γ-extendible, then (G, H_2) is also Γ-extendible.*
(iii) *Suppose that $V(G)$ can be linearly ordered as $v_1, v_2, \ldots, v_n$ such that $d_{G_i}(v_i) \leq k$ $(i = 1, 2, \ldots, n)$, where $G_i = G[\{v_1, v_2, \ldots, v_i\}]$. Then for any group Γ of order at least $k + 1$, (G_{i+1}, G_i) $(i = 1, 2, \ldots, n - 1)$ is Γ-extendible and so G is Γ-colorable.*

An immediate corollary of Proposition 2.2(iii) is given below.

Corollary 2.3 (*Lai and Zhang* [5]). *If G is a multigraph, then each of the following holds:*

(i) $\chi_g(G) \leq \max_{H \subseteq G}\{\delta(H)\} + 1$.
(ii) $\chi_g(G) \leq \Delta(G) + 1$.

Lemma 2.4. *Let G be a graph. Then for any $k \in \mathbb{Z}^+$, $\chi_g(kG) \geq k(\chi_g(G) - 1) + 1$.*

Proof. By the definition of $\chi_g(G)$, there exist a group Γ of order $\chi_g(G) - 1$ and a function $f \in F(G, \Gamma)$ such that there are no (Γ, f)-colorings of G. Let $\Gamma' = \Gamma \times \mathbb{Z}_k$ (where $|\Gamma'| = k(\chi_g(G) - 1)$). Define $f' \in F(kG, \Gamma')$ by $f(e_i) = (f(e), i)$ for $1 \leq i \leq k$ where $\{e_1, e_2, \ldots, e_k\}$ are the k parallel edges in $E(kG)$ corresponding to $e \in E(G)$. There are no (Γ', f')-colorings of kG, since any (Γ', f')-coloring of kG can give rise to a (Γ, f)-coloring of G. Therefore kG is not Γ'-colorable, and $\chi_g(kG) \geq k(\chi_g(G) - 1) + 1$. □

3. Group chromatic number of multigraphs

Brooks proved that for any connected graph G, $\chi(G) \leq \Delta(G) + 1$ where equality holds if and only if either $\Delta(G) = 2$ and G is an odd cycle or $\Delta(G) \geq 3$ and G is a complete graph. Lai and Zhang [5] proved Theorem 1.1 as a strengthening of Brooks' Theorem for the group chromatic number of simple graphs. In this section, we shall extend Theorems 1.1–1.3. We start with two lemmas.

Lemma 3.1. *Let Γ be a group. If S_1 and S_2 are subsets of Γ such that $|S_2| > |S_1|$, then there exist x and y in S_2 such that $S_1x \neq S_1y$.*

Proof. We argue by contradiction and assume that $\forall x, y \in S_2, S_1x = S_1y$. Let $|S_1| = m$. Since $|S_2| > |S_1|$, we can pick $m + 1$ distinct elements $b_1, b_2, \ldots, b_{m+1}$ from S_2. Note that $S_1b_i = S_1b_j$ for $1 \leq i < j \leq m + 1$. Fix $a \in S_1$. Since $ab_i \in S_1b_i = S_1b_1$ $(1 \leq i \leq m + 1)$, $\{ab_i : 1 \leq i \leq m + 1\} \subseteq S_1b_1$. Since $m = |S_1b_1| \geq |\{ab_i : 1 \leq i \leq m + 1\}|$, there exist $1 \leq k < l \leq m + 1$ such that $ab_k = ab_l$. Hence $b_k = b_l$, contrary to the assumption that $b_1, b_2, \ldots, b_{m+1}$ are distinct. □

Lemma 3.2. *Let G be a 2-connected graph whose simplification is neither a cycle nor a complete graph; then there exist three vertices v_1, v_2, v_n in G such that both of the following hold:*

(i) *$v_1v_n, v_2v_n \in E(G)$ and $v_1v_2 \notin E(G)$, and*
(ii) *$G - \{v_1, v_2\}$ is connected.*

The proof of Lemma 3.2 is contained in almost every graph theory textbook which proves Brooks' Theorem. So it is omitted here.

Proof of Theorem 1.3. By Corollary 2.3(ii), we only need to show that when the equality holds, G must be either a kC_n or a kK_n. Let G be a graph such that $\chi_g(G) = \Delta(G) + 1$ and $n = |V(G)|$. By Proposition 2.1(ii), we may assume that G is 2-connected.

Claim 1. *G is regular.*

If G not regular, then $\max_{H \subseteq G}\{\delta(H)\} \leq \Delta(G) - 1$. Therefore, by Corollary 2.3(i), $\chi_g(G) \leq \max_{H \subseteq G}\{\delta(H)\} + 1 \leq \Delta(G)$, a contradiction to $\chi_g(G) = \Delta(G) + 1$.

Claim 2. *G_0 is either a cycle or a complete graph.*

Suppose that G_0 is neither a cycle nor a complete graph. Since $\chi_g(G) = \Delta(G) + 1$, there exist a group Γ of order $\Delta(G)$ and a function $f \in F(G, \Gamma)$ such that G has no (Γ, f)-colorings.

By Lemma 3.2, G has three vertices v_1, v_2, v_n such that $v_1v_n, v_2v_n \in E(G)$, $v_1v_2 \notin E(G)$ and $G - \{v_1, v_2\}$ is connected. Now we arrange the vertices of $G - \{v_1, v_2\}$ in non-increasing order of their distance from v_n, say $v_3, v_4, \ldots, v_n$. Then the list $\{v_1, v_2, \ldots, v_n\}$ is such that each vertex other than v_n is adjacent to at least one vertex following it. Thus each vertex other than v_n is adjacent to at most $\Delta(G) - 1$ vertices preceding it.

Let D be an orientation such that every arc between v_i and v_j is directed from v_j to v_i if $i < j$ and from v_i to v_j otherwise. Define a map $c : V(G) \to \Gamma$ as follows. For $i = 1, 2$, let e_i denote an arc from v_n to v_i. Choose $a_1, a_2 \in \Gamma$ such that $f(e_1)a_1 = f(e_2)a_2$. Define $c(v_i) = a_i$ $(i = 1, 2)$. For v_j $(3 \leq j \leq n)$, let $A_j = \{f(e)c(v_i) : e \in E[v_j, v_i]$ and $i = 1, 2, \ldots, j - 1\}$. If $j < n$, then $|A_j| \leq d_{G[\{v_1, v_2, \ldots, v_j\}]}(v_j) \leq \Delta(G) - 1$ and $\Gamma - A_j \neq \emptyset$; if $j = n$, then $\Gamma - A_n \neq \emptyset$ since $f(e_1)a_1 = f(e_2)a_2$. Hence we can choose $c(v_j) \in \Gamma - A_j$ $(3 \leq j \leq n)$, so that c is a (Γ, f)-coloring of G, contrary to the assumption that G has no (Γ, f)-colorings.

Claim 3. *If $G = kC_n$ or kK_n, then $\chi_g(G) = \Delta(G) + 1$.*

If $G = kC_n$, then by Corollary 2.3(ii) and Lemma 2.4, $\Delta(G) + 1 \geq \chi_g(G) \geq k(3 - 1) + 1 = \Delta(G) + 1$, and so $\chi_g(G) = \Delta(G) + 1$. Similarly, if $G = kK_n$, then $\chi_g(G) = \Delta(G) + 1$.

Claim 4. *If $G_0 = C_n$, then $G = kC_n$, where $k = M(G)$.*

Assume that $G \neq kC_n$. By Claim 1, G is regular. It follows that G must satisfy both $G_0 = C_{2k} = u_1v_1u_2v_2 \cdots u_kv_k$ and $m(u_iv_i) = a$, $m(v_iu_{i+1}) = b$ with $a \neq b$, where $i = 1, 2, \ldots, k$ and the subscripts are taken modulo k. Without loss of generality, assume $a > b$. Let D be an orientation of G such that every arc $e \in E[u_i, v_i]$ is directed from u_i to v_i and every arc $e \in E[v_i, u_{i+1}]$ is directed from v_i to u_{i+1} $(i = 1, 2, \ldots, k$, where subscripts are taken modulo $k)$.

Since $\chi_g(G) = \Delta(G) + 1$, there exist a group Γ of order $\Delta(G) = a + b$ and $f \in F(G, \Gamma)$ such that G has no (Γ, f)-colorings. Let $H = G[\{u_1, v_1, u_2, v_2, \ldots, u_{k-1}, v_{k-1}\}]$. Since H_0 is a path, it follows by Corollary 2.3 that $\chi_g(H) \leq \Delta(H) = a + b$ and then H has a $(\Gamma, f|_H)$-coloring c'. Let $c : V(G) \to \Gamma$ be a function where $c|_{V(H)} = c'$.

Let $\Gamma_1 = \Gamma \setminus \{f(e) : e \in E[v_{k-1}, u_k]\}$ and $\Gamma_2 = \{f(e) : e \in E[v_k, u_1]\}$. Pick $y_0 \in \Gamma \setminus \{f(e) : e \in E[u_k, v_k]\}$. Since $|\{y_0^{-1}x^{-1}c(v_{k-1})c(u_1)^{-1} : x \in \Gamma_1\}| = |\Gamma_1| \geq a > b \geq |\Gamma_2|$, choose $x_0 \in \Gamma_1$ such that $y_0^{-1}x_0^{-1}c(v_{k-1})c(u_1)^{-1} \notin \Gamma_2$. Let $c(u_k) = x_0^{-1}c(v_{k-1})$ and $c(v_k) = y_0^{-1}c(u_k)$. Now c is a (Γ, f)-coloring of G, contrary to the assumption that G has no (Γ, f)-colorings. This completes the proof of Claim 4.

Claim 5. *If $G_0 = K_n$, then $G = kK_n$, where $k = M(G)$.*

Assume that $G \neq kK_n$ and $n \geq 4$. Since $\chi_g(G) = \Delta(G) + 1$, there exist a group Γ of order $\Delta(G)$ and a function $f \in F(G, \Gamma)$ such that there is no (Γ, f)-coloring of G. By Claim 1, G is regular. It follows that there exist $u, v_1, v_2 \in V(G)$ with $m(uv_1) = a$, $m(uv_2) = b$ and $m(v_1v_2) = d$ such that $a < b$. Let $H = G - \{u, v_1, v_2\}$. Let D be an orientation such that arcs in $E[\{u, v_1, v_2\}, V(H)]$ are all directed into H; arcs in $E[u, v_i]$ are all directed from u to v_i $(i = 1, 2)$ and arcs in $E[v_2, v_1]$ are all directed from v_2 to v_1.

Since H is not regular, it follows by Corollary 2.3(i) that $\chi_g(H) \leq \Delta(H) \leq \Delta(G)$. Thus H has a $(\Gamma, f|_H)$-coloring c. For any $v \in \{v_1, v_2, u\}$, define $A_v = \Gamma \setminus \{f(e)c(x) : x \in V(H), e \in E[v, x]\}$. Since $|\Gamma| = \Delta(G)$, $|A_{v_1}| \geq a + d$, $|A_{v_2}| \geq b + d$, and $|A_u| \geq a + b$. Taking a subset if needed, we may assume that $|A_{v_1}| = a + d$, $|A_{v_2}| = b + d$, and $|A_u| = a + b$. Since $|\{f(e) : e \in E[v_2, v_1]\}| \leq d$ and $|A_{v_1}| = a + d$, it follows by Lemma 3.1 that there exist $x_1, x_2 \in A_{v_1}$ such that $\{f(e)x_1 : e \in E[v_2, v_1]\} \neq \{f(e)x_2 : e \in E[v_2, v_1]\}$.

Let c_1 be an extension of c on $G[V(H) \cup v_1]$ such that $c_1(v_1) = x_1$. For any $v \in \{v_2, u\}$, define $A'_v = A_v \setminus \{f(e)x_1 : e \in E[v, v_1]\}$. Note that $|A'_{v_2}| \geq b$ and $|A'_u| \geq b$. If $|A'_{v_2}| > b$, then choose $c_1(u) \in A'_u$ and $c_1(v_2) \in A'_{v_2} \setminus \{f(e)^{-1}c_1(u)\}$, such that c_1 is a (Γ, f)-coloring of G, contrary to the assumption that G has no (Γ, f)-colorings. Thus

$$|A'_{v_2}| = b, \quad \text{and similarly} \quad |A'_u| = b. \tag{1}$$

Assume that there is a $z \in A'_u$ such that $\{f(e)^{-1}z : e \in E[u, v_2]\} \neq A'_{v_2}$. Since $|A'_{v_2}| = b = m(uv_2) \geq |\{f(e)^{-1}z : e \in E[u, v_2]\}|$, we can pick $y \in A'_{v_2} \setminus \{f(e)^{-1}z : e \in E[u, v_2]\}$ and extend c_1 to a map $c_2 : V(G) \mapsto \Gamma$ by assigning $c_2(v_2) = y$ and $c_2(u) = z$. By the choices of y and z, it is routine to verify that c_2 is indeed a (Γ, f)-coloring of G, contrary to the assumption that G has no (Γ, f)-colorings. Hence we may assume that

$$\forall z \in A'_u, \quad \{f(e)^{-1}z : e \in E[u, v_2]\} = A'_{v_2}. \tag{2}$$

Let c_0 be an extension of c on G such that $c_0(v_1) = x_2$. For any $v \in \{v_2, u\}$, define $A''_v = A_v \setminus \{f(e)x_2 : e \in E[v, v_1]\}$. By the choice of x_1 and x_2, $A'_{v_2} \neq A''_{v_2}$. So we can pick $y_0 \in A''_{v_2} \setminus A'_{v_2}$. As G has no (Γ, f)-colorings, it follows by a similar argument to conclude (1) that we must also have $|A''_{v_2}| = |A''_u| = b$.

Since $|A_u| = a + b \leq 2b$, $A''_u \cap A'_u \neq \emptyset$. Take $z_0 \in A''_u \cap A'_u$. Define $c_2(v_2) = y_0$ and $c_2(u) = z_0$. By (2) and since $y_0 \notin A'_{v_2}$, it is routine to verify that c_0 is indeed a (Γ, f)-coloring of G, contrary to the assumption that G has no (Γ, f)-colorings. This completes the proof of Claim 5.

After we have established these claims, it is straightforward to see that Theorem 1.3 now follows from Claim 3 to 5. □

Since $\Delta(G) \leq M(G)\Delta(G_0)$, Corollary 3.3 below follows from Theorem 1.3 immediately.

Corollary 3.3. *For any graph G, $\chi_g(G) \leq M(G)\Delta(G_0) + 1$, with equality if and only if $G = M(G)C_n$ or $G = M(G)K_n$.*

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