Discrete Applied Mathematics 160 (2012) 1837–1844

Contents lists available at SciVerse ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Collapsible graphs and Hamiltonian connectedness of line graphs☆

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ARTICLE INFO

Article history:
Received 16 November 2010
Received in revised form 22 March 2012
Accepted 24 March 2012
Available online 17 April 2012

Keywords:
Thomassen's conjecture
Line graph
Super-Eulerian graph
Collapsible graph
Hamiltonian line graph
Dominating Eulerian subgraph
Edge connectivity

ABSTRACT

Thomassen conjectured that every 4-connected line graph is Hamiltonian. Chen and Lai [Z.-H. Chen, H.-J. Lai, Reduction techniques for super-Eulerian graphs and related topics—an update, in: Ku Tung-Hsin (Ed.), Combinatorics and Graph Theory, vol. 95, World Scientific, Singapore/London, 1995, pp. 53–69, Conjecture 8.6] conjectured that every 3-edge connected, essentially 6-edge connected graph is collapsible. In this paper, we prove the following results. (1) Every 3-edge connected, essentially 6-edge connected graph with edge-degree at least 7 is collapsible. (2) Every 3-edge connected, essentially 5-edge connected graph with edge-degree at least 6 and at most 24 vertices of degree 3 is collapsible which implies that 5-connected line graph with minimum degree at least 6 of a graph with at most 24 vertices of degree 3 is Hamiltonian. (3) Every 3-connected, essentially 11-connected line graph is Hamilton-connected which strengthens the result in [H.-J. Lai, Y. Shao, H. Wu, J. Zhou, Every 3-connected, essentially 11-connected line graph is Hamiltonian, J. Combin. Theory, Ser. B 96 (2006) 571–576] by Lai et al. (4) Every 7-connected line graph is Hamiltonian connected which is proved by a method different from Zhan's. By using the multigraph closure introduced by Ryjáček and Vrána which turns a claw-free graph into the line graph of a multigraph while preserving its Hamilton-connectedness, the results (3) and (4) can be extended to claw-free graphs.

1. Introduction

Unless stated otherwise, we follow [1] for terminology and notations, and we consider finite connected graphs without loop. In particular, we use $\kappa(G)$ and $\kappa'(G)$ to represent the *connectivity* and *edge-connectivity* of a graph G. A graph is *trivial* if it contains no edges. A vertex cut X of G is *essential* if $G - X$ has at least two non-trivial components. For an integer $k > 0$, a graph G is *essentially k-connected* if G does not have an *essential cut* X with $|X| < k$. An edge cut Y of G is *essential* if $G - Y$ has at least two non-trivial components. For an integer $k > 0$, a graph G is *essentially k-edge-connected* if G does not have an essential edge cut Y with $|Y| < k$. In particular, the *essential edge-connectivity* of G, denote by $\lambda'(G)$, is the size of a minimum essential edge-cut. Let $u \in V(G)$ and $d_G(u)$ the degree of u, or simply $d(u)$ if no confusion. For $e = uv \in E(G)$, define $d(e) = d(u) + d(v) - 2$ the edge degree of e, and $\xi(G) = \min\{d(e) : e \in E(G)\}$. Esfahanian in [6] proved that if a connected graph G with $|V(G)| \geq 4$ is not a star $K_{1,n-1}$, then $\lambda'(G)$ exists and $\lambda'(G) \leq \xi(G)$. Thus, a essentially k-edge connected graph has edge-degree at least k. Denote $D_i(G)$ and $d_i(G)$ the set of vertices of degree i and $|D_i(G)|$, respectively. If no confusion, we directly use D_i and d_i for $D_i(G)$ and $d_i(G)$, respectively. For a subgraph $A \subseteq G$, $v \in V(G)$, $N_G(v)$ denotes

☆ The research is supported by NSFC (No. 11171279, 11126321, 11101345).
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doi:10.1016/j.dam.2012.03.028

the set of the neighbors of v in G and $N_G(A)$ denotes the set $\left(\bigcup_{v\in V(A)} N_G(v)\right) \setminus V(A)$. If no confusion arises, we use an edge uv for a subgraph with three elements of $\{u, v, uv\}$. Denote $G[X]$ the subgraph induced by the vertex set X of $V(G)$.

The line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common. From the definition of a line graph, if $L(G)$ is not a complete graph, then a subset $X \subseteq V(L(G))$ is a vertex cut of $L(G)$ if and only if X is an essential edge cut of G. Thomassen in 1986 posed the following conjecture:

Conjecture 1.1 (*Thomassen [18]*)**.** *Every* 4*-connected line graph is Hamiltonian.*

A graph that does not have an induced subgraph isomorphic to $K_{1,3}$ is called a claw-free graph. It is well known that every line graph is a claw-free graph. Matthews and Sumner proposed a seemingly stronger conjecture:

Conjecture 1.2 (*Matthews and Sumner [14]*)**.** *Every* 4*-connected claw-free graph is Hamiltonian.*

Theorem 1.3 (*Zhan [20]*)**.** *Every* 7*-connected line graph is Hamiltonian connected.*

Theorem 1.4 (*Ryjáček [15]*)**.**

(i) *Conjectures* 1.1 and 1.2 *are equivalent.*
(ii) *Every* 7*-connected claw-free graph is Hamiltonian.*

Very recently, an important progress towards the conjectures was submitted by Kaiser and Vrána [9] in which the following theorem is listed:

Theorem 1.5 (*[9]*)**.** *Every* 5*-connected line graph with minimum degree at least* 6 *is Hamiltonian.*

So we clearly have:

Corollary 1.6. *Every* 6*-connected line graph is Hamiltonian.*

Using Ryjáček's line graph closure, the following corollary is obtained:

Corollary 1.7 (*[9]*)**.** *Every* 5*-connected claw-free graph G with minimum degree at least* 6 *is Hamiltonian.*

We list some known partial results on the Hamiltonicity of line graphs and claw-free graphs as follows. Chen et al. in [5] reported that every 4-connected line graph $L(G)$ with $D_3(G) = \emptyset$ is Hamiltonian. Li in [13] proved that every 6-connected claw-free graph with at most 33 vertices of degree 6 is Hamiltonian. Let G be a 6-connected line graph. Hu et al. in [7] showed that if $d_6(G) \leq 29$ or $G[D_6(G)]$ contains at most 5 vertex disjoint K_4's, then G is Hamilton-connected. Let G be a 6-connected claw-free graph. Hu et al. in [8] showed that if $d_6(G) \leq 44$ or $G[D_6(G)]$ contains at most 8 vertex disjoint K_4's, then G is Hamiltonian. Let G be a 6-connected line graph. Zhan in [21] showed that if either $d_6(G) \leq 74$, or $d_6(G) \leq 54$ or $G[D_6(G)]$ contains at most 5 vertex disjoint K_4's, then G is Hamiltonian.

In particular, Lai et al. in [11] considered the following problem: For 3-connected line graphs, can high essential connectivity guarantee the existence of a Hamiltonian cycle? They proved the following theorem:

Theorem 1.8 (*Lai et al. [11]*)**.** *Every* 3*-connected, essentially* 11*-connected line graph is Hamiltonian.*

We shall prove that every 3-connected, essentially 11-connected line graph is Hamilton-connected in Section 4.

Chen and Lai in [4] posed the following conjectures:

Conjecture 1.9 (*Chen and Lai Conjecture 8.6 [4]*)**.** *Every* 3*-edge connected, essentially* 6*-edge connected graph G is collapsible.*

Conjecture 1.10 (*Chen and Lai Conjecture 8.7 [4]*)**.** *Every* 3*-edge connected, essentially* 5*-edge connected graph G is super-Eulerian.*

Now we list the results of the current paper. We prove that (1) Every 3-edge connected, essentially 6-edge connected graph with edge-degree at least 7 is collapsible. (2) Every 3-edge connected, essentially 5-edge connected graph with edge-degree at least 6 and at most 24 vertices of degree 3 is collapsible which implies that 5-connected line graph with minimum degree at least 6 of a graph with at most 24 vertices of degree 3 is Hamiltonian. (3) Every 3-connected, essentially 11-connected line graph is Hamilton-connected which strengthens the result of Lai et al. in [11]. (4) Every 7-connected line graph is Hamilton-connected which is proved by a method different from Zhan's [20].

2. Reductions

Catlin in [2] introduced collapsible graphs. For a graph G, let $O(G)$ denote the set of odd degree vertices of G. A graph G is *eulerian* if G is connected with $O(G) = \emptyset$, and G is *super-eulerian* if G has a spanning Eulerian subgraph. A graph G is *collapsible* if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 (\mathrm{mod}\ 2)$, G has a spanning connected subgraph H_R such that $O(H_R) = R$. Note that when $R = \emptyset$, a spanning connected subgraph H with $O(H) = \emptyset$ is a spanning Eulerian subgraph of G. Thus every collapsible graph is super-Eulerian. Catlin [2] showed that any graph G has a unique subgraph H such that every component of H is a maximally connected collapsible subgraph of G and every non-trivial connected collapsible subgraph of G is contained in a component of H. For a subgraph H of G, the graph G/H is obtained from G by identifying the two ends of each edge in H and then deleting the resulting loops. The contraction G/H is called the *reduction* of G if H is the maximal collapsible subgraph of G, i.e. there is no non-trivial collapsible subgraph in G/H. A graph G is *reduced* if it is the reduction of itself. Let $F(G)$ denote the minimum number of edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. The following summarizes some of the previous results concerning collapsible graphs.

Theorem 2.1. *Let G be a connected graph. Each of the following holds.*

(i) *(Catlin [2]). If H is a collapsible subgraph of G, then G is collapsible if and only if G/H is collapsible; G is super-Eulerian if and only if G/H is super-Eulerian.*
(ii) *(Catlin, Theorem 5 of [2]). A graph G is reduced if and only if G contains no non-trivial collapsible subgraphs. As cycles of length less than 4 are collapsible, a reduced graph does not have a cycle of length less than 4.*
(iii) *(Catlin, Theorem 8 of [2]). If G is reduced and if $|E(G)| \geq 3$, then $\delta(G) \leq 3$, and $2|V(G)| - |E(G)| \geq 4$.*
(iv) *(Catlin [2]). If G is reduced and if $|E(G)| \geq 3$, then $\delta(G) \leq 3$ and $F(G) = 2|V(G)| - |E(G)| - 2$.*
(v) *(Catlin et al. [3]) Let G be a connected reduced graph. If $F(G) \leq 2$, then $G \in \{K_1, K_2, K_{2,t}\}(t \geq 1)$.*

Let G be a connected, essentially 3-edge-connected graph such that $L(G)$ is not a complete graph. The *core* of this graph G, denoted by G_0, is obtained by deleting all the vertices of degree 1 and contracting exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$.

Lemma 2.2 (*Shao [17]*)**.** *Let G be a connected, essentially 3-edge-connected graph G.*

(i) *G_0 is uniquely defined, and $\kappa'(G_0) \geq 3$.*
(ii) *If G_0 is super-Eulerian, then $L(G)$ is Hamiltonian.*

3. Collapsible graphs

In the following lemma, the graph considered may have loops. Noticing that a loop is an edge with two same endpoints. For a graph G and $u \in V(G)$, denote $E_G(u)$ the set of edges incident with u in G. When the graph G is understood from the context, we write E_u for $E_G(u)$ simply.

Lemma 3.1. *Let G be a graph with $\delta(G) \geq 3$, $\xi(G) \geq 6$. Then $|E(G)| \geq 2|V(G)| - \frac{d_3}{5}$.*

Proof. Note that if a component of G has no vertex of degree 3, then the component satisfies the inequality. So we assume that each of the components of G contains some vertices of degree 3. Let $N = N_G(D_3)$, $T = V \setminus (N \cup D_3)$. Note that G is a graph with $\delta(G) \geq 3$, $\xi(G) \geq 6$, then D_3 is an independent set of G and the degree of the vertices in N is at least 5, the degree of each vertex in T is at least 4. We prove this claim by induction on $|T|$.

We first let $|T| = \emptyset$, then each of the vertices in N has degree at least 5. If $|N| > \frac{3}{5}d_3$, we have

$$\begin{aligned}|E(G)| = \frac{\sum id_i}{2} &\geq \frac{3d_3}{2} + \frac{5(|V(G)| - d_3)}{2} = 2|V(G)| - d_3 + \frac{|V(G)|}{2}\\ &= 2|V(G)| - d_3 + \frac{d_3 + |N|}{2} > 2|V(G)| - d_3 + \frac{d_3 + \frac{3}{5}d_3}{2}\\ &= 2|V(G)| - d_3 + \frac{4}{5}d_3\\ &= 2|V(G)| - \frac{d_3}{5}.\end{aligned} \tag{1}$$

Thus, we may assume $|N| \leq \frac{3}{5}d_3$. It is easy to see that

$$\begin{aligned}|E(G)| &\geq 3d_3 = 2d_3 + \frac{6}{5}d_3 - \frac{d_3}{5} \geq 2d_3 + 2|N| - \frac{d_3}{5}\\ &= 2(d_3 + |N|) - \frac{d_3}{5}\\ &= 2|V(G)| - \frac{d_3}{5}.\end{aligned} \tag{2}$$

Now, we assume $|T| = 1$ and $T = \{u\}$. Clearly, $d(u) \geq 4$. We first suppose $d(u) = 2k$ for some $k \geq 2$. Assume that there are l loops on u and let $2k = 2l + 2t$. Now, we delete the l loops of u and label the $2t$ neighbors corresponding the $2t$ edges naturally. Denote the $2t$ neighbors by $N'(u) = \{u_1, u_2, \ldots, u_{2t}\}$ (it is not a set if $G[\{u\} \cup N(u)]$ contains some multi-edges), that is, $N'(u)$ contains vp times if there are p edges between u and v. We construct a graph G' by (i): deleting vertex u and edges uu_i, $i = 1, 2, \ldots, 2t$; (ii): adding new edges $u_1u_2, u_3u_4, \ldots, u_{2t-1}u_{2t}$. It can be seen that $D_3(G) = D_3(G')$, $V(G') = V(G) \setminus \{u\}$, $E(G') = (E(G) \setminus E_u) \cup \{u_1u_2, u_3u_4, \ldots, u_{2t-1}u_{2t}\}$. Hence, $|V(G')| = |V(G)| - 1$, $|E(G')| = |E(G)| - \frac{d(u)}{2}$. Note that the set T of G' is $\emptyset$, then we have $|E(G')| \geq 2|V(G')| - \frac{d_3}{5}$. Therefore,

$$
\begin{aligned}
|E(G)| &= |E(G')| + \frac{d(u)}{2} \\
&\geq 2|V(G')| - \frac{d_3}{5} + \frac{d(u)}{2} = 2(|V(G)| - 1) - \frac{d_3}{5} + \frac{d(u)}{2} \\
&= 2|V(G)| - \frac{d_3}{5} + \left(\frac{d(u)}{2} - 2\right) \\
&\geq 2|V(G)| - \frac{d_3}{5}.
\end{aligned} \tag{3}
$$

Next, we suppose $u \in T$ with l loops, $d(u) = 2k + 1$ and $2k + 1 = 2l + 2t + 1$ for some $k \geq 2$ and similarly $N'(u) = \{u_1, u_2, \ldots, u_{2t+1}\}$. Let $u' \in N$, we first construct G' by adding an new edge uu'. Now, u is in the T of G' and $d_{G'}(u) \geq 6$ is even. Similarly as above, we construct a new graph G'' such that the T of G'' is empty. Note that $\frac{d_{G'}(u)}{2} \geq 3$, then

$$
\begin{aligned}
|E(G')| &= |E(G'')| + \frac{d_{G'}(u)}{2} \\
&\geq 2|V(G'')| - \frac{d_3}{5} + \frac{d_{G'}(u)}{2} = 2(|V(G')| - 1) - \frac{d_3}{5} + \frac{d_{G'}(u)}{2} \\
&= 2|V(G)| - \frac{d_3}{5} + \left(\frac{d_{G'}(u)}{2} - 2\right) \\
&\geq 2|V(G')| - \frac{d_3}{5} + 1.
\end{aligned} \tag{4}
$$

Thus, $|E(G)| = |E(G')| - 1 \geq 2|V(G)| - \frac{d_3}{5}$.

(►) Assume that the claim holds for $1 \leq |T| < m$ and consider $|T| = m \geq 2$ in the following. Note that each of the components of G contains some vertex of degree 3, then there is a vertex u in T which is adjacent to some vertex of N. Clearly, by the argument above, if $d(u) = 2l + 2t$ is even, then, the claim holds by constructing a new graph G' (similarly as the case when $|T| = 1$, i.e. G' is constructed by deleting a vertex u, $l + t$ edges, and adding t new edges) with $|T| = m - 1$ and then by induction, we are done. Assume $d(u)$ is odd. Similarly as the case when $|T| = 1$. It can be seen that $d_{G'}(u)$ is even and $d_{G'}(u) \geq 6$. Then we construct a new graph G'' similar to that of $|T| = 1$, by induction and the argument similar to that of (4), the claim holds. We complete the proof of the claim. □

Let G' be the reduction of G. Note that contraction can not decrease the edge connectivity of G, then G' is either a k-edge connected graph or a trivial graph if G is k-edge connected. Assume that G' is the reduction of a 3-edge connected graph and non-trivial. It follows from Theorem 2.1(v) and G' being 3-edge connected that $F(G') \geq 3$. Then by Theorem 2.1(iv), we have $|E(G')| \leq 2|V(G')| - 5$.

We call a vertex of G' *non-trivial* if the vertex is obtained by contracting a collapsible subgraph of G, and *trivial*, otherwise. Assume that G is a 3-edge connected, essentially $k \geq 4$-edge connected graph. It is easy to see that G' contains no non-trivial vertex of degree i such that $3 \leq i < k$ (otherwise, an essentially edge cut of G with size less than k is found).

Theorem 3.2. *A* 3*-edge connected, essentially* 5*-edge connected graph with edge-degree at least* 6 *and at most* 24 *vertices of degree* 3 *is collapsible.*

Proof. Let G be a 3-edge connected, essentially 5-edge connected graph with edge-degree 6 and at most 24 vertices of degree 3, and G' be the reduction of G. Note that G is essentially 5-edge connected, then the contraction can not product new vertex of degree 3 or 4 by Theorem 2.1(ii) (suppose u is vertex obtained by contracting a non-trivial maximal collapsible connected subgraph of G and $d_{G'}(u) < 5$. By Theorem 2.1(ii), $G' - \{u\}$ contains at least one non-trivial component. It is not difficult to see that G contains an essential edge-cut with size less than than 5, a contradiction), that is, $|D_3(G')| \leq |D_3(G)|$ and G' is 3-edge connected graph with edge-degree 6. By Lemma 3.1, we have $|E(G')| \geq 2|V(G')| - \frac{|D_3(G')|}{5}$, that is, $|E(G')| \geq 2|V(G')| - \frac{|D_3(G')|}{5} \geq 2|V(G')| - 4$ which contradicts with $|E(G')| \leq 2|V(G')| - 5$. Thus, we complete the proof. □

Note that a 3-edge connected, essentially 6-edge connected graph has edge-degree at least 6, then we have the following two corollaries which are the partial results of Conjectures 1.9 and 1.10, respectively:

Corollary 3.3. *A* 3*-edge connected, essentially* 5*-edge connected graph with edge-degree at least* 6 *and at most* 24 *vertices of degree* 3 *is super-Eulerian.*

Corollary 3.4. *A* 3*-edge connected, essentially* 6*-edge connected graph with at most* 24 *vertices of degree* 3 *is collapsible.*

For a graph G, if $L(G)$ is $k \geq 3$-connected, then G is essentially k-edge connected. Clearly, the core of G is 3-edge connected and essentially k-edge connected. Clearly, if the minimum degree of $L(G)$ is k, then the edge degree of G is at least k. Thus, by Lemma 2.2 and Theorem 3.2, we have the following corollaries:

Corollary 3.5. *A* 5*-connected line graph with minimum degree at least* 6 *of a graph with at most* 24 *vertices of degree* 3 *is Hamiltonian.*

From the proof above, it can be seen that Lemma 3.1 plays a key role. Similarly, we pose the following lemma for considering the 3-edge connected graphs with $\xi(G) \geq 7$. The proof of the following is very similar to that of Lemma 3.1, we leave the complete proof to readers and only point the part which is different from the proof of Lemma 3.1.

Lemma 3.6. *Assume that G is a graph with* $\delta(G) \geq 3$, $\xi(G) \geq 7$. *Then* $|E(G)| \geq 2|V(G)|$.

Proof. All the process of the proof for this claim is similar to the proof of Lemma 3.1 excepting the paragraph of (►). In (►), we can take any vertex u of T such that the resulting graph (G' or G'') satisfies the the assumption of Lemma 3.1. Here, the new graphs constructed by the method in proof of Lemma 3.1 may contain some edges of degree 6 which make the induction invalid. So we must take here the vertex $u \in T$ such that $d(u) = \min\{d(u)|u \in T\}$. This choice makes the induction work well in the proof. □

By Lemma 3.6, and the similar argument to that of the proof of Theorem 3.2, we have the following theorem which is another partial result of Conjecture 1.10. Note that Lemma 7 in [7] implies a stronger result: Every 3-edge connected essentially 6-edge-connected graph with edge-degree at least 7 has 2 edge disjoint spanning trees (also 44 edges with edge-degree 6 are allowed), which also implies the following.

Theorem 3.7. *A* 3*-edge connected, essentially* 6*-edge connected graph with edge-degree at least* 7 *is collapsible.*

Similarly as Corollary 3.3, we have the following corollary which is posed by Chen and Lai [4].

Corollary 3.8 (*Chen and Lai Theorem 7.3 of [4]*)**.** *A* 3*-edge connected, essentially* 7*-edge connected graph is collapsible.*

By Lemma 2.2 and Theorem 3.7, we give a weaker result (than the result of [7]):

Corollary 3.9. *A* 6*-connected line graph with minimum degree at least* 7 *is Hamiltonian.*

By Corollary 3.9, the following corollary is clear.

Corollary 3.10 (*Zhan [20]; Chen and Lai Theorem 7.2 of [4]*)**.** *A* 7*-connected line graph is Hamiltonian.*

4. Hamiltonian connectedness of line graph

Lemma 4.1 (*Lai et al. Theorem 2.3(iii) [12]*)**.** *If G is collapsible, then for any pair of vertices* $u, v \in V(G)$, *G has a spanning* (u, v)*-trail.*

A dominating (e_1, e_2)-trail of G is an (e_1, e_2)-trail T of G such that every edge of G is incident with an internal vertex of T.

Lemma 4.2 (*Lai et al. Proposition 2.2 [12]*)**.** *Let G be a graph with* $|E(G)| \geq 3$. *Then* $L(G)$ *is Hamiltonian connected if and only if for any pair of edges* $e_1, e_2 \in E(G)$ *has a dominating* (e_1, e_2)*-trail.*

For a graph G and any pair of edges $e_1, e_2 \in E(G)$, let $G(e_1, e_2)$ denote the graph obtained from G by subdividing both e_1 and e_2, and denote the new vertices by $v(e_1)$ and $v(e_2)$. Thus $V(G(e_1, e_2)) - V(G) = \{v(e_1), v(e_2)\}$. The following lemma is obtained by Lai et al. by combining the definition of collapsible and Lemma 2.9 of [12]; Combining Lemmas 4.1 and 4.2, Zhan also stated the following lemma in (4.3) of [21].

Lemma 4.3 (*Lai et al. Lemma 2.9 [12], Zhan 4.3 of [21]*)**.** *Assume that* G_0 *is the core of G. If* $G_0(e_1, e_2)$ *is collapsible, then* $L(G)$ *is Hamilton-connected.*

A subgraph of G isomorphic to a $K_{1,2}$ or a 2-cycle is called a 2-path or a P_2 subgraph of G. An edge cut X of G is a P_2*-edge cut* of G if at least two components of $G - X$ contain 2-paths. By the definition of a line graph, for a graph G, if $L(G)$ is not a complete graph, then $L(G)$ is essentially k-connected if and only if G does not have a P_2-edge cut with size less than k. Since the core G_0 is obtained from G by contractions (deleting a pendant edge is equivalent to contracting the same edge), every P_2-edge-cut of G_0 is also a P_2-edge-cut of G. Hence, the following lemma is easy:

Lemma 4.4 (*Lai et al. Lemma 2.3 of [11]*). *Let $k > 2$ be an integer, and let G be a connected, essentially 3-edge connected graph. If $L(G)$ is essentially k-connected, then every P_2-edge cut of G_0 has size at least k.*

If V_1, V_2 are two disjoint subsets of $V(G)$, then $[V_1, V_2]_G$ denotes the set of edges in G with one end in V_1 and the other end in V_2. When the graph G is understood from the context, we also omit the subscript G and write $[V_1, V_2]$ for $[V_1, V_2]_G$. If H_1, H_2 are two vertex disjoint subgraphs of G, then we also write $[H_1, H_2]$ for $[V(H_1), V(H_2)]$.

Lemma 4.5 (*Lai et al. Lemma 3.1 of [11]*). *Let G be graph such $L(G)$ is 3-connected and essentially 11-connected, and G' be the reduction of G_0. For each $u, v, w \in V(G')$ such that $P = uvw$ is a 2-path in G', the edge cut $X = [\{u, v, w\}, V(G') \setminus \{u, v, w\}]_{G'}$ is a P_2-edge cut of G' and $|X| \geq 11$.*

Recall that $f(x) = \frac{x-4}{x}$ and $l(u) = f(d(u))$ defined in [11]. The following lemma is a useful property of $f(x)$.

Lemma 4.6 (*Lai et al. Lemma 3.3 of [11]*). *Each of the following holds.*

(i) *$f(x)$ is an increasing function.*
(ii) *If $d(u) \geq k$, then $l(u) \geq f(k)$.*

Lemma 4.7. *Let G be a graph such that $L(G)$ is 3-connected and essentially 11-connected. Then $G_0(e_1, e_2)$ is collapsible.*

Proof. Assume that G_0 is the core of G. Clearly, any P_2-edge cut of G_0 has size at least 11. Let G' be the reduction of $G_0(e_1, e_2)$. If G' is trivial, then, by Lemma 4.3, the assertion holds. By contradiction, suppose G' is non-trivial. Let $v \in D_3, N_{G'}(v) = \{v_1, v_2, v_3\}$, and let $s = |N_{G'}(v) \cap D_2(G')|$. We first show some claims as follows.

Claim 1. $F(G') \geq 3$.

Assume $F(G') \leq 2$. By Theorem 2.1(v), G' is a $K_{2,t}$. It is easy to find an edge cut of G_0 of size 2, which contradicts to the fact that G_0 is 3-edge connected. So the claim holds.

By Theorem 2.1(iv), we have $|E(G')| \leq 2|V(G)| - 5$.

Recall that we call a vertex of G' non-trivial if the vertex is obtained by contracting a non-trivial collapsible subgraph of $G_0(e_1, e_2)$, and trivial, otherwise. Assume that u is a non-trivial vertex of G', and it is the contraction of a maximal collapsible connected subgraph H. We call H the preimage of u and denote $PM(u) = H$.

Claim 2. *Let xyz be a P_2 of G' and $\min\{d_{G'}(x), d_{G'}(y), d_{G'}(z)\} \geq 3$. Then $[G_0[PM(x) \cup PM(y) \cup PM(z)], G_0 - G_0[PM(x) \cup PM(y) \cup PM(z)]]_{G_0}$ is a P_2-edge cut of G_0.*

Note that the degree (in G') of the vertex in $\{x, y, z\}$ is at least 3 and G' contains at most two vertices of degree 2, then it is easy to find a P_2 without vertex of degree 2 in $G' - \{x, y, z\}$ (Note that G' contains no cycles of lengths 3 and 2, then $N_{G'}(x) \cup N_{G'}(y) \cup N_{G'}(z) \setminus \{x, y, z\}$ contains at least 4 vertices of $G' - \{x, y, z\}$. Since there are at most two of $N_{G'}(x) \cup N_{G'}(y) \cup N_{G'}(z) \setminus \{x, y, z\}$ with degree 2, a simple argument shows that $G' - \{x, y, z\}$ contains a P_2 clearly.). Thus, $[G_0[PM(x) \cup PM(y) \cup PM(z)], G_0 - G_0[PM(x) \cup PM(y) \cup PM(z)]]_{G_0}$ is a P_2-edge cut of G_0.

Claim 3. *Let $v \in D_3(G'), N_{G'}(v) = \{v_1, v_2, v_3\}$. For any two vertices $v_i, v_j \in N_{G'}(v) \setminus D_2(G')$, $d_{G'}(v_i) + d_{G'}(v_j) \geq 12$ for $i \neq j$ hold.*

By Claim 2., this claim is clear.

Claim 4. *Each component of $G'[D_3(G')]$ contains at most two vertices.*

Suppose that there is a component in $G'[D_3(G')]$ contains a P_2, say xyz. By Claim 2, $[G_0[PM(x) \cup PM(y) \cup PM(z)], G_0 - G_0[PM(x) \cup PM(y) \cup PM(z)]]_{G_0}$ is a P_2-edge cut of G_0 with size less than 11, a contradiction.

Claim 5. *Suppose that $v \in D_3(G')$ is an isolated vertex of $G'[D_3(G')]$ and $N_{G'}(v) \cap D_2(G') = \emptyset$. Then $l(v_1) + l(v_2) + l(v_3) \geq 1$.*

By Lemma 4.6 and Claim 3, this claim is clear.

Claim 6. *Suppose that $v, w \in D_3(G')$ and $vw \in E(G')$ and $N(vw) \cap D_2(G') = \emptyset$. If v_1, v_2 are the vertices adjacent to v in G' different from w and v_3, v_4 are the vertices adjacent to w in G' different from v, then* (I) *v_1, v_2, v_3, v_4 are mutually distinct vertices, and* (II) *both $l(v_1) + l(v_2) \geq 1$ and $l(v_3) + l(v_4) \geq 1$.*

By Theorem 2.1(ii), (I) is clearly true. By Lemma 4.6 and Claim 3, this claim is clear.

We now turn to prove Lemma 4.7. We first assume that $N_{G'}(v) \cap D_2(G') = \emptyset$ for all $v \in D_3(G')$. By Claims 5 and 6, we have the following inequality.

$$\begin{aligned} d_3 &= \sum_{v\in D_3} 1 \le \sum_{v\in D_3} \sum_{uv\in E, u\notin D_3} l(u) = \sum_{i\ge 4}\sum_{u\in D_i} \sum_{uv\in E, v\in D_3} l(u) \\ &\le \sum_{i\ge 4}\sum_{u\in D_i} i\cdot f(i) = \sum_{i\ge 4}\sum_{u\in D_i}(i-4) = \sum_{i\ge 4}(i-4)\cdot d_i. \end{aligned} \tag{5}$$

Now assume $N_{G'}(D_2(G')) \cap D_3(G') \ne \emptyset$. Notice that $|D_2(G')| \le 2$. It is not difficult to see that at most 4 vertices of degree 3 do not satisfy Claims 5 and 6. Assume that S is the set of the vertices in $N_{G'}(D_2(G')) \cap D_3(G')$ and the vertices in $D_3(G')$ such that one of its neighbors is in $N_{G'}(D_2(G')) \cap D_3(G')$. Assume $|S| = t$. We claim that $N_{G'}(S) \cap D_3(G') = \emptyset$ and $|S| \le 4$. In fact, suppose by the way of contradiction that $N_{G'}(S) \cap D_3(G') \ne \emptyset$ and let $x \in N_{G'}(S) \cap D_3(G')$. By the definition of S, there are two vertices $y, z \in S$ such that $xy \in E(G')$ and yz is a connected component of $G'[D_3(G')]$. By Claim 2 and Lemma 4.5, we induce a contradiction (note that the degree of each vertex of x, y, z is 3, this fact contradicts that $[G_0[PM(x) \cup PM(y) \cup PM(z)], G_0 - G_0[PM(x) \cup PM(y) \cup PM(z)]]_{G_0}$ should be a P_2-edge-cut of G_0 by Claim 2) and thus $N_{G'}(S) \cap D_3(G') = \emptyset$. By an argument similar to the proof of Claim 2, we have $|S| \le 4$. If $|S| \ge 5$, $by |D_2(G')| \le 2$, we can find x, y and z which satisfy the conditions of Claim 2.

Then, we have

$$\begin{aligned} d_3 &= t + \sum_{v\in D_3\setminus S} 1 \le t + \sum_{v\in D_3\setminus S} \sum_{uv\in E, u\notin D_3\setminus S} l(u) = t + \sum_{i\ge 4}\sum_{u\in D_i} \sum_{uv\in E, v\in D_3\setminus S} l(u) \\ &\le t + \sum_{i\ge 4}\sum_{u\in D_i} i\cdot f(i) = t + \sum_{i\ge 4}\sum_{u\in D_i}(i-4) = t + \sum_{i\ge 4}(i-4)\cdot d_i \\ &\le 4 + \sum_{i\ge 4}(i-4)\cdot d_i. \end{aligned} \tag{6}$$

On the other hand, by Claim 1, we have

$$\begin{aligned} 10 &\le 2(2|V(G')| - |E(G')|) = 4|V(G')| - 2|E(G')| = 4|V(G')| - \sum_i id_i \\ &= \sum_i (4-i)d_i = (4-2)d_2 + \sum_{i\ge 3}(4-i)d_i \le 4 + d_3 - \sum_{i\ge 4}(i-4)d_i \\ &\le 4+4 = 8, \end{aligned} \tag{7}$$

a contradiction. Thus, $G' = K_1$ and $G_0(e_1, e_2)$ is collapsible. □

By Lemmas 4.3 and 4.7, we have the following theorem:

Theorem 4.8. *Every 3-connected, essentially 11-connected line graph is Hamilton-connected.*

Next, we consider the Hamiltonicity of 7-connected line graph. If $L(G)$ is 7-connected, then G is essentially 7-edge connected and G_0 is 3-edge connected, essentially 7-edge connected. By an argument very similar to that of Lemma 4.7 and Theorem 4.8(moreover, it is easier than the argument of 4.7 and 4.8), we have the following theorems:

Lemma 4.9. *Let G be 3-connected, essentially 7-edge connected graph. Then $G_0(e_1, e_2)$ is collapsible.*

Thus, by Lemma 4.3, we have

Theorem 4.10. *Every 7-connected line graph is Hamilton-connected.*

In [16], Ryjáček and Vrána introduced a closure named multigraph closure which turns a claw-free graph into the line graph of a multigraph while preserving its Hamilton-connectedness. Using Theorem 9 in [16], Theorem 4.8 can be extended to claw-free graph.

Corollary 4.11. *Every 3-connected, essentially 11-connected claw-free graph is Hamilton-connected.*

5. Open problem

It is well known that the line graph of the graph obtained by subdividing each edge of the Petersen graph exactly once is a 3-connected claw-free graph without a Hamiltonian cycle. So Lai et al. conjectured that the minimum essential connectivity that guarantees the existence of a Hamiltonian cycle in a 3-connected line graph is 4. We investigated the 3-connected and essentially 4-connected line graphs in [19], in that note we pointed out that the conjecture is incorrect since a counterexample can be obtained by subdividing a perfect matching of a snark.

In [10], Kužel and Xiong show that every 4-connected line is Hamiltonian if and only if it is Hamilton-connected. By Theorem 4.8, it is natural to consider the following problem: What is the minimum integer $k \geq 5$ such that a 3-connected, essentially k-connected line graph graph is Hamiltonian if and only if it is Hamilton-connected?

Acknowledgments

We would like to thank the anonymous referees for their kind help and valuable suggestions.

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