# Spanning Eulerian Subgraphs in Generalized Prisms*

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**Abstract**

For a graph $G$ with vertices labeled $1, 2, \cdots, n$ and a permutation $\alpha$ in $S_n$, the symmetric group on $\{1, 2, \cdots, n\}$, the *$\alpha$-generalized prism* over $G$, $\alpha(G)$, consists of two copies of $G$, say $G_x$ and $G_y$, along with the edges $(x_i, y_{\alpha(i)})$, for $1 \leq i \leq n$. In [10], the importance of building large graphs by using generalized prisms is indicated. A graph $G$ is *supereulerian* if it has a spanning eulerian subgraph. In this note, we consider results of the form that if $G$ has property $P$, then for any $\alpha \in S_{|V(G)|}$, $\alpha(G)$ is supereulerian. As a result, we obtain a few properties of $G$ which implies that for any $\alpha \in S_{|V(G)|}$, $\alpha(G)$ is supereulerian. Also, while the permutations are restricted, the related result is discussed.

**Keywords:** generalized prisms, supereulerian, eulerian subgraphs

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# 1 Introduction

We use [1] for terminology and notation not defined here, and consider loopless finite graphs only. For a graph $G$, let $O(G)$ denote the set of all vertices in $G$ with odd degrees. An *eulerian* graph is a connected graph $G$ with $O(G) = \emptyset$. An eulerian subgraph $H$ of a graph $G$ is *spanning* if $V(H) = V(G)$, and a graph is called *supereulerian* if it has a spanning eulerian subgraph. Then $K_1$ is both an eulerian and supereulerian graph. The collection of all supereulerian graphs will be denoted by $\mathscr{SL}$.

Let $S_n$ denote the permutation group of degree $n$. For a labeled graph $G$ with $V(G)=\{1, 2, \cdots, n\}$ and a permutation $\alpha$ in $S_n$, the symmetric group on $\{1, 2, \cdots, n\}$, the *$\alpha$-generalized prism* over $G$, $\alpha(G)$($\alpha(G)$ also called a permutation graph), consists of two copies of $G$, say $G_x$ and $G_y$ with vertex sets $V(G_x) = \{x_1, x_2, \cdots, x_n\}$ and $V(G_y) = \{y_1, y_2, \cdots, y_n\}$, along with the permutation edges $(x_i, y_{\alpha(i)})$, for $1 \leq i \leq n$. Generalized prisms were introduced by Chartrand and Harary[5] who were interested in finding those which are planar. Other properties of generalized prisms which have been examined include crossing number[13], chromatic number[2], [7], [8], edge-chromatic number[6], [14], and cut frequency vectors[11]. In [10], the importance of building large graphs by using generalized prisms is indicated.

In [9], Klee studied the Hamiltonian properties of generalized prisms. In this note we investigate sufficient conditions for the supereulerian properties of generalized prisms and consider results of the form that if $G$ has property $P$, then for any $\alpha \in S_{|V(G)|}$, $\alpha(G)$ is supereulerian.

Determining whether a graph is a supereulerian graph has been shown to be a NP-Completely problem in [12]. In 1988, Catlin P. A. presented a *contraction method* to determine whether a graph is a supereulerian graph, which interested many researchers. In the next section, we will review Catlin's contraction method first.

# 2 Collapsible graphs and reduced graphs

A graph $G$ is *collapsible* if for every set $R \subseteq V(G)$ with $|R|$ even, there is a spanning connected subgraph $H_R$ of $G$, such that $O(H_R) = R$. Thus $K_1$ is both supereulerian and collapsible. Denote the family of collapsible graphs by $\mathscr{CL}$. Let $G$ be a collapsible graph and let $R = \emptyset$. Then by the definition, $G$ has a spanning connected subgraph $H$ with $O(H) = \emptyset$, and so $G$ is supereulerian. Therefore, we have $\mathscr{CL} \subset \mathscr{SL}$.

For a graph $G$ with a connected subgraph $H$, the contraction $G/H$ is the graph obtained from $G$ by replacing $H$ by a vertex $v_H$, such that the number of edges in $G/H$ joining any $v \in V(G)-V(H)$ to $v_H$ in $G/H$ equals the number of edges joining $v$ in $G$ to $H$. The subgraph $H$ is called the *preimage* of $v_H$. $v_H$ is *nontrivial* if $E(H) \neq \emptyset$, otherwise $v_H$ is *trivial*.

In [3], Catlin showed that every graph $G$ has a unique collection of pairwise disjoint maximal collapsible subgraphs $H_1, H_2, ..., H_c$. The contraction of $G$ obtained from $G$ by contracting each $H_i$ into a vertex$(1 \leq i \leq c)$, is called the *reduction* of $G$. A graph is *reduced* if it is the reduction of some other graph.

**Theorem 1** (Catlin, Theorem 8 of [3]) Let $H$ be a collapsible subgraph of a graph $G$, then $G \in \mathscr{SL}$ if and only if $G/H \in \mathscr{SL}$.

**Corollary 1** Graph $G$ is collapsible if and only if the reduction of $G$ is $K_1$.

Let $F(G)$ denote the minimum number of extra edges that must be added to $G$ so that the resulting graph has two edge-disjoint spanning trees.

**Theorem 2** (Catlin, Han and Lai, Theorem 1.3 of [4]) Let $G$ be a connected graph. If $F(G) \leq 2$, then either $G$ is collapsible, or the reduction of $G$ is a $K_2$ or a $K_{2,t}$ for some integer $t \geq 1$.

# 3 Main results

**Definition 1** Let $k \geq 0$ be an integer. $G \in \mathscr{F}_k$ if and only if for any $S \subseteq V(G)$ with $|S| = 2k$, $G$ has a connected spanning subgraph $H$ such that $O(H) = S$. Let $\mathscr{F} = \bigcup_{k \geq 1} \mathscr{F}_k$.

**Observation 1** $\mathscr{CL} = \bigcap_{k \geq 0} \mathscr{F}_k$, $\mathscr{SL} = \mathscr{F}_0$.

**Theorem 3** Let $G$ be a connected graph and $|V(G)| = n$. If $G \in \mathscr{F}$ then for any $\alpha \in S_n$, $\alpha(G) \in \mathscr{SL}$.

**Proof:** for any $\alpha \in S_n$, let $G_x$ and $G_y$ denote the two copies of $G$ in $\alpha(G)$. By the assumption, $G_x \in \mathscr{F}$, then there exists a integer $k > 0$, such that for any $S_x \subseteq V(G_x)$ with $|S_x| = 2k$, $G_x$ has a connected spanning subgraph $H_x$ with $O(H_x) = S_x$. Let $S_x = O(H_x) = \{v_{i_1}, v_{i_2}, \cdots, v_{i_{2k}}\}$. In $\alpha(G)$, let $S_y = \{v_{\alpha(i_1)}, v_{\alpha(i_2)}, \cdots, v_{\alpha(i_{2k})}\}$. Since $G_y \cong G \in \mathscr{F}$, $G_y$ has a connected spanning subgraph $H_y$ such that $O(H_y) = S_y$. Let $E_k = \{v_{i_j} v_{\alpha(i_j)} | j = 1, 2, \cdots, 2k\}$. Hence $\alpha(G)[E(H_x) \cup E(H_y) \cup E_k]$ is the spanning eulerian subgraph in $\alpha(G)$. Thus $\alpha(G) \in \mathscr{SL}$. □

Conversely, that for any $\alpha \in S_n$, $\alpha(G) \in \mathscr{SL}$, does not imply $G \in \mathscr{F}$. The following is a counterexample. In the figure 1, the graph $G$ is $K_4$ adding a vertex of degree one. Since $K_4 \in \mathscr{CL}$, then the reduction of $G$ is

$K_2$. Let $G^*$ denote the graph obtained by contracting the two copies of $G$ in $\alpha(G)$. Then 4-cycle is the spanning eulerian subgraph of $G^*$. Hence by Theorem 1, for any $\alpha \in S_{|V(G)|}$, $\alpha(G)$ is supereulerian. But $G \notin \mathscr{F}$ since for any even subset $S \subseteq V(G)$, whenever $S$ does not contain the vertex of degree one, $G$ cannot have a spanning connected subgraph with $O(H) = S$.

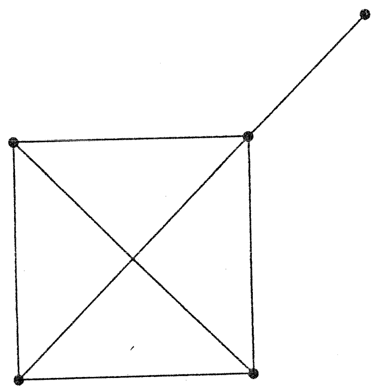


Figure 1: Graph $G$

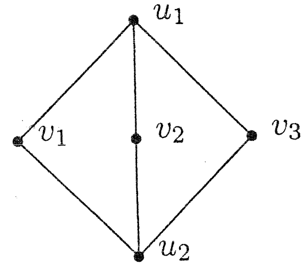



Figure 2: Graph $K_{2,3}$

**Lemma 1** $K_{2,t} \in \mathscr{F}_1$, where $t \geq 3$ is an odd integer.

**Proof:** Let $(X, Y)$ be a bipartition of $K_{2,t}$, where $X = \{u_1, u_2\}$ and $Y = \{v_1, v_2, \cdots, v_t\}$(As an example, $K_{2,3}$ is shown in Fig. 2). To show $K_{2,t} \in \mathscr{F}_1$, we only need to show for arbitrary distinct two vertices $u, v \in V(K_{2,t})$, $K_{2,t}$ has a spanning eulerian subgraph $H$ with $O(H) = \{u, v\}$.

**Case 1** $u, v \in X$

Let $u = u_1$ and $v = u_2$. Since $t$ is an odd integer, $K_{2,t}$ is a spanning eulerian subgraph which odd vertex set is $\{u, v\}$.

**Case 2** $u, v \in Y$

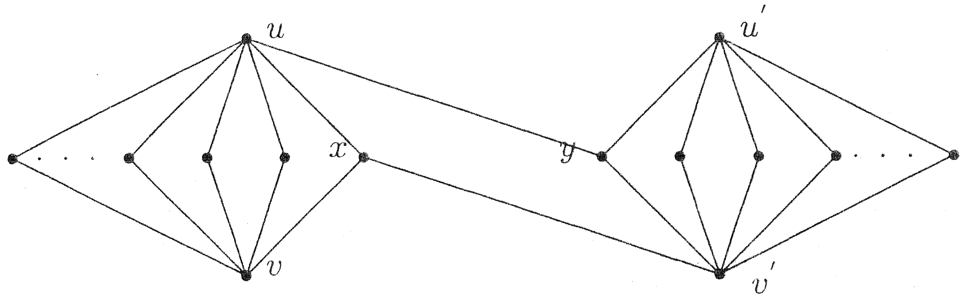
Let $u = v_i$ and $v = v_j$, $1 \leq i \leq 2$, $1 \leq j \leq t$. Since $i \neq j$, $K_{2,t} - u_1v_i - u_2v_j$ is a spanning eulerian subgraph which odd vertex set is $\{u, v\}$.

**Case 3** $u \in X, v \in Y$

Let $u = u_i$ and $v = v_j$, then $K_{2,t} - u_iv_j$ is a spanning eulerian subgraph which odd vertex set is $\{u, v\}$, $1 \leq i \leq 2$, $1 \leq j \leq t$.

**Case 4** $v \in X, u \in Y$

The result is obtained similarly.□




Graph $G_x^{'}$ Graph $G_y^{'}$

Figure 3: One case of $\alpha(G)^*$

**Theorem 4** Let $G$ be a graph with $|V(G)| \geq 2$ and $F(G) \leq 2$ . If $G$ has at most one cut edge, then for any $\alpha \in S_{|V(G)|}$, $\alpha(G)$ is supereulerian.

**Proof:** Let $\alpha \in S_{|V(G)|}$ be a permutation. Let $G^{'}$ be the reduction of $G$. $\alpha(G)^*$ denotes the graph obtained by contracting the two copies of $G$ in $\alpha(G)$. By Theorem 1, $\alpha(G) \in \mathscr{SL}$ if and only if $\alpha(G)^* \in \mathscr{SL}$. By Theorem 2, $G^{'}$ must be $K_1$ or $K_2$ or $K_{2,t}$ for some integer $t \geq 1$. $G$ has at most one cut edge implies $t \geq 2$.

Let $G_x$ and $G_y$ be two copies of $G$, $G_x^{'}$ and $G_y^{'}$ the reductions of $G_x$ and $G_y$, respectively. For every $v \in V(G^{'})$, let $H_v$ denote the preimage of $v$.

**Case 1** $G^{'}$ is $K_1$.

Since 2-cycle is collapsible, then for any $\alpha \in S_{|V(G)|}$, $\alpha(G) \in \mathscr{CL}$ by corollary 1. Thus $\alpha(G)$ is supereulerian.

**Case 2** $G^{'}$ is $K_2$.

Thus 4-cycle is the spanning eulerian subgraph of $\alpha(G)^*$. Hence by Theorem 1, for any $\alpha \in S_{|V(G)|}$, $\alpha(G)$ is supereulerian.

**Case 3** $G^{'}$ is $K_{2,t}$ for some odd integer $t \geq 3$.

We choose vertex $u \in V(G_x^{'})$ such that for every $v \in V(G_x^{'})$, $|V(H_u)| \geq |V(H_v)|$. Select vertex $v \in V(G_x^{'})$ such that $v \neq u$. Thus we can pick two distinct vertices $u^{'} \in V(G_y^{'})$ and $v^{'} \in V(G_y^{'})$. There exist four vertices $x_1 \in V(H_u)$, $x_2 \in V(H_v)$, $y_1 \in V(H_{u^{'}})$, $y_2 \in V(H_{v^{'}})$, such that $\alpha(x_1) = y_1$ and $\alpha(x_2) = y_2$. By Lemma 1, $G^{'} \in \mathscr{F}_1$, then there exists an open eulerian trail $L_x$ in $G_x^{'}$ whose origin is $u$ and whose terminus is $v$. Similarly in $G_y^{'}$ there exists an open eulerian trail $L_y$ whose origin is $u^{'}$ and whose terminus is $v'$. Note that edges $e_1 = (x_1, y_1)$ and $e_2 = (x_2, y_2)$ are also edges of $\alpha(G)^*$. Thus $\alpha(G)^*[E(L_x) \cup E(L_y) + e_1 + e_2]$ is a spanning eulerian subgraph of $\alpha(G)^*$. Hence by Theorem 1, for any $\alpha \in S_{|V(G)|}$, $\alpha(G)$ is supereulerian.

**Case 4** $G'$ is $K_{2,t}$ for some even integer $t \geq 2$.

Note that in this case, $G'$ is supereulerian. Let $E'$ denotes the set of permutation edges of $\alpha(G)$, $D_x$ and $D_y$ the set of all vertices with degree 2 of $G'_x$ and $G'_y$, respectively.

**Subcase 4.1** As shown in figure 3, there exists some vertex $y \in D_y$ such that one of $u$ and $v$(say, $u$) is adjacent to $y$, e.g., $e_1 = uy \in E'$.

Since $|V(H_u)|+|V(H_v)| = |V(H_{u'})|+|V(H_{v'})|$, there exists some vertex $x \in D_x$ which is adjacent to one of $u'$ and $v'$, say $v'$, e.g., $e_2 = xv' \in E'$. Thus $\alpha(G)^*[(E(G'_x) - ux) \cup (E(G'_y) - yv') + e_1 + e_2]$ is a spanning eulerian subgraph of $\alpha(G^*)$. Hence by Theorem 1, for any $\alpha \in S_{|V(G)|}$, $\alpha(G)$ is supereulerian.

**Subcase 4.2** In $\alpha(G)^*$, for any $w \in D_x$ and any $w' \in D_y$, $uw' \notin E'$, $vw' \notin E'$, $u'w \notin E'$ and $v'w \notin E'$.

As shown in figure 4, one of $uu'$ and $uv'$ must be in $E'$, say, $uu'$ $E'$. For every $x \in D_x$, there is $y \in D_y$ such that $xy \in E'$. Thus $\alpha(G)^*[(E(G'_x) - ux) \cup (E(G'_y) - u'y) + uu' + xy]$ is a spanning eulerian subgraph of $\alpha(G)^*$. Hence by Theorem 1, for any $\alpha \in S_{|V(G)|}$, $\alpha(G)$ is supereulerian. □

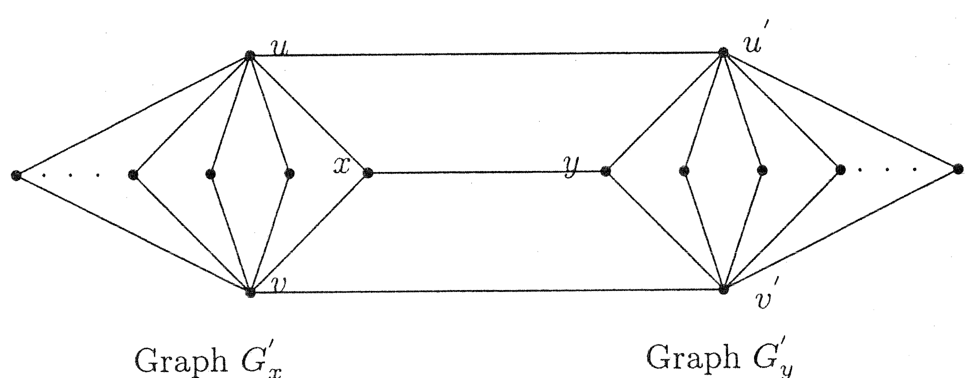



Figure 4: Another case of $\alpha(G)^*$

In general, a supereulerian graph $G$ does not imply that for any $\alpha \in S_{|V(G)|}$, $\alpha(G)$ is supereulerian. The Peterson graph is a counterexample since it can be regarded as a $\alpha(G)$ when $G$ is a 5-cycle. But if the permutations are restricted, we can get the following result.

**Theorem 5** Let $G \in \mathscr{SL}$ and $\alpha \in S_{|V(G)|}$. If there exist two vertices $u, v \in V(G)$, $uv \in E(G)$, such that $(\alpha(u), \alpha(v)) \in E(\alpha(G))$, then $\alpha(G) \in \mathscr{SL}$.

**Proof:** Let $G_x$ and $G_y$ denote the two copies of $G$ in $\alpha(G)$. By the assumption, let $e_x = uv \in E(G_x)$ and $e_y = (\alpha(u), \alpha(v)) \in E(G_y)$. Since $G$ is supereulerian, then there exist spanning eulerian subgraph $H_x$ and $H_y$ in $G_x$ and $G_y$ respectively. If $e_x \in E(H_x)$ and $e_y \in E(H_y)$, then $\alpha(G)[(E(H_x) - e_x) \cup (E(H_y) - e_y) + (u, \alpha(u)) + (v, \alpha(v))]$ is a spanning eulerian subgraph in $\alpha(G)$. If $e_x \notin E(H_x)$ and $e_y \notin E(H_y)$, then $\alpha(G)[(E(H_x) \cup (E(H_y)) + (u, \alpha(u)) + (v, \alpha(v))]$ is a spanning eulerian subgraph in $\alpha(G)$. If one of $e_x$ and $e_y$ is in $E(H_x) \cup E(H_y)$ and the other is not, say, $e_x \in E(H_x)$ and $e_y \notin E(H_y)$, then $\alpha(G)[(E(H_x) - e_x) \cup (E(H_y)) + (u, \alpha(u)) + (v, \alpha(v))]$ is a spanning eulerian subgraph in $\alpha(G)$. This completes the proof. □

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