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Degree condition and Z_3-connectivity

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ABSTRACT

Let G be a 2-edge-connected simple graph on $n \geq 3$ vertices and A an abelian group with $|A| \geq 3$. If a graph G^* is obtained by repeatedly contracting nontrivial A-connected subgraphs of G until no such a subgraph left, we say G can be A-reduced to G^*. Let G_5 be the graph obtained from K_4 by adding a new vertex v and two edges joining v to two distinct vertices of K_4. In this paper, we prove that for every graph G satisfying $\max\{d(u), d(v)\} \geq \frac{n}{2}$ where $uv \notin E(G)$, G is not Z_3-connected if and only if G is isomorphic to one of twenty two graphs or G can be Z_3-reduced to K_3, K_4 or K_4^- or G_5. Our result generalizes the former results in [R. Luo, R. Xu, J. Yin, G. Yu, Ore-condition and Z_3-connectivity, European J. Combin. 29 (2008) 1587–1595] by Luo et al., and in [G. Fan, C. Zhou, Ore condition and nowhere zero 3-flows, SIAM J. Discrete Math. 22 (2008) 288–294] by Fan and Zhou.
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1. Introduction

Graphs in this paper are finite and may have multiple edges without loops. Terminology and notation not defined here are from [1]. Let H be a subgraph of a graph G and u a vertex of G. Denote by $d_H(u)$ the degree of u in H. When $H = G$, we write $d(u)$ for $d_G(u)$. Let H_1 and H_2 be two subgraphs of G such that $V(H_1) \cap V(H_2) = \emptyset$. Denote by $e_G(H_1, H_2)$ (or simply $e(H_1, H_2)$) the number of edges with one end vertex in H_1 and the other one in H_2. If $V(H_1) = \{a\}$, we use $e_G(a, H_2)$ (or simply $e(a, H_2)$) instead of $e_G(H_1, H_2)$. For simplicity, if V_1, V_2 are two subsets of $V(G)$ with $V_1 \cap V_2 = \emptyset$, we use $e_G(V_1, V_2)$ for $e_G(G[V_1], G[V_2])$. We similarly define $e(V_1, V_2)$ and $e(a, V_2)$. A simple graph G satisfies the *Ore-condition* [10] if for every $uv \notin E(G)$, $d(u) + d(v) \geq |V(G)|$. A vertex v is a k^+-*vertex* if $d(v) \geq k$. For simplicity, a 3-cycle on three vertices u, v and w is denoted by uvw.

Let G be a graph. For an orientation D of a graph G and for a vertex $v \in V(G)$, denote by $E^+(v)$ (or $E^-(v)$, respectively) the set of edges with tails (or heads, respectively) at v. It is known [5] that group connectivity is independent of the orientation of G. The subscript D may be omitted when D is understood from the context.

Let A denote a nontrivial abelian group with identity element 0, and let $A^* = A - \{0\}$. Define $F(G, A) = \{f : E(G) \to A\}$ and $F(G, A^*) = \{f : E(G) \to A^*\}$. For an $f \in F(G, A)$, the boundary of f is a mapping $\partial f : V(G) \to A$ defined by $\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e)$, for each $v \in V(G)$.

Tutte [12] first introduced the theory of nowhere-zero flows. The concept of group connectivity was introduced by Jaeger et al. in [5], where nowhere-zero flows were successfully generalized to group connectivity. We give these definitions below.

Let G be an undirected graph and A an abelian group with identity 0. A mapping $b : V(G) \to A$ is an *A-valued zero-sum mapping* on G if $\sum_{v \in V(G)} b(v) = 0$. Denote by $\mathcal{Z}(G, A)$ all A-valued zero-sum mappings on G. A graph G is *A-connected* if for each $b \in \mathcal{Z}(G, A)$, there is an $f \in F(G, A^*)$ such that $b = \partial f$. A graph G admits a *nowhere-zero A-flow* if there exists an $f \in F(G, A^*)$ such that $\partial f(v) \equiv 0$ for G.

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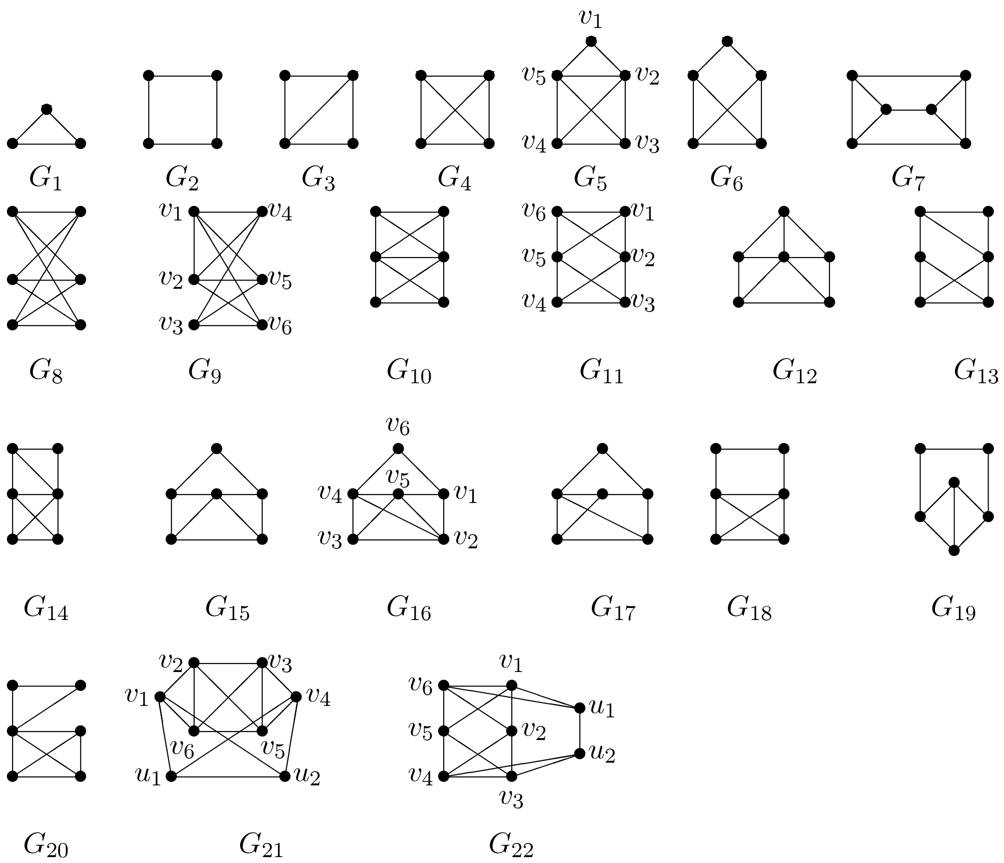

Fig. 1. Exceptional graphs for the main theorem.

A *contraction* of a graph G is the graph G' obtained from G by contracting a set of edges and deleting any loops generated in the process. When H is a subgraph of G, the contraction of G obtained by contracting the edges in H and deleting resulting loops is denoted by G/H. Note that each component of H becomes a vertex of G/H. A graph G is *A-reduced* if no nontrivial subgraph of G is A-connected. We say that a graph G_0 is an *A-reduction* of G if G_0 is A-reduced and if G_0 can be obtained from G by contracting all maximally A-connected subgraphs of G [7]. It is known that (Corollary 2.3 of [7]) the A-reduction of a graph is A-reduced and an A-reduction of a reduced graph is itself.

The following two conjectures on nowhere-zero flows and group connectivity are well-known.

Conjecture 1.1 (*Tutte, [12,15]*). *Every 4-edge-connected graph admits nowhere-zero Z_3-flow.*

Conjecture 1.2 (*Jaeger et al., [5]*). *Every 5-edge-connected graph is Z_3-connected.*

In order to approach these two conjectures, nowhere-zero 3-flows and Z_3-connectivity have been studied extensively. More recently, degree conditions are used to ensure the existence of nowhere-zero flows and group connectivity of graphs. For the literature for group connectivity, the readers can see the survey [8], and the results [14,13,16] and others. In particular, Fan and Zhou [4,3] investigated sufficient degree conditions for nowhere-zero Z_3-flows. Luo et al. [9] extended the result of Fan and Zhou [4] by characterizing all Z_3-connected graphs satisfying the Ore-condition.

Theorem 1.3 (*Luo et al. [9]*). *Let G be a simple graph satisfying the Ore-condition with at least three vertices. The graph G is not Z_3-connected if and only if G is one of G_i in Fig. 1, where $1 \leq i \leq 12$.*

Motivated by Conjectures 1.1 and 1.2 and Theorem 1.3, we will further investigate Z_3-connectivity by a given degree condition. To simplify the notation, for an integer $n \geq 3$, we define $\mathcal{F}$ to be the set of all simple 2-edge-connected graphs on n vertices such that $G \in \mathcal{F}$ if and only if $\max\{d(u), d(v)\} \geq \frac{n}{2}$ for every $uv \notin E(G)$. In this paper, we prove the following result.

Theorem 1.4. *Let $G \in \mathcal{F}$ on $n \geq 3$ vertices. The graph G is not Z_3-connected if and only if one of the following holds:*

(1) *G is isomorphic to one of 22 graphs in Fig. 1; or*
(2) *G can be Z_3-reduced to one of G_1, G_3, G_4 and G_5.*

Theorem 1.4 generalized the result of Luo et al. [9]. If a graph G satisfies the Ore-condition, then $\max\{d(u), d(v)\} \geq \frac{n}{2}$ for every pair of nonadjacent vertices u and v and so G satisfies the hypothesis of Theorem 1.4. Note that each of G_i, where $13 \leq i \leq 22$, contains a pair of nonadjacent vertices with the sum of their degree less than $|V(G_i)|$. Thus, G is isomorphic to none of $G_{13}, \ldots, G_{22}$. We now show that G cannot be Z_3-reduced to G_j for each $j \in \{1, 3, 4, 5\}$. Suppose otherwise that G is Z_3-reduced to G_j, where $j \in \{1, 3, 4, 5\}$. Let H be a nontrivial Z_3-connected subgraph of G and v_H be a vertex of G_i which H is contracted to. Since every Z_3-connected graph has at least 5 vertices and v_H has at most four neighbors in G_j, H contains at least one vertex u such that $d_G(u) \leq |V(H)| - 1$ and $e(u, G - V(H)) = 0$. If G_j has two vertices v_{H_1} and v_{H_2} such that two nontrivial Z_3-subgraphs H_1 and H_2 are contracted to, respectively, pick $u_1 \in V(H_1)$ and $u_2 \in V(H_2)$ satisfying $d(u_k) \leq |V(H_k)| - 1$ for $k = 1, 2$, and $u_1u_2 \notin E(G)$. If G has only one Z_3-connected subgraph H, pick a vertex u_1 with $d(u_1) \leq |V(H)| - 1$ such that $e(u, G - V(H)) = 0$, and $u_2 \in V(G) - V(H)$, then $u_1u_2 \notin E(G)$. In both cases, it is easy to see that $d(u_1) + d(u_2) < n$ and G does not satisfy the Ore-condition. This tells us that if G satisfies the Ore-condition, then G cannot be Z_3-reduced to none of G_1, G_3, G_4 and G_5. So, Theorem 1.4 extends Theorem 1.3.

As G_i admits a nowhere-zero 3-flow for each $i \in \{1, 2, 3, 5, 8, 11\}$, the argument above implies that G_j does not admit a nowhere-zero 3-flow if and only if $j \in \{4, 6, 7, 9, 10, 12\}$ and so the Fan's result follows from Theorem 1.4.

We organize this paper as follows. We establish several lemmas in Section 2. We prove Theorem 1.4 for small cases when $n \leq 8$ in Section 3 and the case when $n \geq 9$ in Section 4.

2. Lemmas

To simplify the notation, throughout the rest of this paper, we use $Z_3 = \{0, 1, 2\}$, and so equality concerning elements in Z_3 is to mean congruence modulo 3. We first state the Turán theorem.

Theorem 2.1 (*Turán, [11]*). *Let G be a simple graph on n vertices. If $|E(G)| \geq \frac{n^2}{4}$, then G contains a triangle or $G \cong K_{m,m}$, where m is a positive integer.*

Lemma 2.2 (*Lai, [6]*). *Let G be a graph and A an abelian group with $|A| \geq 3$. Then each of the following holds:*

(1) *K_1 is A-connected;*
(2) *if $e \in E(G)$ and if G is A-connected, then G/e is A-connected, and*
(3) *if H is a subgraph of G and if both H and G/H are A-connected, then G is A-connected.*

One notes that K_4 is not Z_3-connected. A nontrivial Z_3-connected simple graph G has $|V(G)| \geq 5$. Denote by C_n the cycle of length n. For every $n \geq 3$, we define $W_n = C_n + w$, where w is the center. A wheel W_n is *even* (or *odd*) if n is even (or odd).

Lemma 2.3 (*[2,5,6,9]*). *Let A be an abelian group. Then each of the following holds:*

(1) *both K_n and K_n^- are Z_3-connected if $n \geq 5$;*
(2) *C_n is A-connected if and only if $|A| \geq n + 1$;*
(3) *$K_{m,n}$ is Z_3-connected if $m \geq n \geq 4$;*
(4) *W_{2k} is Z_3-connected, where $k \geq 2$;*
(5) *if G is not Z_3-connected, then none of any spanning subgraph of G is Z_3-connected; and*
(6) *let G be a simple graph and H a nontrivial Z_3-connected subgraph of G. Then $|V(H)| \geq 5$.*

Let G be a graph and let u, v, w be three vertices of G with $uv, uw \in E(G)$. $G_{[uv,uw]}$ is defined to be the graph obtained from G by deleting two edges uv and uw and adding one edge vw. It is clear that $d_{G_{[uv,uw]}}(u) = d(u) - 2$.

Lemma 2.4 (*[2,6]*). *Let A be an abelian group. Let G be a graph and let u, v, w be three vertices of G with degree $d(u) \geq 4$ and $uv, uw \in E(G)$. If $G_{[uv,uw]}$ is A-connected, then so is G.*

Let A be an abelian group. Let H be a connected subgraph of G and let $V_1 = V(H), V_2 = V(G) - V(H)$. From the proof [8, Proposition 3.2], we obtain the following lemma.

Lemma 2.5 (*Lai, [6]*). *Let $b \in \mathcal{Z}(G, A)$. If there is a mapping $f \in F(G, A^*)$ such that $\partial f(v) = b(v)$, then define $b' : V_2 \to A$ by*

$$b'(v) = \begin{cases} b(v), & \text{if } v \in V_2 - N(H), \\ b(v) - \sum\limits_{e \in E^-(v) \cap E(V_1, V_2)} f(e) + \sum\limits_{e \in E^+(v) \cap E(V_1, V_2)} f(e) & \text{if } v \in N(H) \cap V_2. \end{cases}$$

Then for such a $b' \in \mathcal{Z}(G - H, A)$, there is a mapping $f' : G - H \to A^*$ such that $\partial f'(v) = b'(v)$ for each $v \in V_2$.

Lemma 2.6. *Both Γ_1 and Γ_2 in Fig. 2 are Z_3-connected.*

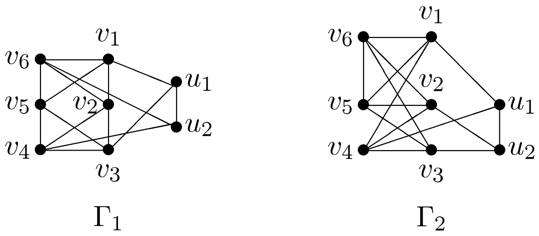

Fig. 2. Two Z_3-connected graphs.

Proof. Let $\Gamma = \Gamma_2$ and $\Gamma' = \Gamma_{[v_2v_5, v_2v_6]}$. It is easy to verify that Γ' can be Z_3-reduced to K_1 which is Z_3-connected. By Lemma 2.4, G is Z_3-connected.

Let $\Gamma = \Gamma_1$ and $\Gamma' = \Gamma_{[v_2v_3, v_2v_4]}$. Then Γ' contains a 2-cycle (v_3, v_4). We contract this 2-cycle to a new vertex v^* and then we get another 2-cycle (v^*, v_5). We contract this 2-cycle into another new vertex v^{**}. In this time, we get an even wheel W_4 induced by $v^{**}, v_6, v_1, u_1, u_2$ with the center at v^{**}. We contract this W_4 into one vertex and also get a 2-cycle. Contracting this 2-cycle, finally we get a K_1 which is Z_3-connected. By Lemma 2.3(2) and (4), and by Lemma 2.4, Γ_1 is Z_3-connected. $\square$

The following lemma is from the survey on group connectivity and group coloring by Lai et al. [8].

Lemma 2.7. *Let G be a graph and $v \in V(G)$ with $d_G(v) = 2$. Then G is Z_3-connected if and only if $G - v$ is Z_3-connected.*

Lemma 2.8. *None of G_{16}, G_{19}, G_{21} and G_{22} is Z_3-connected.*

Proof. We shall use the same notation for the labeling of the vertices of these graphs as in Fig. 1. Recall that K_4 does not have a nowhere-zero 3-flow, and so cannot be Z_3-connected.

Since $G_{16} - \{v_1, v_6\}$ is a K_4, which is not Z_3-connected, by Lemma 2.7, G_{16} is also not Z_3-connected.

Since G_{19} can be contracted to K_4, and since K_4 does not have a nowhere-zero Z_3-flow, by Lemma 2.2(2), G_{19} is no Z_3-connected.

We now show that G_{21} is not Z_3-connected. Suppose otherwise that G_{21} is Z_3-connected. By the definition, for a $b \in \mathcal{Z}(G_{21}, Z_3)$ by $b(u_1) = b(u_2) = 0$, $b(v_1) = b(v_3) = b(v_5) = 1$ and $b(v_2) = b(v_4) = b(v_6) = 2$, there is an $f \in \mathcal{Z}(G_{21}, Z_3)$ such that $\partial f = b$. Recall that group connected is independent of orientations. We assume that u_1u_2 is oriented from u_1 to u_2; u_1v_1 is from v_1 to u_1; u_1v_4 from v_4 to u_1; u_2v_1 from u_2 to v_1; u_2v_4 from u_2 to v_4. If $f(u_1u_2) = \lambda \in Z_3^*$, then $f(v_1u_1) = f(v_4u_1) = f(u_2v_1) = f(u_2v_4) = \mu \in Z_3 - \{0, \lambda\}$.

Note that $f(u_2v_1) = f(v_1u_1)$ and $f(u_2v_4) = f(v_4u_1)$. By Lemma 2.5, there is a mapping $f' : V(G) - \{u_1, u_2\} \to Z_3^*$ such that $\partial f'(v_i) = b(v_i)$, where $1 \le i \le 6$.

We assume that v_6v_1 is oriented from v_6 to v_1, v_1v_2 is from v_1 to v_2; v_3v_4 is from v_3 to v_4; v_4v_5 is from v_4 to v_5. $b(v_1) = 1$ implies that $f'(v_6v_1) = 1$ and $f'(v_1v_2) = 2$; $b(v_4) = 2$ implies that $f'(v_3v_4) = 2$ and $f'(v_4v_5) = 1$. Let $G^* = G_{21} - \{u_1, u_2, v_1, v_4\}$. By Lemma 2.5, there is a $b'' \in \mathcal{Z}(G^*, Z_3)$ with $b''(v_i) = 0,\ i = 2, 3, 5, 6$, which implies that K_4 admits nowhere-zero Z_3-flow. This contradiction proves that G_{21} is not Z_3-connected.

It remains to show that G_{22} is not Z_3-connected. Suppose otherwise that G_{22} is Z_3-connected. By the definition, for a $b \in \mathcal{Z}(G_{22}, Z_3)$ with $b(v_i) = 2, i = 1, 2, \ldots, 6$ and $b(u_j) = 0, j = 1, 2$, there is an $f \in F(G_{22}, Z_3^*)$ such that $\partial f = b$. Assume that u_1u_2 is oriented from u_2 to u_1; u_1v_1 is from u_1 to v_1; u_1v_6 from u_1 to v_6; u_2v_3 from v_3 to u_2; v_4u_2 from v_4 to u_2.

Let $f(u_1u_2) = \lambda \in Z_3^*$. Then $f(u_1v_1) = f(u_1v_6) = f(u_2v_3) = f(u_2v_4) = \mu \in Z_3 - \{0, \lambda\}$. Let $G' = G_{22} - \{u_1, u_2\}$ and define $b' : V(G') \to Z_3$ by $b'(v_1) = b(v_1) - \mu = 2 - \mu$; $b'(v_2) = b(v_2) = 2$; $b'(v_3) = b(v_3) + \mu = 2 + \mu$; $b'(v_4) = b(v_4) + \mu = 2 + \mu$; $b'(v_5) = b(v_5) = 2$ and $b'(v_6) = b(v_6) - \mu = 2 - \mu$. It is easy to see that $b'(v_3) = b'(v_4) = 0$ or $b'(v_1) = b'(v_6) = 0$ depends on $\mu = 1$ or $\mu = 2$. By symmetry of G', we assume that $\mu = 1$. In this case, $b'(v_1) = 1, b'(v_2) = 2, b'(v_3) = 0, b'(v_4) = 0, b'(v_5) = 2$ and $b'(v_6) = 1$.

Lemma 2.5 shows that for such a b', there is an $f' \in F(G', Z_3^*)$ with $\partial f' = b'$. Note that $b'(v_3) = 0$ and $b'(v_4) = 0$. All edges incident with v_3 are assumed to be oriented either into or from v_3, f' achieves 1 or 2 at these edges. In this case, all edges incident with v_4 must be oriented either from or into v_4, f' achieves 1 or 2 at these edges. In all cases, $G' - \{v_3, v_4\}$ is a $K_4 - v_2v_5$ with vertex set $\{v_1, v_2, v_5, v_6\}$ and $b'(v_1) = b'(v_6) = 1, b'(v_2) = b'(v_5) = 2$. We assume, without loss of generality, that two edges incident with v_2 (v_5) are oriented from v_2 (v_5). Since $b'(v_2) = b'(v_5) = 2$, f' achieves 1 on these four edges. f' cannot achieve any non-zero element of Z_3 on an edge v_1v_6 no matter how v_1v_6 is oriented. This contradiction proves that G_{22} is not Z_3-connected. $\square$

From Lemma 2.8 and Theorem 1.3, we obtain the following lemma.

Lemma 2.9. *None of $G_1, G_2, \ldots, G_{22}$ is Z_3-connected.*

Proof. Theorem 1.3 states that none of G_i, where $1 \le i \le 12$, is Z_3-connected. By Lemma 2.8, none of G_{16}, G_{19}, G_{21} and G_{22} is Z_3-connected. Since G_{13}, G_{14}, G_{18} and G_{20} are spanning subgraphs of G_{10}, G_{15} is a spanning subgraph of G_{12} and G_{17} is a spanning subgraph of G_{16}. By Lemma 2.3(5), none of $G_{13}, G_{14}, G_{15}, G_{17}, G_{18}$ and G_{20} is Z_3-connected. $\square$

3. The case when $n \leq 8$

Throughout this section, we assume that $G \in \mathcal{F}$ on n vertices. Define

$$X_G = \left\{u \in V(G) : d(u) < \frac{n}{2}\right\}. \tag{1}$$

Throughout the rest of this section, we assume that $X = X_G$. For simplicity, we define $Y = V(G) - X$. The following fact is straightforward.

Lemma 3.1. *(1) $G \in \mathcal{F}$ if and only if $G[X]$ is a complete subgraph of G.*
(2) If $G[Y]$ is Z_3-connected and $e(X, Y) \geq |X| + 1$, then G is Z_3-connected.

Lemma 3.2. *If G is not Z_3-connected and if $5 \leq n \leq 8$, then either $1 \leq |X| \leq \lfloor \frac{n}{2} \rfloor - 1$ or G is one of $G_7, G_8, G_9, G_{10}, G_{11}$ and G_{12}.*

Proof. Suppose otherwise that $|X| \geq \lfloor \frac{n}{2} \rfloor$. By Lemma 3.1, $d_{G[X]}(x) = |X| - 1$. Since G is connected, G has a vertex $x_0 \in X$ adjacent to a vertex not in X, and so $d(x_0) \geq |X| \geq \lfloor \frac{n}{2} \rfloor$. When n is even, $d(x_0) \geq \frac{n}{2}$ and this contradicts the definition of X. Thus, n is odd. If $|X| \geq \lfloor \frac{n}{2} \rfloor + 1$, since G is 2-edge connected, there is a vertex $x \in X$ such that $d(x) \geq \lfloor \frac{n}{2} \rfloor + 1 \geq \frac{n}{2}$. This contradiction shows that $|X| = \lfloor \frac{n}{2} \rfloor$. Then $|Y| = \lceil \frac{n}{2} \rceil$. In this case $|Y| = |X| + 1$ and for each vertex $x \in X$, $e(x, Y) \leq 1$. It implies that there is at least one vertex $y \in Y$ such that $d(y) \leq \lfloor \frac{n}{2} \rfloor$. This contradiction establishes $|X| \leq \lfloor \frac{n}{2} \rfloor - 1$.

If $X = \emptyset$, then $d(u) \geq \frac{n}{2}$ for each vertex $u \in V(G)$. In this case, G satisfies the Ore-condition, and G is one of $G_7, G_8, G_9, G_{10}, G_{11}$ and G_{12} by Theorem 1.3. □

Lemma 3.3. *Suppose that $3 \leq n \leq 5$. Then G is not Z_3-connected if and only if G is G_i in Fig. 1, where $1 \leq i \leq 6$.*

Proof. Since no simple graph of order at most 4 is Z_3-connected, $G \in \{G_1, G_2, G_3, G_4\}$. Thus, we may assume that $n = 5$. By Lemma 3.2, $|X| \leq 1$. If $X = \{x\}$, then $d(x) = 2$ and for each $y \in V(G) - X$, $d(y) \geq 3$, and so $G \in \{G_5, G_6\}$. Hence we assume that $X = \emptyset$. By Theorem 1.3, G is Z_3-connected or $G \in \{G_1, G_2, G_3, G_4\}$. □

Lemma 3.4. *Suppose that $n = 6$. Then G is not Z_3-connected if and only if G is G_i in Fig. 1, where $7 \leq i \leq 20$.*

Proof. By Lemma 3.2, $|X| \leq 2$. If $X = \emptyset$, then G is G_i, $7 \leq i \leq 12$, from Theorem 1.3. If $|X| = 2$, then as $\kappa'(G) \geq 2$, $d(v) = 2$ for each $v \in X$. Thus, $e(v, G - X) = 1$ for each $v \in X$. Thus there are at most two vertices $u_1, u_2 \in Y$ such that $d_{G[Y]}(u_i) = 2$, for $i = 1, 2$. In this case, $G \in \{G_{18}, G_{19}, G_{20}\}$.

Hence $X = \{v\}$. As $\kappa'(G) \geq 2$, $d(v) = 2$, and so $d_G(y) \geq 3$ for each $y \in Y$. By Lemma 2.7, G is Z_3-connected if and only if $G - v$ is. By Lemma 3.3, if $G - v$ has at most one vertex of degree 2, then $G \in \{G_{13}, G_{14}, G_{16}, G_{17}\}$. Hence we assume that $G - v$ has exactly two vertices of degree 2. Note that if $G - v$ has 3 vertices of degree 4, then $\delta(G - v) \geq 3$, which implies that G contains a K_5^- which is Z_3-connected, a contradiction. Since the number of odd degree vertices must be even, $G - v$ has exactly one vertex of degree 4. This forces that $G = G_{15}$. □

Lemma 3.5. *Suppose that $n = 7$. G is not Z_3-connected if and only if G is Z_3-reduced to K_3.*

Proof. If G is Z_3-reduced to K_3, by Lemma 2.2, G is not Z_3-connected. Thus, assume that G is not Z_3-connected. By Lemma 3.2 and Theorem 1.3, $0 < |X| \leq 2$. Suppose first that $X = \{v\}$. Then $d(v) \leq 3$ and for each vertex u of $G[Y]$, $d_{G[Y]}(u) \geq 3$. This means that $G[Y]$ satisfying the Ore-condition with $n = 6$. If $G[Y]$ is not Z_3-connected, by Theorem 1.3, then $G[Y]$ is one of $G_7, G_8, \ldots, G_{12}$. On the other hand, $G[Y]$ has at least three 4^+-vertices while each of $G_7, \ldots, G_{12}$ has at most two 4^+-vertices. This contradiction proves that $G[Y]$ is Z_3-connected and so is G, a contradiction.

Thus, we assume that $X = \{x_1, x_2\}$. Then $d(x_1) \leq 3$ and $d(x_2) \leq 3$. We first assume that $e(\{x_1, x_2\}, Y) \leq 2$. In this case, $d(x_1) = d(x_2) = 2$ and $e(\{x_1, x_2\}, Y) = 2$ since G is 2-edge connected. Moreover, $G^* = G - \{x_1, x_2\}$ contains at least three 4^+-vertices. It follows that G^* is K_5 or K_5^- which is Z_3-connected by Lemma 2.3(1). So G can be Z_3-reduced to K_3. Thus, $e(\{x_1, x_2\}, Y) \geq 3$. In the remainder of the proof we will use the following claim.

Claim. *Suppose that $e(\{x_1, x_2\}, Y) \geq 3$. If $u_1, u_2 \in Y$ such that $e(\{u_1, u_2\}, \{x_1, x_2\}) = 0$, then G is Z_3-connected.*

Let $G^* = G[Y] = G - \{x_1, x_2\}$. Then G^* has a degree sequence $d_1 \leq d_2 \leq d_3 \leq d_4 \leq d_5$ with $d_1 \geq 2$, $d_2 \geq 2$, $d_4 = d_5 = 4$. Thus, $G[Y]$ satisfies the Chvátal-condition and G^* contains a Hamilton cycle $C = y_1y_2y_3y_4y_5y_1$.

When $u_1u_2 = y_iy_{i+1}$, where the subscript i is taken modulo 5, G^* is isomorphic to K_5 or K_5^- which is Z_3-connected by Lemma 2.3(1). By Lemma 3.1, G is Z_3-connected.

Thus, we assume, without loss of generality, that $y_1 = u_1, y_3 = u_2$. Since $d_{G^*}(y_1) = d_{G^*}(y_3) = 4$, $y_1y_3, y_1y_4, y_3y_5 \in E(G^*)$. If either $y_2y_5 \in E(G^*)$ or $y_2y_4 \in E(G^*)$, then G^* contains an even wheel W_4. By Lemma 2.3(4), G^* is Z_3-connected and so is G. If both $y_2y_5 \notin E(G^*)$ and $y_2y_4 \notin E(G^*)$, then $x_1y_2, x_2y_2 \in E(G)$ and $e(y_i, \{x_1, x_2\}) \geq 1$, where $i = 4, 5$, since for each $y \in Y$, $d(y) \geq 4$. In this case, $G_{[y_5y_1, y_5y_3]}$ contains a 2-cycle. Contract this 2-cycle and recursively contract any new 2-cycle obtained in the process, finally we get a K_1 which is Z_3-connected. By Lemmas 2.2 and 2.4, G is Z_3-connected. So far, we have proved our claim.

Recall that G is not Z_3-connected. By Claim, let $e(\{x_1, x_2\}, Y) = 4$ and $|(N(x_1) \cup N(x_2)) \cap Y| = 4$. It follows that there exists $y^* \in Y$ such that $d_{G^*}(y) = 4$ and for each $y \in Y - \{y^*\}$, $d_{G^*}(y) \geq 3$ and hence $d_{G^*-y^*}(y) \geq 2$. By the Ore's Theorem, the subgraph induced by $Y - \{y^*\}$ is a 4-cycle. In this case, G^* contains an even wheel W_4 with the center at y^*. By Lemma 2.3(4), G^* is Z_3-connected and so is G, a contradiction. □

Lemma 3.6. *Suppose that $n = 8$. G is not Z_3-connected if and only if G can be Z_3-reduced to K_3 or K_4 or K_4^- or G is G_{22} or G_{21}.*

Proof. We shall use the same notation for the labeling of the vertices of the graphs in Fig. 1. If G can be Z_3-reduced to K_3 or K_4 or K_4^- or G is G_{22} or G_{21}, by Lemmas 2.2 and 2.9, G is not Z_3-connected. Thus, assume that G is not Z_3-connected. Let $d_1 \leq d_2 \leq \cdots \leq d_{|Y|}$ be a degree sequence of $G[Y]$. By Lemma 3.2 and Theorem 1.3, $0 < |X| \leq 3$.

Case 1. $X = \{x_1, x_2, x_3\}$.

It follows that $d_{G[X]}(x_i) = 2$ and $e(x_i, G-X) \leq 1$ for each x_i, $i = 1, 2, 3$. Since G is 2-edge connected, $3 \geq e(X, G-X) \geq 2$. If $|N(X) \cap Y| = 1$ or $|N(X) \cap Y| = 3$, then $G[Y] \in \mathcal{F}$ with $|V(G[Y])| = 5$. Since $G[Y]$ contains at least two 4^+-vertices, by Lemma 3.3, $G[Y]$ is Z_3-connected. When $e(X, G - X) = 3$, G can be Z_3-reduced to K_4. When $e(X, G - X) = 2$, G can be Z_3-reduced to K_4^-. Assume that $|N(X) \cap Y| = 2$. Then $d_1 \geq 2, d_2 \geq 3$ and $d_5 \geq d_4 \geq d_3 \geq 4$. Thus, $G[Y]$ satisfies the Chvátal-condition and $G[Y]$ is a Hamilton cycle $C = y_1y_2y_3y_4y_5y_1$. Since $|N(X) \cap Y| = 2$, there are two adjacent vertices y_i, y_{i+1} with $e(\{y_i, y_{i+1}\}, X) = 0$. In this case, $G[Y]$ contains an even wheel W_4 induced by $y_1, \ldots, y_5$ with the center vertex at y_i. By Lemma 2.3(4), $G[Y]$ is Z_3-connected and hence G can be Z_3-reduced to K_4^- since G is not Z_3-connected.

Case 2. $X = \{x_1, x_2\}$.

Since G is 2-edge connected, $4 \geq e(X, G-X) \geq 2$. Suppose first that $|N(X) \cap Y| = 4$. Then $d_1 \geq 3, d_2 \geq 3, d_3 \geq 3, d_4 \geq 3, d_6 \geq d_5 \geq 4$. Thus, $G[Y] \in \mathcal{F}$. Since G is not Z_3-connected, by Lemma 3.4, $G[Y]$ is one of G_i, where $7 \leq i \leq 20$. Since each vertex of $G[Y]$ is a 3^+-vertex and $G[Y]$ has at least two 4^+-vertices, $G[Y]$ is one of G_9, G_{10} and G_{11}. If $G[Y]$ is G_{11}, then G is isomorphic to Γ_1 or G_{22}. By Lemmas 2.6 and 3.1, G is G_{22}. Assume then that $G[Y]$ is G_{10}. By Lemmas 2.3(5), 2.6 and 3.1, Γ_1 is not a subgraph of G. Thus, G_{22} is a subgraph of G, that is, G is obtained from G_{22} by adding an edge v_2v_5 in Fig. 1. In this case, let $G' = G_{[v_3v_2, v_3v_5]}$. Then G' can be Z_3-reduced to K_1 which is Z_3-connected. By Lemmas 2.2 and 2.4, G is Z_3-connected, a contradiction. Thus, $G[Y]$ is G_9. Then G is isomorphic to Γ_2. By Lemma 2.6, G is Z_3-connected, a contradiction.

Suppose that $|N(X) \cap Y| = 3$. In this case, $d_1 \geq 2, d_2 \geq 3, d_3 \geq 3$ and $d_6 \geq d_5 \geq d_4 \geq 4$. It is easy to see that $G[Y] \in \mathcal{F}$. By Lemmas 3.1 and 3.4, $G[Y]$ is G_{16} with three vertices of degree 4. In this case, we assume, without loss of generality, that $x_1v_1, x_1v_6, x_2v_6, x_2v_3 \in E(G)$. Let $G' = G_{[v_3v_5, v_3v_2]}$. Then G' can be Z_3-reduced to K_1 which is Z_3-connected. By Lemmas 2.2 and 2.4, G is Z_3-connected, a contradiction.

Suppose then that $|N(X) \cap Y| = 2$. In this case, $d_1 \geq 2, d_2 \geq 2$ and $d_6 \geq d_5 \geq d_4 \geq d_3 \geq 4$. If $d_2 \geq 3$, then $G[Y] \in \mathcal{F}$. Thus, $d_1 = d_2 = 2$ and $d_6 \geq \cdots \geq d_3 \geq 4$. Let $y_1, y_2 \in Y$ such that $d_{G[Y]}(y_1) = d_{G[Y]}(y_2) = 2$. If $y_1y_2 \notin E(G[Y])$, then $G[Y] \in \mathcal{F}$. On the other hand, if $G[Y] \in \mathcal{F}$, since $G[Y]$ contains four 4^+-vertices, by Lemma 3.4, $G[Y]$ is Z_3-connected. Thus, we assume that $d_{G[Y]}(y_1) = d_{G[Y]}(y_2) = 2$ and $y_1y_2 \notin E(G[Y])$. In this case, G is G_{21}.

Case 3. $X = \{x\}$.

By the hypothesis, $2 \leq d(x) \leq 3$. In this case, $d_1 \geq 3, d_2 \geq 3, d_3 \geq 3$ and $d_7 \geq d_6 \geq d_5 \geq d_4 \geq 4$. Then $G[Y]$ satisfies the Chvátal-condition and $G[Y]$ has a Hamilton cycle $y_1y_2 \cdots y_7y_1$.

Suppose first that $d_7 \geq 5$. We assume, without loss of generality, that $d(y_1) = d_7$. Since $|Y| = 7$, there are y_j, y_{j+1} such that $y_1y_j, y_1y_{j+1} \in E(G[Y])$, where $j \neq 2, j+1 \neq 7$. Let $G' = G[Y]_{[y_1y_j, y_1y_{j+1}]}$. It follows that G' contains a 2-cycle (y_j, y_{j+1}). We contract this 2-cycle into a new vertex and recursively contract any new 2-cycle obtained in the process. Let G'' be the resulting graph from $G[Y]$. Then $|V(G'')| \leq 6$ and $\delta(G'')| \geq 2$. $\delta(G'') = 2$ if and only if $d(x) = 2$, $xy_j, xy_{j+1} \in E(G)$, $d(y_j) = 4$, $d(y_{j+1}) = 4$, $d_{G''}(v_H) = 2$, $d_{G''}(y_1) = d(y_1) - 2$, $d_{G''}(v) = 4$ for $v \in V(G'') - \{v_H, y_1\}$ and $|V(G'')| = 6$. Thus, $G'' \in \mathcal{F}$. If $|V(G'')| \leq 5$, by Lemmas 3.1 and 3.3, G'' is one of G_i, where $1 \leq i \leq 6$. We claim that G'' is not one of G_i, where $1 \leq i \leq 6$. It is easy to see that when $u \notin \{v_H, y_1\}$, $d_{G''}(u) \geq 3$. Thus, G'' is not one of G_1, G_2 and G_3. When $|V(G'')| = 4$, G'' has at least one 4^+-vertex, which implies that G'' is not G_4. When $|V(G'')| = 5$, G^* has at least two 4^+-vertices and no vertex of degree 2. This shows that G'' is not one of G_5 and G_6. This contradiction shows that $|V(G'')| = 6$. Since G'' has at least four 4^+-vertices, by Lemma 3.4, G'' is Z_3-connected and so is G, a contradiction.

Thus, $d_7 = 4$. Since the number of vertices of odd degree is even, $d(x) = 2$. Let $N(x) = \{u_1, u_2\}$ such that $d_{G[Y]}(u_1) = d_{G[Y]}(u_2) = 3$. If $u_1u_2 \in E(G^*)$, then $G' = G - x \in \mathcal{F}$. By Lemma 3.5, G' is Z_3-connected or G' can be Z_3-reduced to K_3. Since G is not Z_3-connected, by Lemma 2.2, G' is not Z_3-connected. So G' can be Z_3-reduced to K_3, which is contrary to the fact that each vertex of G' is 3^+-vertex.

Thus, $u_1u_2 \notin E(G')$. Then $u_2 \notin N(u_1)$. Let $G'' = G' - u_1$. Then $|V(G'')| = 6$ and G'' has two vertices of degree 4 and four vertices of degree 3. It implies that $G'' \in \mathcal{F}$. By Lemma 3.4, G'' is G_9 or G_{11}. When $G'' = G_9$, by symmetry, G' is $G'' \cup \{u_2v_4, u_2v_5, u_2v_6\}$ or $G'' \cup \{u_2v_3, u_2v_5, u_2v_6\}$. In both cases, let $G^* = G'_{[v_6v_1, v_6v_2]}$. When G'' is G_{11}, by symmetry, $G' = G'' \cup \{u_2v_1, u_2v_3, u_2v_4\}$. Let $G'_{[v_2v_3, v_2v_4]}$. We contract all 2-cycle obtained in the process and G^* is Z_3-reduced to K_1, which is Z_3-connected. By Lemma 2.4, G' is Z_3-connected and so is G, a contradiction. □

4. The proof of Theorem 1.4

Throughout this section, we assume that $G \in \mathcal{F}$ on $n \geq 9$ vertices and $X = X_G$. We argue by contradiction, and assume that there exists a graph $G \in \mathcal{F}$ such that

$$G \text{ is a counterexample to Theorem 1.4} \tag{2}$$

subject to (2)

$$|V(G)| \text{ is minimized.} \tag{3}$$

In order to complete the proof of Theorem 1.4, we establish some lemmas. The following Lemmas 4.1 and 4.2, Corollary 4.3, Lemmas 4.4–4.10 have the same hypotheses of Theorem 1.4. By Lemmas 2.2 and 2.3(1), the following lemma is straightforward.

Lemma 4.1. *Let H be a maximal nontrivial Z_3-connected subgraph of G and let $G^* = G/H$. Then*

(1) *If $|V(H) \cap X| \geq 2$, then $X \subseteq V(H)$.*
(2) *For each vertex $v \in V(G) - V(H)$, $e(v, H) \leq 1$. Moreover, for each vertex $v \in V(G) - (V(H) \cup X)$, $d_{G^*}(v) > \frac{|V(G^*)|}{2}$.*

Lemma 4.2. *If $n \geq 9$, then G does not contain a nontrivial Z_3-connected subgraph H.*

Proof. Suppose that our lemma fails and let H be a maximal Z_3-connected subgraph of G. Denote $G^* = G/H$ and let v_H be the vertex of G^* obtained by contracting H.

We claim that $G^* \in \mathcal{F}$. By Lemma 3.1, it is sufficient to show that X_{G^*} is a complete subgraph of G^*. If $|V(H) \cap X| \geq 2$, by Lemma 4.1, $X \subseteq V(H)$ and for each vertex $v \in V(G^*) - \{v_H\}$, $d_{G^*}(v) \geq \frac{|V(G^*)|}{2}$. Thus, $X_{G^*} \subseteq \{v_H\}$ and $G^* \in \mathcal{F}$. Thus, assume that $|V(H) \cap X| \leq 1$. If $|V(H) \cap X| = 1$, then $|X| \leq 4$, for otherwise the subgraph induced by $V(H) \cup X$ is Z_3-connected, contrary to the choice of H. In this case, $v_H \in X_{G^*}$ and $X_{G^*} \subseteq X$. Thus, X_{G^*} is a complete subgraph of G^*. By Lemma 4.1, $G^* \in \mathcal{F}$.

It remains for us to show that $V(H) \cap X = \emptyset$. Let $k = |V(H)|$. We claim that $k \leq \frac{n}{2}$. Suppose otherwise that $k > \frac{n}{2}$. If $v \in V(G) - (V(H) \cup X)$, then $d_G(v) \geq \frac{n}{2}$. Since $k > \frac{n}{2}$, $|V(G) - V(H)| < \frac{n}{2}$. Thus, v has at least two neighbors in H. This contradicts to that $e(v, H) \leq 1$ by Lemma 4.1(2). This contradiction proves that $V(G) = (V(H) \cup X)$. Thus, G is Z_3-connected or G can be Z_3-reduced to one of G_1, G_3, G_4 and G_5, contrary to (2).

Thus, $k \leq \frac{n}{2}$. In this case, $d_{G^*}(v_H) \geq k\frac{n}{2} - k(k-1)$. When $k \leq \frac{n}{2}$ and $k \geq 1$,

$$k\frac{n}{2} - k(k-1) - \frac{n-k+1}{2} = (k-1)\left(\frac{n}{2} - k\right) + \frac{k-1}{2} \geq 0.$$

Thus, $d_{G^*}(v_H) \geq \frac{n-k+1}{2}$. This means that $X_{G^*} \subseteq X$ and hence X_{G^*} is a complete subgraph of G^* and $G^* \in \mathcal{F}$.

By the choice of G, G^* is Z_3-connected or G^* is isomorphic to G_i, where $1 \leq i \leq 22$, or G^* can be Z_3-reduced to one of G_1, G_3, G_4 and G_5. If G^* is Z_3-connected, by Lemma 2.2 G is Z_3-connected, contrary to (2). If G^* can be Z_3-reduced to one of G_1, G_3, G_4 and G_5, so is G, contrary to (2). If G^* is one of G_i, where $1 \leq i \leq 22$, let $D = \{v : d(v) \leq 4\}$. $n \geq 9$ implies that if $v \in D$, then $v \in X$. Moreover, all vertices of D except one vertex form a $K_{|D|-1}$ in G_i (v_H may be in D). It means that G^* is one of G_1, G_3, G_4 and G_5. Thus, G can be Z_3-reduced to one of G_1, G_3, G_4 and G_5, contrary to (2). $\square$

When $|X| \geq 5$, $G[X]$ is a Z_3-connected subgraph. We obtain the following corollary immediately from Lemma 4.2.

Corollary 4.3. $|X| \leq 4$.

A K_4^- of G is a *distinguished* K_4^- if it is induced by the union of two triangles uu_1u_2 and u_1u_2w with $u \notin X$ and the vertex u is called a *distinguished* vertex of it. For such a distinguished K_4^- of G, define $G' = G_{[uu_1,uu_2]}$ and let $G_0 = G'/H$ be a Z_3-reduction of G', where H is Z_3-connected and contains a 2-cycle (u_1, u_2). In order to prove that $G_0 \in \mathcal{F}$, by Lemma 4.1, we only need to show that X_{G_0} is a complete subgraph of G_0. By Lemma 4.1, we only consider whether u, v_H and x are in X_{G_0}, where $x \in X$ in the following lemmas.

Lemma 4.4. *Suppose that $n \geq 9$ and $G_0 = G'/H$ is a Z_3-reduction of $G' = G_{[uu_1,uu_2]}$, where H is Z_3-connected. Then each of the following holds.*

(1) *If $|V(H)| \geq 5$ and $u \notin V(H)$, then $d_{G_0}(u) \geq \frac{|V(G_0)|}{2}$, and*
(2) *G_0 is 2-edge-connected.*

Proof. (1) When $|V(H)| \geq 5$, $|V(G_0)| \leq n - 4$ and $d_{G_0}(u) \geq \frac{n}{2} - 2 \geq \frac{|V(G_0|)}{2}$.

(2) It is sufficient to show that G' is 2-edge-connected. Suppose otherwise that G' is not 2-edge-connected. We define G'' as follows. $G'' = G'$ if G' is not connected; $G'' = G' - e$ if G' has a cut edge $e = xy$. Let F_1 and F_2 be the two components of G'' such that $u \in V(F_1)$ and $u_1, u_2 \in V(F_2)$.

Suppose that G' is not connected. Since $n \geq 9$, $d(u) \geq 5$ implies that $d_{F_1}(u) \geq 3$. Assume first that both F_1 and F_2 contain a vertex not in $X \cup \{u\}$. Then F_1 contains a vertex $v \in V(G) - (X \cup \{u\})$. Since $d(v) \geq \frac{n}{2}$, $|V(F_1)| \geq \frac{n}{2} + 1$. Similarly, $|V(F_2)| \geq \frac{n}{2}$. Thus, $n \geq |V(F_1)| + |V(F_2)| \geq n + 1$, a contradiction.

Thus, either F_1 or F_2 does not contain any vertex in $V(G) - (X \cup \{u\})$. In the former case, since F_1 does not contain any vertex in $V(G) - (X \cup \{u\})$, $V(F_1) \subseteq X \cup \{u\}$. Note that G' is not connected, $V(F_1) = X \cup \{u\}$. Thus, each vertex in F_2 is in $V(G) - X$. Since $d_{F_2}(u_1) \geq \frac{n}{2} - 1 \geq 4$, u_1 has a neighbor $z \in V(F_2)$ such that $e(z, F_1) = 0$. From $d_{F_2}(z) \geq 5$, $|V(F_2)| \geq 6$. Then

F_2 contains at most two vertices of degree at least $\max\{\frac{n}{2}-1, 4\}$ and others has degree at least $\max\{\frac{n}{2}, 5\}$. Theorem 1.3 shows that F_2 is Z_3-connected, contrary to Lemma 4.2. In the later case, for each vertex v in $F_1 - u$, $d(v) \geq \frac{n}{2}$ and $d_{F_1}(u) \geq \frac{n}{2} - 2$. Applying Theorem 1.3 to F_1, similarly, F_1 is Z_3-connected, contrary to Lemma 4.2.

Suppose then that G' has a cut edge $e = xy$. Assume that both F_1 and F_2 contain a vertex not in $X \cup \{u\}$. We claim that $|V(F_1)| \geq \frac{n}{2}+1$. If F_1 contains such a vertex v and $v \neq x$, then $d_{F_1}(v) \geq \frac{n}{2}$ and $|V(F_1)| \geq \frac{n}{2}+1$. If F_1 contains only one such a vertex and $v = x$, then $d_{F_1}(v) \geq \frac{n}{2} - 1$. Since $n \geq 9$, $d_{F_1}(v) \geq 4$. Note that $|X| \leq 4$. When each neighbor of v is in X, we have $d_{F_1}(v) = 4$, $|X| = 4$ and $n = 8, 9$, F_1 contains a K_5 which is Z_3-connected by Lemma 2.3(1), contrary to Lemma 4.2. Thus, v has a neighbor v' not in X. If $v' \neq u$, $e(v', F_2) = 0$ and $d_{F_1}(v') \geq \frac{n}{2}$ and $|V(F_1)| \geq \frac{n}{2}+1$; if $v' = u$, then $d_{F_1}(v) = 4$, $|X| = 3$ and $e(u, X) \geq 2$. Thus, F_1 contains an even wheel W_4 which is Z_3-connected by Lemma 2.3(4), contrary to Lemma 4.2.

Suppose that F_2 contains a vertex z not in X. When $z \notin \{y, u_1, u_2\}$ or $z \in \{u_1, u_2\} - y$ where $y \in \{u_1, u_2\}$, $d_{F_2}(z) \geq \frac{n}{2} - 1$ and $|V(F_2)| \geq \frac{n}{2}$. In this case, $n \geq |V(F_1)| + |V(F_2)| \geq n + 1$, a contradiction. Thus, $z = y = u_2$ and $u_1 \in X$ and $V(F_2) - z \subseteq X$. Since $d_{F_2}(z) \geq \frac{n}{2} - 2 \geq 3$, $|X| \geq 3$. On the other hand, $|X| \leq 4$. Then $F_2 = K_4$ or K_5^-. By Lemmas 2.3(1) and 4.2, $F_2 = K_4$, $d(z) = 5$ and $n = 9, 10$. Each vertex $(\neq u)$ in F_1 has degree at least $\max\{\frac{n}{2}, 5\}$ and $|V(F_1)| = 5, 6$. Since G is simple, $|V(F_1)| = 6$. Theorem 1.3 proves that F_1 is Z_3-connected, contrary to Lemma 4.2.

It remains that one of F_1 and F_2 does not contain any vertex in $V(G) - (X \cup \{u\})$. If F_1 does not contain any vertex in $V(G) - (X \cup \{u\})$, then $d_{F_1}(u) \geq 3$ and $|V(F_1)| \geq 4$. Note that $G[X]$ is a complete graph. Since xy is a cut edge G, $y \notin X$. This implies that each vertex in F_2 is in $V(G) - X$ and has degree at least $\max\{\frac{n}{2} - 1, 4\}$ except one when $y \in \{u_1, u_2\}$. By Theorem 1.3, F_2 is Z_3-connected, contrary to Lemma 4.2. The proof is similar for the case when F_2 does not contain any vertex in $V(G) - (X \cup \{u\})$. $\square$

Lemma 4.5. *Suppose that $n \geq 9$. If G contains a distinguished K_4^- and $X \cap V(K_4^-) = \emptyset$, then $G_0 \in \mathcal{F}$ or G_0 is one of G_1, G_3, G_4 and G_5.*

Proof. Our proof is divided in to two parts. In first part, we show that if G satisfies the hypothesis of our lemma, we find a distinguished K_4^-, which is the union of two triangles uu_1u_2 and wu_1u_2 and $V(K_4^-) \cap X = \emptyset$ such that $G' = G_{[uu_1,uu_2]}$ and $G_0 = G'/H$ such that either $|V(H)| \geq 5$ or $d_{G_0}(u) \geq \frac{|V(G_0)|}{2}$; in second part, we show $G_0 \in \mathcal{F}$. Let K be the given subgraph of G such that such a K_4^- is a subgraph of K, $V_1 = V(K) = \{v_1, v_2, v_3, v_4\}$, and $\{v_1v_2, v_2v_3, v_1v_3, v_2v_4, v_3v_4\} \subseteq E(K)$.

Case 1. $v_1v_4 \in E(G)$.

In this case, the subgraph induced by V_1 is a K_4. We claim that there is a vertex $v_0 \notin V_1$ such that $e(v_0, V_1) \geq 2$. Suppose otherwise that for each vertex $v \notin V_1$, $e(v, V_1) \leq 1$. Then $n-4 \geq e(V_1, V(G)-V_1) = d(v_1)+d(v_2)+d(v_3)+d(v_4)-12 \geq 2n - 12$, which implies that $n \leq 8$. This contradicts that $n \geq 9$. Thus, we assume that $e(v_0, V_1) \geq 2$. It follows from Lemmas 2.3 and 4.2 that $e(v_0, V_1) = 2$. We assume, without loss of generality, that $v_0v_1, v_0v_2 \in E(G)$.

In this case, we further claim that there is one vertex $u_0 \in V(G) - (\{v_0\} \cup V_1)$ such that $e(u_0, V_1) \geq 2$ for otherwise we have $n - 5 \geq d(v_1) + d(v_2) + d(v_3) + d(v_4) - 3 - 3 - 4 - 4 \geq 4\lceil\frac{n}{2}\rceil - 14$. When n is even, this inequality implies that $n \leq 8$; when n is odd; this inequality implies $n \leq 7$. Both cases contradicts assumption that $n \geq 9$. Thus, when $n \geq 9$, such a vertex u_0 exists. Note that $d(v_3) \geq 5$ and $d(v_4) \geq 5$. We define $\widetilde{G}$ as follows. If $u_0v_3 \notin E(G)$, let $\widetilde{G} = G_{[v_3v_1,v_3v_2]}$; If $u_0v_4 \notin E(G)$, let $\widetilde{G} = G_{[v_4v_1,v_4v_2]}$. Thus, assume that $u_0v_3, u_0v_4 \in E(G)$. If $u_0, v_0 \in X$, by Lemma 3.1, then $u_0v_0 \in E(G)$. In this case, $\widetilde{G} = G_{[v_4v_1,v_4v_2]}$. Thus, we say $u_0 \notin X$ or $v_0 \notin X$. If $u_0 \notin X$, let $\widetilde{G} = G_{[u_0v_3,u_0v_4]}$; if $v_0 \notin X$, let $\widetilde{G} = G_{[v_0v_1,v_0v_2]}$. Let $G_0 = \widetilde{G}/H$ be a Z_3-reduction of $\widetilde{G}$, where H is Z_3-connected with $|V(H)| \geq 5$ and contains a 2-cycle. By Lemma 4.4, G_0 is 2-edge connected.

Case 2. $v_1v_4 \notin E(G)$.

We claim that there is a vertex $v_0 \notin V_1$ such that either $e(v_0, \{v_1, v_2, v_3\}) \geq 2$ or $e(v_0, \{v_2, v_3, v_4\}) \geq 2$. Suppose otherwise that for each vertex $v \notin V_1$, both $e(v, \{v_1, v_2, v_3\}) \leq 1$ and $e(v, \{v_2, v_3, v_4\}) \leq 1$. Then $N(v_2) \cap N(v_3) - \{v_1, v_4\} = \emptyset$ and $|N(v_2) \cup N(v_3)| = |N(v_2)| + |N(v_3)| - |N(v_2) \cap N(v_3)| \geq n - 2$. It follows that $n - |N(v_2) \cup N(v_3)| \leq 2$. Since $|N(v_2) \cap N(v_4) - \{v_3\}| \leq 0$ and $|N(v_3) \cap N(v_4) - \{v_2\}| \leq 0$, $N(v_4) \subseteq (V(G) - (N(v_2) \cup N(v_3) \cup \{v_1\})) \cup \{v_2, v_3\}$ and $d(v_4) \leq 4$ and hence $n \leq 8$, contrary to that $n \geq 9$. By symmetry, assume that there exists v_0 such that $v_0v_3, v_0v_4 \in E(G)$ or $v_0v_2, v_0v_3 \in E(G)$.

We prove here for the case when $v_0v_3, v_0v_4 \in E(G)$. The proof for the case when $v_0v_2, v_0v_3 \in E(G)$ is similar. Suppose first that $v_2v_0 \in E(G)$. By Lemmas 2.3 and 4.2, $v_0v_1 \notin E(G)$. If $v_0 \notin X$, then we get a K_4 induced by v_2, v_3, v_4 and v_0, that is Case 1. Thus, assume that $v_0 \in X$. We claim that there is no vertex $w \notin V_1 \cup \{v_0\}$ such that $wv_1 \in E(G)$ and $wv_4 \in E(G)$. Otherwise, suppose such a vertex exists. If $w \notin X$, let $\widetilde{G} = G_{[wv_1,wv_4]}$ and let $G_0 = \widetilde{G}/H$, which contains a K_5^- and $|V(H)| \geq 5$. Thus, $v_0, w \in X$, by Lemma 3.1, $wv_0 \in E(G)$. In this case, let $\widetilde{G} = G_{[v_2v_3,v_2v_4]}$. Thus, for each vertex w, either $wv_1 \notin E(G)$ or $wv_4 \notin E(G)$. Similarly, for each vertex w, either $wv_0 \notin E(G)$ or $wv_1 \notin E(G)$. We claim that there is a vertex u_0 such that $e(u_0, \{v_0\} \cup V_1) \geq 2$. Otherwise, we have $n-5 \geq d(v_1)+d(v_2)+d(v_3)+d(v_4)+d(v_0)-3-2-4-4-3 \geq 2n-13+d(v_0)-3$. Since $d(v_0) \geq 3$, $n \leq 8$, contrary to that $n \geq 9$. Thus, such a vertex u_0 exists. If $u_0v_1 \in E(G)$, then $u_0v_3 \in E(G)$ by symmetry. When $u_0 \notin X$, then let $\widetilde{G} = G_{[u_0v_1,u_0v_3]}$; when $u_0 \in X$, then $u_0v_0 \in E(G)$ and let $\widetilde{G} = G_{[v_2v_4,v_3v_4]}$. If $u_0v_1 \notin E(G)$, let $\widetilde{G} = G_{[v_1v_2,v_1v_3]}$.

Suppose then that $v_2v_0 \notin E(G)$. In this case, $v_0v_1 \notin E(G)$ for otherwise G contains an even wheel W_4 with the center at v_3, which is Z_3-connected by Lemma 2.3(4), contrary to Lemma 4.2. We claim that there is a vertex $u_1 \notin \{v_0\} \cup V_1$ such that

$e(u_1, V_1) \geq 2$. Otherwise, we have $n-5 \geq d(v_1)+d(v_2)+d(v_3)+d(v_4)-3-3-4-2 \geq 2n-12$, which implies $n \leq 7$, contrary to that $n \geq 9$. Thus, such a vertex u_1 exists. If $v_0 \notin X$, let $\widetilde{G} = G_{[v_0v_3, v_0v_4]}$. Thus, assume that $v_0 \in X$. If $u_1 \in X$, then $v_0u_1 \in E(G)$ by Lemma 3.1. If $u_1v_4 \in E(G)$, let $\widetilde{G} = G_{[v_1v_2, v_1v_3]}$; if $u_1v_4 \notin E(G)$, let $\widetilde{G} = G_{[v_4v_2, v_4v_3]}$. Thus, $u_1 \notin X$. In this case, if $v_1u_1 \notin E(G)$, let $\widetilde{G} = G_{[v_1v_2, v_1v_3]}$. Thus, $u_1v_1 \in E(G)$. Let $u_1v_j \in E(G)$ for $j = 2, 3, 4$. If $u_1v_4 \notin E(G)$, let $\widetilde{G} = G_{[u_1v_1, u_1v_j]}$. Thus $u_1 \notin X$ and $u_1v_1, u_1v_4 \in E(G)$. In this case, we claim that there is a vertex $u_2 \notin \{u_1, v_0\} \cup V_1$ such that $e(u_2, V_1) \geq 2$. Otherwise, we have $n-6 \geq d(v_1)+d(v_2)+d(v_3)+d(v_4)-3-3-4-4 \geq 2n-14$, which implies that $n \leq 8$, contrary to that $n \geq 9$. Thus such a vertex u_2 exists. Similarly, we have $u_2 \notin X$ and $u_2v_1, u_2v_4 \in E(G)$. Define $G' = G_{[u_1v_1, u_1v_4]}$ and then define $\widetilde{G} = G'_{[u_2v_1, u_2v_4]}$. In all cases above, let $G_0 = \widetilde{G}/H$, where H is the maximal Z_3-subgraph containing the 2-cycle in $\widetilde{G}$. It is easy to see that $|V(H)| \geq 5$. So far we have completed the first part of our proof.

From now on we show the second part of our proof. For simplicity, we assume that $\widetilde{G} = G_{[uu_1, uu_2]}$ with $uu_1, uu_2 \in E(G)$. From our definition of $\widetilde{G}$, let $G_0 = \widetilde{G}/H$ be a Z_3-reduction of $\widetilde{G}$, where H is Z_3-connected, contains a 2-cycle (u_1, u_2) and $|V(H)| \geq 5$. By Lemma 4.4, we only consider whether v_H and $x \in X$ are in X_{G_0}.

Suppose that $V(H) \cap X \neq \emptyset$. If $|V(H) \cap X| \geq 2$, by Lemma 4.1, $X \subseteq V(H)$. Thus, X_{G_0} contains at most v_H, that is, $X_{G_0} \subseteq \{v_H\}$. If $|V(H) \cap X| = 1$, then $v_H \in X$ and $X_{G_0} \subseteq X$. In both cases, by Lemma 4.4, $G_0 \in \mathcal{F}$.

Thus, we assume that $V(H) \cap X = \emptyset$. Suppose that $k = |V(H)| \leq \frac{n}{2} - 1$. Since

$$k\frac{n}{2} - k(k-1) - 2 - \frac{n-k+1}{2} = (k-1)\left(\frac{n}{2} - k\right) + \frac{k-5}{2} \geq 0,$$

$d_{G_0}(v_H) \geq \frac{n-k+1}{2}$. It follows that $v_H \notin X_{G_0}$, $X_{G_0} \subseteq X$ and hence $G_0 \in \mathcal{F}$.

Suppose that $k \geq \frac{n}{2}$. We claim that there is no vertex $v \in V(G) - (V(H) \cup X)$. Suppose otherwise such a vertex v exists. It follows that $d_{G_0}(v) = d_G(v) \geq \frac{n}{2}$, which implies that $e(v, H) \geq 2$, contrary to Lemma 4.1. This contradiction shows that $V(G) = X \cup V(H)$. It follows that $V(G_0) = X \cup \{v_H\}$ and hence $u \in V(H)$. By Corollary 4.3, $|X| \leq 4$.

When $|X| = 4$, by Lemma 4.4, $e_{G_0}(v_H, X) \geq 2$. We claim that $e_{G_0}(v_H, X) = 2$. Otherwise G_0 contains a K_5^- which is Z_3-connected. By Lemmas 2.2 and 2.4, G is Z_3-connected, contrary to (2). Thus, G_0 is G_5. When $2 \leq |X| \leq 3$, $2 = e_{G_0}(v_H, X) \leq |X|$ since $V(H) \cap X = \emptyset$. G_0 is one of G_1, G_3 and G_4. □

Lemma 4.6. *If $n \geq 9$ and $|X| = 1$, then G contains a distinguished K_4^-. Moreover, $G_0 \in \mathcal{F}$ or G_0 is one of G_1, G_3, G_4 and G_5.*

Proof. Define $G^* = G - X$ and $X = \{x\}$. Assume that $d_G(x) = t \geq 2$. It follows that

$$\sum_{v \in V(G^*)} d_{G^*}(v) \geq t\left(\frac{n}{2} - 1\right) + (n-t-1)\frac{n}{2} = \frac{n^2-n}{2} - t.$$

Since $t \leq \frac{n-1}{2}$, $|E(G^*)| \geq (\frac{n-1}{2})^2$. By Theorem 2.1, G^* contains a triangle or is isomorphic to $K_{m,m}$. In the later case, since $n \geq 9$, $m \geq 5$. By Lemma 2.3, G^* is Z_3-connected. Since G is 2-edge connected, by Lemma 2.2, G is Z_3-connected, contrary to (2). In the former case, let $v_1v_2v_3$ be a triangle of G^*.

We claim that there is a vertex $u \in V(G) - \{v_1, v_2, v_3\}$ such that $e(u, \{v_1, v_2, v_3\}) \geq 2$. Suppose otherwise that for each vertex $u \in V(G) - \{v_1, v_2, v_3\}$, $e(u, \{v_1, v_2, v_3\}) \leq 1$. In this case, $n - 3 \geq d(v_1) + d(v_2) + d(v_3) - 6 \geq 3(\frac{n}{2}) - 6$, which implies that $n \leq 6$, contrary to that $n \geq 9$. We assume, without loss of generality, that $uv_1, uv_2 \in E(G)$.

If $u \neq x$, G^* contains a distinguished K_4^- induced by v_1, v_2, v_3 and u with distinguished vertex u. By Lemma 4.5, $G_0 \in \mathcal{F}$ or G_0 is one of G_1, G_3, G_4 and G_5. Thus, $u = x$ and hence G contains a distinguished K_4^- induced by v_1, v_2, v_3 and x with distinguished vertex v_3. If $xv_3 \in E(G)$, define $G' = G_{[v_3v_1, v_3v_2]}$ and let G_0 be a Z_3-reduction of G'. In this case, $v_3x \in E(G_0)$ and $X_{G_0} \subseteq \{v_3, x\}$ and X_{G_0} is a complete subgraph of G_0. If $v_3x \notin E(G_0)$, we claim that there is a vertex $u_0 \notin \{x, v_1, v_2, v_3\}$ such that $e(u_0, \{v_1, v_2, v_3\}) \geq 2$. Suppose otherwise that such a vertex does not exist. Then $n - 4 \geq d(v_1) + d(v_2) + d(v_3) - 6 - 2 \geq 3(\frac{n}{2}) - 8$, which implies that $n \leq 8$, contrary to that $n \geq 9$. Thus, such a vertex u_0 exists and $u_0 \notin X$. So the distinguished K_4^- induced by v_1, v_2, v_3 and u_0 with the distinguished vertex u_0 is as required. By Lemma 4.5, $G_0 \in \mathcal{F}$ or G_0 is one of G_1, G_3, G_4 and G_5. □

In order to prove Lemma 4.10, we establish the following two lemmas.

Lemma 4.7. *Suppose that $n \geq 9$ and $X = \{x_1, x_2, \ldots, x_t\}$, where $2 \leq t \leq 4$. If $\sum_{i<j} |N(x_i) \cap N(x_j) - X| \geq 2$, then G contains a distinguished K_4^-. Moreover, $G_0 \in \mathcal{F}$ or G_0 is one of G_1, G_3, G_4 and G_5.*

Proof. Suppose first that $X = \{x_1, x_2\}$ and $y_1, y_2 \in N(x_1) \cap N(x_2)$. Then G contains a distinguished K_4^- induced by x_1, x_2, y_1 and y_2 with distinguished vertex y_1. Let $G' = G_{[y_1x_1, y_1x_2]}$ and $G_0 = G'/H$ be a Z_3-reduction of G', where H is Z_3-connected. If $y_1y_2 \in E(G)$, then $y_1v_H \in E(G_0)$ or $y_1 = v_H$ in G_0. Moreover, $X_{G_0} \subseteq \{y_1, v_H\}$ and $G_0 \in \mathcal{F}$. Thus, $y_1y_2 \notin E(G)$.

If $|V(H)| = 3$ and $d(x_1) + d(x_2) \leq 6$, let $G^* = G - \{x_1, x_2\}$. Since $n \geq 9$, $\sum_{v \in V(G^*)} d(v) \geq (n-4)\frac{n}{2} + 2(\frac{n}{2} - 2) > \frac{(n-2)^2}{2}$. By Theorem 2.1, G^* contains a K_3 with vertex set $\{v_1, v_2, v_3\}$. We claim that there is a vertex $v \notin \{v_1, v_2, v_3, x_1, x_2\}$ such that $e(v, \{v_1, v_2, v_3\}) \geq 2$. Suppose otherwise that such a vertex does not exist. Then $n-3 \geq d(v_1)+d(v_2)+d(v_3)-6 \geq 3(\frac{n}{2})-6$, which $n \leq 6$, contrary to that $n \geq 9$. Thus, G^* contains a distinguished K_4^- induced by v_1, v_2, v_3 and v with the distinguished vertex v. By Lemma 4.5, $G_0 \in \mathcal{F}$ or G_0 is one of G_1, G_3, G_4 and G_5.

Suppose that $|V(H)| = 3$ and $d(x_1) + d(x_2) \geq 7$. In this case, $d_{G_0}(v_H) \geq \frac{n}{2} - 2 + 1 = \frac{n-|V(H)|+1}{2}$. Thus $X_{G_0} \subseteq \{y_1\}$. Thus, $G_0 \in \mathcal{F}$.

Now we assume that $|V(H)| = 4$. Let $v_5 \in V(H) - (X \cup \{y_2\})$. Since $n \geq 9$, $d_{G_0}(v_H) = d(y_2) + d(v_5) + d(x_1) + d(x_2) - 12 \geq n - 6 \geq \frac{n-3}{2}$ as $d(x_1) + d(x_2) \geq 6$. Thus, $d_{G_0}(v_H) \geq \frac{|V(G_0)|}{2}$. By Lemma 4.4, $X_{G_0} \subseteq \{y_1\}$ and $G_0 \in \mathcal{F}$. When $|V(H)| \geq 5$, $X_{G_0} \subseteq \{v_H\}$. Thus, $G_0 \in \mathcal{F}$.

Suppose then that $X = \{x_1, x_2, x_3\}$. As in the proof of Lemma 4.6, there is at least one vertex $u \notin X$ such that $e(u, X) \geq 2$. We choose $y \in \{u : e(u, X) \geq 2 \text{ and } u \notin X\}$ such that $e(y, X)$ is maximum and let $z \in N(x_a) \cap N(x_b) - (X \cup \{y\})$, where $a, b \in \{1, 2, 3\}$. Without loss of generality, we assume that $yx_1, yx_2 \in E(G)$. In this case, G contains a distinguished K_4^- induced by x_1, x_2, x_3 and u with the distinguished vertex u. Define $G' = G_{[yx_1, yx_2]}$ and $G_0 = G'/H$ be a Z_3-reduction of G', where H is Z_3-connected and contains a 2-cycle (x_1, x_2). If $\sum_{i<j} |N(x_i) \cap N(x_j) - X| \geq 3$, then $|V(H)| \geq 5$ and hence $G_0 \in \mathcal{F}$. If $e(y, X) = 3$, then v_H is adjacent to u or $v_H = u$ in G_0. Thus, $X_{G_0} \subseteq \{v_H, u\}$ and hence $G_0 \in \mathcal{F}$. If $e(X, G - X) \geq 5$, then $d_{G_0}(v_H) \geq \frac{n}{2} - 2 + 1 \geq \frac{|V(G_0)|}{2}$. Thus, $X_{G_0} \subseteq \{y\}$ and $G_0 \in \mathcal{F}$.

Thus, $\sum_{i<j} |N(x_i) \cap N(x_j) - X| = 2$, $e(X, G - X) = 4$, $e(y, X) = 2$ and $e(z, X) = 2$. In this case, let $G^* = G - X$. Then $\sum_{v \in V(G^*)} d_{G^*}(v) \geq (n-5)\frac{n}{2} + 2(\frac{n}{2} - 2) > \frac{(n-3)^2}{2}$. By Theorem 2.1, G^* contains a triangle $v_1v_2v_3$. We claim that there is a vertex $u \in V(G^*)$ such that $e(u, \{v_1, v_2, v_3\}) \geq 2$ for otherwise we have $n - 6 \geq d_{G^*}(v_1) + d_{G^*}(v_2) + d_{G^*}(v_3) - 6 \geq \frac{3n}{2} - 6 - 4$, which implies that $n \leq 8$, contrary to that $n \geq 9$. Thus, we obtain the distinguished K_4^- induced by v_1, v_2, v_3 and u with the distinguished vertex u. By Lemma 4.5, $G_0 \in \mathcal{F}$ or G_0 is one of G_1, G_3, G_4 and G_5.

Suppose that $X = \{x_1, x_2, x_3, x_4\}$. By Lemmas 2.3 and 4.2, for each vertex $u \in V(G) - X$ such that $e(u, X) \leq 2$. Assume that $\sum_{i<j} |N(x_i) \cap N(x_j) - X| \geq 2$ and $y \in N(x_i) \cap N(x_j) - X$, where $i \neq j$, $i, j \in \{1, 2, 3, 4\}$. In this case, we get a distinguished K_4 induced by x_i, x_j, x_k and y with the distinguished vertex y, where $k \in \{1, 2, 3, 4\} - \{i, j\}$. Let $G' = G_{[yx_i, yx_j]}$ and $G_0 = G'/H$ be a Z_3-reduction of G', where H is Z_3-connected. In this case, $|V(H)| \geq 5$. Thus, $d_{G_0}(y) \geq \frac{|V(G_0)|}{2}$. It follows that $X_{G_0} \subseteq \{v_H\}$. Thus, $G_0 \in \mathcal{F}$. $\square$

Lemma 4.8. *Suppose that $n \geq 9$ and G contains a triangle $v_1v_2v_3$ where $v_i \notin X$ for $i = 1, 2, 3$. If $e(X, \{v_1, v_2, v_3\}) \geq |X| + 2$, where $2 \leq |X| \leq 3$, then G contains a distinguished K_4^-. Moreover, $G_0 \in \mathcal{F}$.*

Proof. Let $X = \{x_1, x_2\}$. We assume, without loss of generality, that $e(v_1, X) = \min_{i \in \{1,2,3\}} e(v_i, X)$. If $e(v_1, X) = 0$, then $e(X, \{v_1, v_2, v_3\}) = 4$ and $e(v_2, X) = e(v_3, X) = 2$. In this case, we get a distinguished K_4^- induced by v_3, x_1, x_2 and v_2 with the distinguished vertex v_2. Define $G' = G_{[v_2v_3, v_2x_1]}$. Then $v_2v_H \in E(G_0)$, $\{x_1, x_2\} \subseteq V(H)$ and $X_{G_0} \subseteq \{v_H, v_2\}$. Thus, $G_0 \in \mathcal{F}$. If $e(v_1, X) \geq 1$, we assume, without loss of generality, that $v_1x_1, v_2x_2, x_1v_3, x_2v_3 \in E(G)$. In this case, we get a distinguished K_4^- induced by v_1, v_2, v_3 and x_2 with the distinguished vertex v_1. Define $G' = G_{[v_1v_3, v_1v_2]}$ and $G_0 = G'/H$ be a Z_3-reduction of G', where H is Z_3-connected. In this case, $v_1v_H \in E(G_0)$ or $v_H = v_1$. Thus, $X_{G_0} \subseteq \{v_1, v_H\}$ and $G_0 \in \mathcal{F}$.

Let $X = \{x_1, x_2, x_3\}$. We assume, without loss of generality, that $e(v_1, X) = \min_{i \in \{1,2,3\}} e(v_i, X)$. If $e(v_1, X) = 0$, we assume, without loss of generality, that $e(v_2, X) = 3$. In this case, G contains an even wheel W_4 induced by X and v_2, v_3 with the center at v_2, which is Z_3-connected, contrary to Lemma 4.2. Thus, $e(v_1, X) \geq 1$. In this case, we may assume that $x_3v_2, x_3v_3 \in E(G)$ and hence we get a distinguished K_4^- induced by v_1, v_2, v_3 and x_2 with the distinguished vertex v_1. Define $G' = G_{[v_1v_2, v_1v_3]}$ and $G_0 = G'/H$ a Z_3-reduction of G', where $\{x_1, x_2, x_3, v_2, v_3\} \subseteq V(H)$. In this case $|V(H)| \geq 5$. By Lemma 4.4, $X_{G_0} \subseteq \{v_H\}$ and $G_0 \in \mathcal{F}$. $\square$

Lemma 4.9. *If $n \geq 9$ and $|X| \geq 2$, then G contains a triangle T such that $V(T) \cap X = \emptyset$.*

Proof. Suppose then that $X = \{x_1, x_2, \ldots, x_s\}$, where $2 \leq s \leq 4$. By Corollary 4.3, $G[X]$ is a complete subgraph of G. Let $G^* = G - X$. Let $d_G(x_k) = t_k 1 \leq k \leq |X|$. By Lemma 4.7, let $\epsilon = \sum_{i<j} |N(x_i) \cap N(x_j) - X| \leq 1$. Since $t_k \leq \frac{n-1}{2}$ for $1 \leq k \leq |X|$,

$$\begin{aligned}
\sum_{v \in V(G^*)} d_{G^*}(v) &\geq ((n - |X| - \epsilon) - (t_1 + \cdots + t_{|X|} - 2(|E(G[X])| + \epsilon)))\frac{n}{2} \\
&\quad + (t_1 + \cdots + t_{|X|} - 2(|E(G[|X|])| + \epsilon))\left(\frac{n}{2} - 1\right) + \epsilon\left(\frac{n}{2} - 2\right) \\
&= \frac{n(n - |X|)}{2} - (t_1 + \cdots + t_{|X|}) + 2|E(G[X])| \\
&\geq \frac{(n - |X|)^2}{2} + \frac{|X|^2 - |X|}{2},
\end{aligned}$$

which implies that $|E(G^*)| > \frac{(n-|X|)^2}{4}$ since $2 \leq |X| \leq 4$. By Theorem 2.1, G^* contains a triangle T. $\square$

Lemma 4.10. *If $n \geq 9$ and $2 \leq |X| \leq 4$, then G contains a distinguished K_4^-. Moreover, $G_0 \in \mathcal{F}$ or G_0 is one of G_1, G_3, G_4 and G_5.*

Proof. By Lemma 4.9, G contains a triangle $T = v_1v_2v_3$ such that $V(T) \cap X = \emptyset$. We claim that there is a vertex $u \notin X \cup V(T)$ such that the K_4^- induced by v_1, v_2, v_3 and u is distinguished. Suppose otherwise that such a vertex does not exist. When $X = \{x_1, x_2\}$, by Lemma 4.8, $e(X, T) \le 3$. Thus, $n - 5 \ge d(v_1) + d(v_2) + d(v_3) - 6 - 3 \ge 3\frac{n}{2} - 9$, which implies that $n \le 8$, contrary to our assumption that $n \ge 9$. When $X = \{x_1, x_2, x_3\}$, by Lemma 4.8, $e(X, T) \le 4$. Thus, $n - 6 \ge d_G(v_1) + d_G(v_2) + d_G(v_3) - 6 - 4 \ge 3(\frac{n}{2}) - 10$, which implies that $n \le 8$, a contradiction. In both cases, G contains a distinguished K_4^- induced by v_1, v_2, v_3 and u. By Lemma 4.5, $G_0 \in \mathcal{F}$ or G_0 is one of G_1, G_3, G_4 and G_5.

Let $X = \{x_1, x_2, x_3, x_4\}$. We claim $e(T, X) \le 3$. Suppose otherwise that $e(T, X) \ge 4$. Note that $e(v, X) \le 2$ for each vertex $v \in V(T)$ by Lemma 2.3(1). We assume, without loss of generality, that v_1 is a vertex of T such that $e(v_1, X) = \min_{v \in V(T)} e(v, X)$. If there is a vertex, say x_1, in X such that $x_1v_2, x_1v_3 \in E(G)$, then G contains a distinguished K_4 induced by v_1, v_2, v_4 and x_1 with the distinguished vertex v_1. Thus, each vertex of X has one neighbor in T. We may assume $v_1x_1, v_1x_2, v_2x_3, v_3x_4 \in E(G)$ and hence G contains a distinguished K_4^- induced by v_1, x_1, x_2, x_3 with the distinguished vertex v_1. In both cases, define $G' = G_{[v_1v_2, v_1v_3]}$. Let $G_0 = G'/H$ a Z_3-reduction of G', where H is Z_3-connected. In this case, $|V(H)| \ge 4$ (H has 2-cycle). It implies that $d_{G_0}(v_1) \ge \frac{|V(G_0)|}{2}$ and $X_{G_0} \subseteq \{v_H\}$. Thus, $G_0 \in \mathcal{F}$.

We now claim that there exist $1 \le i < j \le 3$ such that $u \in N(v_i) \cap N(v_j) - (V(T) \cup X)$. Otherwise we have $n - 7 \ge d_G(v_1) + d_G(v_2) + d_G(v_3) - 6 - 3 \ge 3\frac{n}{2} - 9$. It implies that $n \le 6$, contrary to that $n \ge 9$. Then G contains a distinguished K_4^- induced by u, v_1, v_2 and v_3 such that $\{u, v_1, v_2, v_3\} \cap X = \emptyset$. By Lemma 4.5, $G_0 \in \mathcal{F}$ or G_0 is one of G_1, G_3, G_4 and G_5. □

Proof of Theorem 1.4. Assume that G is one of $G_1, \ldots, G_{22}$ or G can be Z_3-reduced to G_i, where $i \in \{1, 3, 4, 5\}$. We will show that G is not Z_3-connected. By Lemma 2.9, none of $G_1, \ldots, G_{22}$ is Z_3-connected. Assume that G can be Z_3-reduced to G_i for $i \in \{1, 3, 4, 5\}$. We claim that G is not Z_3-connected. Suppose otherwise that G is Z_3-connected. Let $X \subset E(G)$ such that $G_i = G/X$. By Lemma 2.2(2), G_i is Z_3-connected, contrary to Lemma 2.9.

Conversely, assume that G is not Z_3-connected. By contradiction, suppose that G satisfies (2) and (3). By Lemmas 3.3–3.6, $n \ge 9$. By Corollary 4.3, $|X| \le 4$. By Lemmas 4.6 and 4.10, G contains a K_4^- which is the union of two triangles uv_1v_2 and v_1v_2w. Let $G' = G_{[uv_1, uv_2]}$ and let $G_0 = G'/H$, where H is a Z_3-connected subgraph of G' and contains a 2-cycle (v_1, v_2). Then either $G_0 \in \mathcal{F}$ and $|V(G_0)| < |V(G)|$ or G_0 is one of G_1, G_3, G_4 and G_5. In the former case, by the choice of G, G_0 is Z_3-connected or G_0 is one of G_i, where $1 \le i \le 22$, or G_0 can be Z_3-reduced to one of G_1, G_3, G_4 and G_5. If G_0 is Z_3-connected, by Lemma 2.4, G is Z_3-connected, contrary to (2).

Assume that G_0 is one of G_i, where $1 \le i \le 22$. Note that $n \ge 9$. If $d(v) \le 4$, then $v \in X$. Let $D = \{v \in V(G) : d(v) \le 4\}$. Since G is connected, all vertices of degree at most 4 in G_i except v_H are in D, where $1 \le i \le 22$. It implies that G contains a complete graph $K_{|D|-1}$. Thus, G_0 is one of G_1, G_3, G_4 and G_5. This means that G can be Z_3-reduced to G_1, G_3, G_4 and G_5.

Suppose that G_0 can be Z_3-reduced to one of G_1, G_3, G_4 and G_5. If $u \in V(H)$, then G can be Z_3-reduced to one of G_1, G_3, G_4 and G_5. Thus, assume that $u \notin V(H)$, that is, v_H and u are two different vertices of G_0. Since $u \notin X$ and $n \ge 9$, $d(u) \ge 5$ and $d_{G_0}(u) \ge 3$. This implies that G_0 cannot be Z_3-reduced to G_1. One notes that all vertices of G_i, where $3 \le i \le 5$ have degree less than 5. Since $n \ge 9$, $d_G(v) \ge 5$ for each vertex $v \in V(G) - X$. Thus, $d_{G'}(v) \ge 5$ for each vertex $v \in V(G') - (X \cup \{u, v_H\})$. It follows that each vertex in G_i, $i = 3, 4, 5$, is v_H or u or belongs to X_G.

When G_0 is G_3 or G_5, v_H is the vertex of degree 2 in G_i, $i = 3, 5$. By Corollary 4.3, H does not contains any vertex in X_G. When G_0 is G_3, $d_{G_0}(u) = 3$, which implies that $d_G(u) = 5$ and $n = 9$ or 10. Thus, $6 \le |V(H)| \le 7$. When G_0 is G_5, $d_{G_0}(u) = 4$, which implies that $d_G(u) = 6$, $n = 11$ or 12 and $7 \le |V(H)| \le 8$. In both cases, $V(H) \cap X_G = \emptyset$ and $e(H, G - V(H)) = 4$. Let $H^* = H - v_1v_2$. Then H^* is a subgraph of G. When G_0 is G_3, by computing the sum of degrees of all vertices in H^*, H^* contains at most one vertex of degree 3 and at least one vertex of degree 5^+; when G_0 is G_5, by computing the sum of degrees of all vertices in H^*, H^* contains at most one vertex of degree 4 and all others of degree 5^+. This means that H^* satisfies the Ore-condition. By Theorem 1.3, H^* is Z_3-connected or H^* is one of G_i, where $1 \le i \le 12$. In the later case, for each case, H^* contains at least one 5^+-vertex while G_i has no 5^+-vertex, a contradiction. In the former case, we contract H^* in G, G/H^* contains a 2-cycle (v_{H^*}, u) and we continue to contract 2-cycles. Eventually, we obtain a K_1 which is Z_3-connected. By Lemma 2.4, G is Z_3-connected, contrary to (2).

Thus, assume that G_0 can be Z_3-reduced to G_4. Let $V(G_4) = \{w_1, w_2, w_3, w_4\}$, $w_1 = v_H$, $w_2 = u$. Since $d_{G_4}(w_2) = d_{G_0}(u) = 3$, $d_G(u) = 5$. This implies that $9 \le n \le 10$. Thus $6 \le |V(H)| \le 7$. $w_3, w_4 \in X_G$. By Lemma 3.1, H contains at most one vertex of X_G.

If H contains exactly one vertex x of X_G, then $xw_3, xw_4 \in E(G)$. Since $d_G(x) \le 4$, $d_H(x) \le 2$. Let $G^* = G - \{x, w_2, w_3, w_4\}$. Then for each vertex z of G^*, $d_{G^*}(z) \ge 3$ and $|V(G^*)| \le 6$. Thus, G^* satisfies the Ore-condition. By Theorem 1.3, G^* is Z_3-connected or G^* is G_i, where $1 \le i \le 12$. Since G^* has either at least four 4^+-vertices or three 4^+-vertices and at least one 5^+-vertex, G^* is none of G_i, $1 \le i \le 12$. Thus, G^* is Z_3-connected. It implies that G is Z_3-connected, contrary to (2).

Thus, H contains no vertex in X_G. If H contains one vertex x such that $xw_2, xw_3, xw_4 \in E(G)$, let $G^* = G - \{w_3, w_4\}$. It is easy to verify that G^* satisfies the Ore-condition. If H has no such a vertex, let $G^* = G - \{w_2, w_3, w_4\}$. In this case, let $xw_4 \in E(G)$. Then either $xw_3 \in E(G)$ or $xw_3 \notin E(G)$. In both cases, G^* contains at most one 3^+-vertex and others are 4^+-vertices. It is easy to see that $|V(G^*)| \le 7$ and G^* is 2-edge-connected. By Theorem 1.3, G^* is Z_3-connected or G^* is one of G_i, where $1 \le i \le 12$. Since G contains at least one 5^+-vertex and four 4^+-vertices or at least two 5^+-vertices and three 4^+-vertices, G^* is not one of G_i, $1 \le i \le 12$. Thus, G^* is Z_3-connected. Since G/H^* contains 2-cycles, G can be Z_3-reduced to K_1 which is Z_3-connected. By Lemma 2.4, G is Z_3-connected, contrary to (2). □

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