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Hamiltonicity of 3-connected line graphs[☆]

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ABSTRACT

Thomassen conjectured that every 4-connected line graph is Hamiltonian. Lai et al. conjectured [H. Lai, Y. Shao, H. Wu, J. Zhou, Every 3-connected, essentially 11-connected line graph is Hamiltonian, J. Combin. Theory Ser. B 96 (2006) 571–576] that every 3-connected, essentially 4-connected line graph is Hamiltonian. In this note, we first show that the conjecture posed by Lai et al. is not true and there is an infinite family of counterexamples; we show that 3-connected, essentially 4-connected line graph of a graph with at most 9 vertices of degree 3 is Hamiltonian; examples show that all conditions are sharp.

1. Introduction

Unless stated otherwise, we follow [1] for terminology and notation, and we consider finite connected graphs without loop (i.e. multiple edge (*multigraph*) is allowed). In particular, we use $\kappa(G)$ and $\lambda(G)$ to represent the *connectivity* and *edge-connectivity* of a graph G. A graph is *trivial* if it contains no edges. A vertex (edge) cut X of G is *essential* if $G - X$ has at least two non-trivial components. For an integer $k > 0$, a graph G is *essentially* k-(*edge*)-*connected* if G does not have an *essential (edge-)cut* X with $|X| < k$. A graph G is *cyclically* k-*edge-connected* if G has no edge-cut F of size $|F| < k$ such that at least two of the components of $G - F$ contain at least one cycle. The chromatic index $\chi'(G)$ of G is the minimum number of colors needed to color the edges of G in such a way that no two adjacent edges are assigned the same color. This definition implies the inequality $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ denotes the maximum degree of G. Vizing' Theorem [2] shows that if G is a connected graph, then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. Vizing' Theorem allows us to classify connected graphs into two classes according to their chromatic indices. More precisely, a graph G is of class one if $\chi'(G) = \Delta(G)$, and of class two if $\chi'(G) = \Delta(G) + 1$. A cubic graph G is a *snark* if it satisfies the following conditions: (1) G is of class two; (2) $g(G) \geq 5$, where $g(G)$ is the girth of G; (3) G is cyclically 4-edge-connected.

The line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common. From the definition of a line graph, if $L(G)$ is not a

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complete graph, then a subset $X \subseteq V(L(G))$ is a vertex cut of $L(G)$ if and only if X is an essential edge cut of G. Thomassen in 1986 posed the following conjecture (see [3–5] for the known results on the conjecture).

Conjecture 1.1 (*Thomassen [6]*). *Every* 4*-connected line graph is Hamiltonian.*

Lai et al. in [7] considered the following problem. For 3-connected line graphs, can high essential connectivity guarantee the existence of a Hamiltonian cycle? They proved the following theorem.

Theorem 1.2 (*Lai et al. [7]*). *Every* 3*-connected, essentially* 11*-connected line graph is Hamiltonian.*

It is well known that the line graph of the graph obtained by subdividing each edge of the Petersen graph exactly once is a 3-connected graph without a Hamiltonian cycle. So they conjectured that the minimum essential connectivity that guarantees the existence of a Hamiltonian cycle is 4.

Conjecture 1.3 (*Lai et al. [7]*). *Every* 3*-connected, essentially* 4*-connected line graph is Hamiltonian.*

However, their conjecture is not always true for 3-connected, essentially 4-connected line graphs. In this note, we show there is an infinite family of counterexamples for Conjecture 1.3; we show that 3-connected, essentially 4-connected line graph of a graph with at most 9 vertices of degree 3 is Hamiltonian; examples show that all conditions are sharp.

2. Reductions

Catlin in [8] introduced collapsible graphs. For a graph G, let $O(G)$ denote the set of odd degree vertices of G. A graph G is *Eulerian* if G is connected with $O(G) = \emptyset$, and G is *super-Eulerian* if G has a spanning Eulerian subgraph. A graph G is *collapsible* if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph H_R such that $O(H_R) = R$. Note that when $R = \emptyset$, a spanning connected subgraph H with $O(H) = \emptyset$ is a spanning Eulerian subgraph of G. Thus every collapsible graph is super-Eulerian. Catlin [8] showed that any graph G has a unique subgraph H such that every component of H is a maximally collapsible subgraph of G and every non-trivial collapsible subgraph of G is contained in a component of H. For a subgraph H of G, the graph G/H is obtained from G by identifying the two ends of each edge in H and then deleting the resulting loops. The contraction G/H is called the *reduction* of G if H is the maximal collapsible subgraph of G. For $v \in V(G/H)$ and $G_1 \subset G/H$, denote $PM(v) = H_1$ if v is obtained by contracting a subgraph H_1 of G and $PM(G_1) = H_2$ if G_1 is obtained by contracting a subgraph H_2 of G. A graph G is *reduced* if it is the reduction of itself. Let $F(G)$ denote the minimum number of edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. The following summarizes some of the former results concerning collapsible graphs.

Theorem 2.1. *Let G be a connected graph. Each of the following holds.*

(i) *(Catlin [8]) If H is a collapsible subgraph of G, then G is collapsible if and only if G/H is collapsible; G is super-Eulerian if and only if G/H is super-Eulerian.*
(ii) *(Catlin, Theorem 5 of [8]) A graph G is reduced if and only if G contains no non-trivial collapsible subgraphs. As cycles of length less than 4 are collapsible, a reduced graph does not have a cycle of length less than 4.*
(iii) *(Catlin [9]) If G is reduced and if $|E(G)| \geq 3$, then $\delta(G) \leq 3$ and $F(G) = 2|V(G)| - |E(G)| - 2$.*
(iv) *(Catlin et al. [10]) Let G be a connected reduced graph. If $F(G) \leq 2$, then $G \in \{K_1, K_2, K_{2,t}\}$ $(t \geq 1)$.*

Let G be a connected, essentially 3-edge-connected graph such that $L(G)$ is not a complete graph. The *core* of this graph G, denoted by G_0, is obtained by deleting all the vertices of degree 1 and contracting exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$.

Lemma 2.2 (*Shao [11]*). *Let G be a connected, essentially* 3*-edge-connected graph G.*

(i) *G_0 is uniquely defined, and $\kappa'(G_0) \geq 3$.*
(ii) *If G_0 is super-Eulerian, then $L(G)$ is Hamiltonian.*

3. Hamiltonicity of 3-connected line graphs

Let G' be the reduction of G. Since contraction does not decrease the edge connectivity of G, G' is either a k-edge connected graph or a trivial graph if G is k-edge connected. Assume that G' is the reduction of a 3-edge-connected graph and non-trivial. It follows from Theorem 2.1(iv) and G' is 3-edge connected that $F(G') \geq 3$. Then by Theorem 2.1(iii), we have $|E(G')| \leq 2|V(G')| - 5$. Denote by $D_i(G)$ and $d_i(G)$ the set of vertices of degree i and $|D_i(G)|$, respectively. For $X \subset V(G)$, denote $[X, V(G) \setminus X]$ the set of edges with one endvertex contained in X and the other one contained in $V(G) \setminus X$. Moreover, we also use $[G[X], G[V(G) \setminus X]]$ for the set $[X, V(G) \setminus X]$ if there is no confusion, where $G[X]$ denotes the subgraph induced by vertex set X.

Lemma 3.1. *Let G be a reduced* 3*-edge-connected non-trivial graph. Then $d_3 \geq 10$.*

Proof. Since $F(G) \geq 3$, we have

$$4|V(G)| - 10 \geq 2|E(G)| = \sum id_i \geq 3d_3 + 4(|V(G)| - d_3) = 4|V(G)| - d_3.$$

Thus, $d_3 \geq 10$. □

A subgraph of G isomorphic to a $K_{1,2}$ or a 2-cycle is called a 2-path or a P_2 subgraph of G. An edge cut X of G is a *P_2-edge-cut* of G if at least two components of $G - X$ contain 2-paths. By the definition of a line graph, for a graph G, if $L(G)$ is not a complete graph, then $L(G)$ is essentially k-connected if and only if G does not have a P_2-edge-cut with size less than k. Since the core G_0 is obtained from G by contractions (deleting a pendant edge is equivalent to contracting the same edge), every P_2-edge-cut of G_0 is also a P_2-edge-cut of G.

Lemma 3.2. *Let G be a 3-edge-connected graph. If $L(G)$ is essentially 4-connected, then $L(G)$ is 4-connected.*

Proof. Since G is 3-edge connected, the minimum degree of G is at least 3. Thus, the minimum degree of $L(G)$ is at least 4. Noticing that $L(G)$ is essentially 4-connected. Thus, there is no vertex cut with less than 4 vertices, that is, $L(G)$ is 4-connected. □

Corollary 3.3. *Let G be a 3-edge-connected graph. If $L(G)$ is essentially 4-connected, then G is essentially 4-edge connected.*

Lemma 3.4. *Let G be a 3-edge-connected graph with at most 9 vertices of degree 3. If $L(G)$ is essentially 4-connected, then G is collapsible.*

Proof. Let G' be the reduction of G. If G' is trivial, we are done. Assume, G' is non-trivial. Note that G contains at most 9 vertices of degree 3. By Lemma 3.1, there is a non-trivial vertex of degree 3 in G', say u. Then $|E(PM(u))| \geq 2$, and so $[PM(u), V(G)-PM(u)]$ is an essential edge-cut with three edges in G. It contradicts to Corollary 3.3. Thus, G is collapsible. □

Note that Petersen graph is not collapsible. Then all conditions of Lemma 3.4 are sharp.

Theorem 3.5. *Let $L(G)$ be a 3-connected, essentially 4-connected line graph of the graph G. If $d_3(G) \leq 9$, then $L(G)$ is Hamiltonian.*

Proof. Let G be a graph with at most 9 vertices of degree 3 such that $L(G)$ is 3-connected, essentially 4-connected. Then by Lemma 2.2, the core of G is 3-edge-connected with at most 9 vertices of degree 3. By Lemma 3.4, the core of G is collapsible. By Lemma 2.2, $L(G)$ is Hamiltonian. □

We shall show that all conditions of Theorem 3.5 are sharp.

We first show that the condition "3-connected" is sharp by the following example. Let u, v be the vertices of degree $2k+3$ in $K_{2,2k+3}$. Denote by $K'_{2,2k+3}$ the graph obtained by subdividing all edges incident with u. Clearly, $L(K'_{2,2k+3})$ is 2-connected, essentially $(2k+3)$-connected, but it is not Hamiltonian.

Second, let P' be the graph obtained by subdividing each edge of the Petersen graph exactly once. We add at least two pendant edges on each vertex of degree 3 in P', and denote the resulting graph by P''. Clearly, $L(P'')$ is a 3-connected, essentially 3-connected graph without a Hamiltonian cycle, then the condition "essentially 4-connected" is sharp.

Third, the following example shows that the condition "$d_3(G) \leq 9$" in Theorem 3.5 is sharp: Petersen graph P has a perfect matching M with five edges. We construct a new graph P' by subdividing the five edges in M. Clearly, the resulting graph P' contains no dominating circuit (the dominating circuit of P' implies a Hamiltonian cycle of P). Thus, $L(P')$ is not Hamiltonian. It is not difficult to see that $L(P')$ is 3-connected, essentially 4-connected (this example is a special case of the following counterexamples; see the detailed proof below).

We will construct an infinite family of counterexamples for Conjecture 1.3. Two known results are needed.

Lemma 3.6 (*Fleischner and Jackson Corollary 1 [12]*). *A cubic graph is cyclically 4-edge connected if and only if it is essentially 4-edge connected.*

Theorem 3.7 (*Petersen's Theorem, Corollary 5.4 [1]*). *Any bridgeless cubic graph has a perfect matching.*

Now let us construct an infinite family of counterexamples for Conjecture 1.3. Let G be a snark. Noticing that G has a perfect matching M. We construct a new graph G' by subdividing the edges in M, i.e., replacing each edge of M by a path of length 2. Note that G is clearly non-Hamiltonian (otherwise, it will be of class one), then G' has no dominating circuit. Therefore $L(G')$ is not Hamiltonian. By Lemma 3.6, $L(G')$ is 3-connected, essentially 4-connected (otherwise, $L(G')$ is 3-connected, essentially 3-connected. Therefore, an essential-cut with three vertices of $L(G')$ induces an essential edge-cut of G by contracting one of the edge of each P_2 added by the subdivision, where a contraction of an edge is obtained by identifying the two ends of and deleting the resulting loops).

What is the minimum integer k such that every 3-connected, essentially k-connected line graph has a Hamiltonian cycle? The problem is still open. By the above remark, we have $5 \leq k \leq 11$. In particular, the next candidate will be $k = 5$.

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