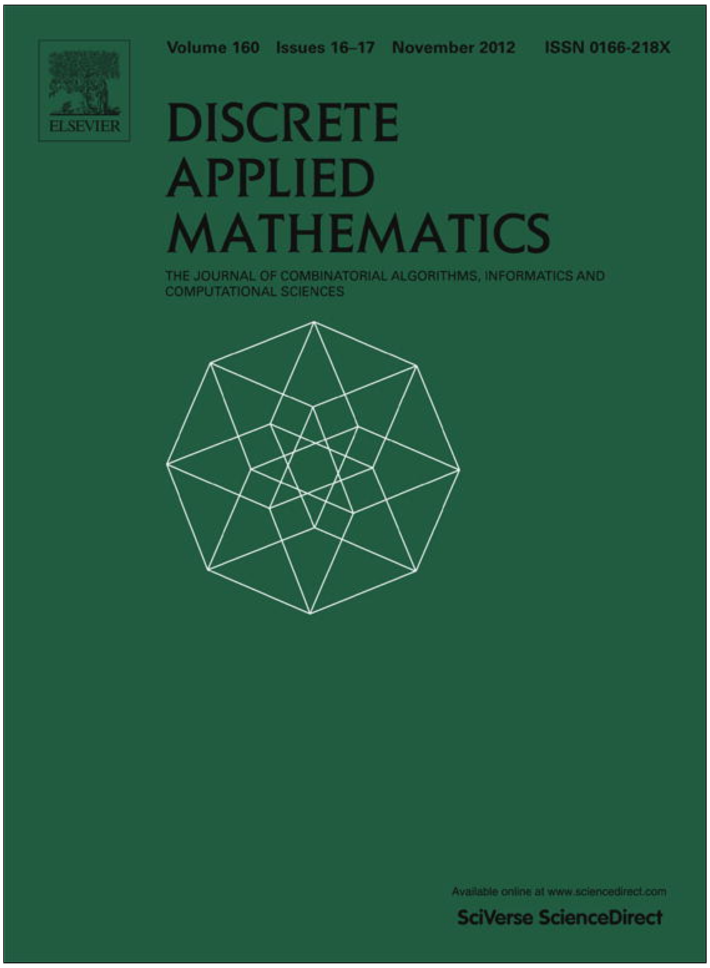
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Characterization of removable elements with respect to having k disjoint bases in a matroid

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ABSTRACT

The well-known spanning tree packing theorem of Nash-Williams and Tutte characterizes graphs with k edge-disjoint spanning trees. Edmonds generalizes this theorem to matroids with k disjoint bases. For any graph G that may not have k-edge-disjoint spanning trees, the problem of determining what edges should be added to G so that the resulting graph has k edge-disjoint spanning trees has been studied by Haas (2002) [11] and Liu et al. (2009) [17], among others. This paper aims to determine, for a matroid M that has k disjoint bases, the set $E_k(M)$ of elements in M such that for any $e \in E_k(M)$, $M - e$ also has k disjoint bases. Using the matroid strength defined by Catlin et al. (1992) [4], we present a characterization of $E_k(M)$ in terms of the strength of M. Consequently, this yields a characterization of edge sets $E_k(G)$ in a graph G with at least k edge-disjoint spanning trees such that $\forall e \in E_k(G)$, $G - e$ also has k edge-disjoint spanning trees. Polynomial algorithms are also discussed for identifying the set $E_k(M)$ in a matroid M, or the edge subset $E_k(G)$ for a connected graph G.

1. Introduction

The number of edge-disjoint spanning trees in a network, when modeled as a graph, often represents certain strength of the network [8]. The well-known spanning tree packing theorem of Nash-Williams [18] and Tutte [23] characterizes graphs with k edge-disjoint spanning trees, for any integer $k > 0$. For any graph G, the problem of determining which edges should be added to G so that the resulting graph has k edge-disjoint spanning trees has been studied; see [11,17], among others. However, it has not been fully studied that for an integer $k > 0$, if a graph G has k edge-disjoint spanning trees, what kind of edge $e \in E(G)$ has the property that $G - e$ also has k-edge-disjoint spanning trees. The research of this paper is motivated by this problem. In fact, we will consider the problem that, if a matroid M has k disjoint bases, what kind of element $e \in E(M)$ has the property that $M - e$ also has k disjoint bases.

We consider finite graphs with possible multiple edges and loops, and follow the notation of Bondy and Murty [1] for graphs, and Oxley [19] or Welsh [24] for matroids, except otherwise defined. Thus for a connected graph G, $\omega(G)$ denotes the number of components of G. For a matroid M, we use ρ_M (or ρ, when the matroid M is understood from the context) to denote the rank function of M, and $E(M)$, $\mathcal{C}(M)$ and $\mathcal{B}(M)$ to denote the ground set of M, and the collections of the circuits and the bases of M, respectively. Furthermore, if M is a matroid with $E = E(M)$, and if $X \subset E$, then $M - X$ is the restricted matroid of M obtained by deleting the elements in X from M, and M/X is the matroid obtained by contracting elements in X from M. As in [19,24], we use $M - e$ for $M - \{e\}$ and M/e for $M/\{e\}$.

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The *spanning tree packing number* of a connected graph G, denoted by $\tau(G)$, is the maximum number of edge-disjoint spanning trees in G. A survey on spanning tree packing number can be found in [20]. By definition, $\tau(K_1) = \infty$. For a matroid M, we similarly define $\tau(M)$ to be the maximum number of disjoint bases of M. Note that by definition, if M is a matroid with $\rho(M) = 0$, then for any integer $k > 0$, $\tau(M) \geq k$. The following theorems are well known.

Theorem 1.1 (*Nash-Williams [18] and Tutte [23]*)**.** *Let G be a connected graph with $E(G) \neq \emptyset$, and let $k > 0$ be an integer. Then $\tau(G) \geq k$ if and only if for any $X \subseteq E(G)$, $|E(G - X)| \geq k(\omega(G - X) - 1)$.*

Theorem 1.2 (*Edmonds [9]*)**.** *Let M be a matroid with $\rho(M) > 0$. Then $\tau(M) \geq k$ if and only if $\forall X \subseteq E(M)$, $|E(M) - X| \geq k(\rho(M) - \rho(X))$.*

Let M be a matroid with rank function r. For any subset $X \subseteq E(M)$ with $\rho(X) > 0$, the *density* of X is

$$d_M(X) = \frac{|X|}{\rho_M(X)}.$$

When the matroid M is understood from the context, we often omit the subscript M. We also use $d(M)$ for $d(E(M))$. Following the terminology in [4], the *strength* $\eta(M)$ and the *fractional arboricity* $\gamma(M)$ of M are respectively defined as

$$\eta(M) = \min\{d(M/X) : \rho(X) < \rho(M)\}, \quad \text{and} \quad \gamma(M) = \max\{d(X) : \rho(X) > 0\}.$$

Thus Theorem 1.2 above indicates that

$$\tau(M) = \lfloor \eta(M) \rfloor. \tag{1}$$

For an integer $k > 0$ and a matroid M with $\tau(M) \geq k$, we define $E_k(M) = \{e \in E(M) : \tau(M - e) \geq k\}$. Likewise, for a connected graph G with $\tau(G) \geq k$, $E_k(G) = \{e \in E(G) : \tau(G - e) \geq k\}$. Using Theorem 1.1, Gusfield proved that high edge-connectivity of a graph would imply high spanning tree packing number.

Theorem 1.3 (*Gusfield [10]*)**.** *Let $k > 0$ be an integer, and let $\kappa'(G)$ denote the edge-connectivity of a graph G. If $\kappa'(G) \geq 2k$, then $\tau(G) \geq k$.*

The next result strengthens Gusfield's theorem, and indicates a sufficient condition for a graph G to satisfy $E_k(G) = E(G)$.

Theorem 1.4 (*Theorem 1.1 of [5]*)**.** *Let $k > 0$ be an integer, and let $\kappa'(G)$ denote the edge-connectivity of a graph G. Then $\kappa'(G) \geq 2k$ if and only if $\forall X \subseteq E(G)$ with $|X| \leq k$, $\tau(G - X) \geq k$. In particular, if $\kappa'(G) \geq 2k$, then $E_k(G) = E(G)$.*

A natural question is to characterize all graphs G with the property $E_k(G) = E(G)$. More generally, for any graph G with $\tau(G) \geq k$, we are to determine the edge subset $E_k(G)$. These questions can be presented in terms of matroids in a natural way. The main purpose of this paper is to characterize $E_k(M)$, for any matroid with $\tau(M) \geq k$. The next theorem is our main result.

Theorem 1.5. *Let M be a matroid and $k > 0$ be an integer. Each of the following holds.*

(i) *Suppose that $\tau(M) \geq k$. Then $E_k(M) = E(M)$ if and only if $\eta(M) > k$.*
(ii) *In general, $E_k(M)$ equals the maximal subset $X \subseteq E(M)$ such that $\eta(M|X) > k$.*

For a connected graph G with $M(G)$ denoting its cycle matroid, let $\eta(G) = \eta(M(G))$ and $\gamma(G) = \gamma(M(G))$. Then Theorem 1.5, when applied to cycle matroids, yields the corresponding theorem for graphs.

Corollary 1.6. *Let G be a connected graph and $k > 0$ be an integer. Each of the following holds.*

(i) *If $\tau(G) \geq k$, $E_k(G) = E(G)$ if and only if $\eta(G) > k$.*
(ii) *In general, $E_k(G)$ equals the maximal subset $X \subseteq E(G)$ such that every component of $\eta(G[X]) > k$.*

In the next section, we shall discuss properties of the strength and the fractional arboricity of a matroid M, which will be useful in the proofs of our main results. We will prove a decomposition theorem in Section 3, which will be applied in the characterizations of $E_k(M)$ and $E_k(G)$ in Section 4. In the last section, we shall develop polynomial algorithms to locate the sets $E_k(M)$ and $E_k(G)$.

2. Strength and fractional arboricity of a matroid

Both parameters $\eta(M)$ and $\gamma(M)$, and the problems related to uniformly dense graphs and matroids (defined below) have been studied by many; see [4,2,3,6,7,13–15,15,21,22], among others. From the definitions of $d(M)$, $\eta(M)$ and $\gamma(M)$, we immediately have, for any matroid M with $\rho(M) > 0$,

$$\eta(M) \leq d(M) \leq \gamma(M). \tag{2}$$

As in [4], a matroid M satisfying $\eta(M) = \gamma(M)$ is called a *uniformly dense matroid*. Both $\eta(M)$ and $\gamma(M)$ can also be described by their behavior in some parallel extension of the matroid. For an integer $t > 0$, let M_t denote matroid obtained from M by replacing each element $e \in E(M)$ by a parallel class of t elements. See p. 252 of [16]. This matroid M_t is usually referred as *the t-parallel extension* of M. For $X \subseteq E(M)$, we use X_t to denote both the matroid $(M|X)_t$ and the set $E((M|X)_t)$.

Theorem 2.1 (*Theorem 4 of [4], and Lemma 1 of [16]*). *Let M be a matroid and let $s \ge t > 0$ be integers. Then we have the following.*

(i) $\eta(M) \ge \frac{s}{t}$ *if and only if* $\eta(M_t) \ge s$.
(ii) $\gamma(G) \le \frac{s}{t}$ *if and only if* $\gamma(M_t) \le s$.
(iii) $t\eta(M) = \eta(M_t)$.
(iv) $t\gamma(M) = \gamma(M_t)$.

Theorem 2.2 (*Theorem 6 of [4]*). *Let M be a matroid. The following are equivalent.*

(i) $\eta(M) = d(M)$.
(ii) $\gamma(M) = d(M)$.
(iii) $\eta(M) = \gamma(M)$.
(iv) $\eta(M) = \frac{s}{t}$, *for some integers $s \ge t > 0$, and M_t, the t-parallel extension of M, is a disjoint union of s bases of M.*
(v) $\gamma(M) = \frac{s}{t}$, *for some integers $s \ge t > 0$, and M_t, the t-parallel extension of M, is a disjoint union of s bases of M.*

For each integer $k > 0$, define

$$\mathcal{T}_k = \{M : \tau(M) \ge k\}.$$

Proposition 2.3. *The matroid family $\mathcal{T}_k$ satisfies the following properties.*

(C1) *If $\rho(M) = 0$, then $M \in \mathcal{T}_k$.*
(C2) *If $M \in \mathcal{T}_k$ and if $e \in E(M)$, then $M/e \in \mathcal{T}_k$.*
(C3) *Let $X \subseteq E(M)$ and let $N = M|X$. If $M/X \in \mathcal{T}_k$ and if $N \in \mathcal{T}_k$, then $M \in \mathcal{T}_k$.*

Proof. Recall that the bases of the contraction M/X has the following form; see, for example, Corollary 3.1.9 of by [19].

$$\mathcal{B}(M/X) = \{B' \subseteq E - X : B' \cup B_X \in \mathcal{B}(M)\}, \quad \text{where } B_X \in \mathcal{B}(M|X). \tag{3}$$

Since $\rho(M) = 0$, $\eta(M) = \infty$, (C1) follows from the definition of η immediately.

If e is a loop of M, then e is not in any basis of M and so by (3), $M/e = M - e$. Thus $\tau(M/e) = \tau(M - e) = \tau(M) \ge k$. Therefore $M/e \in \mathcal{T}_k$.

Suppose that e is not a loop. Let $B_1, \ldots, B_k$ be disjoint bases of M. We assume that $\forall i \in \{1, 2, \ldots, k\}$, if $e \notin B_i$, then $C_i = C_M(e, B_i)$ is the unique circuit of $B_i \cup e$. Since e is not a loop, $\exists e_i \in C_i - e$. Define $B_i' = B_i \cup e - e_i$, if $e \notin B_i$; $B_i' = B_i$, if $e \in B_i$. It follows that $B_1', B_2', \ldots, B_k'$ are bases of M such that for any $i \ne j$, $B_i \cap B_j = e$. Note that if $X = \{e\}$, then $B_X = \{e\} \in \mathcal{B}(M|X)$. It follows from (3) that $B_i' - e$ is a basis of M/e, and all $\{B_i' - e\}$ are disjoint. Hence $M/e \in \mathcal{T}_k$. This proves (C2).

Let $B_1'', B_2'', \ldots, B_k''$ be disjoint bases of N and $B_1', B_2', \ldots, B_k'$ be disjoint bases of M/N. By (3), $B_1' \cup B_1'', B_2' \cup B_2'', \ldots, B_k' \cup B_k''$ are disjoint bases of M, and so $M \in \mathcal{T}_k$. □

Lemma 2.4. *Let M be a matroid with $\rho(M) > 0$, and let $l \ge 1$ be a fractional number. Each of the following holds.*

(i) *(Lemma 10 of [4]) If $X \subset E(M)$ and if $\eta(M|X) \ge \eta(M)$, then $\eta(M/X) = \eta(M)$.*
(ii) *(Theorem 17 of [4]) If $X \subset E(M)$ and if $d(X) = \gamma(M)$, then $\eta(M|X) = \gamma(M|X) = d(X) = \gamma(M)$.*
(iii) *A matroid M is uniformly dense if and only if $\forall X \subseteq E(M)$, $d(X) \le \eta(M)$.*
(iv) *A matroid M is uniformly dense if and only if for any restriction N of M, $\eta(N) \le \eta(M)$.*
(v) *If $d(M) \ge l$, then there exists a subset $X \subseteq E(M)$ with $\rho(X) > 0$ such that $\eta(M|X) \ge l$.*

Proof. (iii) If $\forall X \subseteq E(M)$, $d(X) \le \eta(M)$, then in particular, $d(M) \le \eta(M)$. It follows from (2) that $d(M) = \eta(M)$, and so by Theorem 2.2, M is uniformly dense. Conversely, suppose that there exists an $X \subseteq E(M)$ with $d(X) > \eta(M)$. Then by (2), $\gamma(M) \ge d(X) > \eta(M)$, contrary to the assumption that M is uniformly dense.

(iv) By (iii) of this lemma, if M is uniformly dense, then for any restriction N, $\eta(N) \le d(E(N)) \le \eta(M)$. On the other hand, if M is not uniformly dense, then $\gamma(M) > \eta(M)$. By the definition of $\gamma(M)$, there exists an $X \subset E(M)$ such that $d(X) = \gamma(M)$. It follows from (ii) of this lemma that $\eta(M|X) = d(X) = \gamma(M) > \eta(M)$, contrary to the assumption. Hence M must be uniformly dense.

(v) By (2), $\gamma(M) \ge d(M) \ge l$. By definition of $\gamma(M)$, there exists a subset $X \subseteq E(M)$ with $\rho(X) > 0$, such that $d(X) = \gamma(M)$. Let $N = M|X$. By (ii) of this lemma, $\eta(N) = \gamma(N) = d(N) = \gamma(M) \ge d(M) \ge l$. □

For each rational number $l > 1$, define

$$\mathcal{S}_l = \{M : \eta(M) \geq l\}. \tag{4}$$

Corollary 2.5. *Let $p > q > 0$ be integers and let $l = \frac{p}{q}$. The matroid family $\mathcal{S}_l$ satisfies the following properties.*

(C1) *If $\rho(M) = 0$, then $M \in \mathcal{S}_l$.*
(C2) *If $M \in \mathcal{S}_l$ and if $e \in E(M)$, then $M/e \in \mathcal{S}_l$.*
(C3) *Let $X \subseteq E(M)$ and let $N = M|X$. If $M/X \in \mathcal{S}_l$ and if $N \in \mathcal{S}_l$, then $M \in \mathcal{S}_l$.*

Proof. As (C1) and (C2) follow from the definition of η, it suffices to prove (C3) only. Since $l = \frac{p}{q}$, and since both $\eta(M/X) \geq \frac{p}{q}$ and $\eta(M|X) \geq \frac{p}{q}$, it follows from Theorem 2.1 that $M_q/(X_q) = (M/X)_q \in \mathcal{T}_p$ and $M_q|X_p = (M|X)_q \in \mathcal{T}_p$. By Proposition 2.3(C3), $M_q \in \mathcal{T}_p$, and so by Theorem 2.1, $M \in \mathcal{S}_l = \mathcal{S}_{\frac{p}{q}} = \{M : \tau(M_q) \geq p\}$. This verifies (C3). □

Lemma 2.6. *Let M be a matroid with $\tau(M) \geq k$. Suppose that $X \subseteq E(M)$ satisfies $\eta(M|X) \geq k$. Then $E_k(M|X) \subseteq E_k(M)$.*

Proof. Let $N = M|X$. It is trivial if $E_k(N) = \emptyset$. Assume $E_k(N) \neq \emptyset$. Let $e \in E_k(N)$. Then $\tau(N - e) \geq k$. By definition of contraction, $(M - e)/(N - e) = M/N$. Since $M \in \mathcal{T}_k$, by Proposition 2.3(C2), $M/N \in \mathcal{T}_k$. Since $N - e \in \mathcal{T}_k$ and $(M - e)/(N - e) \in \mathcal{T}_k$, by Proposition 2.3(C3), $M - e \in \mathcal{T}_k$. Therefore $e \in E_k(M)$. □

Lemma 2.7. *Let M be a matroid, and N be a restriction of M. If $M/N, N \in \mathcal{T}_k$, and if both $E_k(N) = E(N)$ and $E_k(M/N) = E_k(M/N)$, then $E_k(M) = E(M)$.*

Proof. Let $e \in E(M)$. There are two cases to be considered.

Case 1: $e \in E(M) - E(N) = E(M/N)$. Since $E_k(M/N) = E(M/N)$, $\tau(M/N - e) \geq k$. But $(M - e)/N = M/N - e \in \mathcal{T}_k$, and $N \in \mathcal{T}_k$, by Proposition 2.3(C3), $M - e \in \mathcal{T}_k$. Hence $e \in E_k(M) \subseteq E(M)$.

Case 2: $e \in E(N)$. Since $E_k(N) = E(N)$, $\tau(N - e) \geq k$. Note that $(M - e)/(N - e) \cong M/N \in \mathcal{T}_k$. By Proposition 2.3(C3), $M - e \in \mathcal{T}_k$, and so $e \in E_k(M) \subseteq E(M)$.

As for any $e \in E(M)$, $e \in E_k(M)$, we have $E_k(M) = E(M)$. □

3. A decomposition theorem

Throughout this section, we assume that M is a matroid with $\rho(M) > 0$. A subset $X \subseteq E(M)$ is an η-*maximal subset* and $M|X$ is an η-*maximal restriction* if for any subset $Y \subseteq E(M)$ with Y properly contains X, we always have $\eta(M|Y) < \eta(M|X)$.

Lemma 3.1. *If $X \subseteq E(M)$ is an η-maximal subset, then X is a closed set in M.*

Proof. Let $\eta(M|X) = \frac{s}{t}$ for some integers $s \geq t > 0$. It follows from Theorem 2.1(i) that $M|X$ has s bases $B_1, B_2, \ldots, B_s$ such that every elements of X lies in at most t of these bases. Suppose that X is not closed. Then there exists an $e \in cl_M(X) - X$, and so $r(X \cup e) = \rho(X)$. Thus $B_1, B_2, \ldots, B_s$ are also bases of $M|(X \cup e)$, and every element in $X \cup e$ lies in at most t of these bases. By Theorem 2.1(i), $\eta(M|(X \cup e)) \geq \frac{s}{t} = \eta(M|X)$, contrary to the assumption that X is an η-maximal subset. □

Lemma 3.2. *Let $W, W' \subset E(M)$ be subsets of $E(M)$, and let $l \geq 1$ be an integer. If $\eta(M|W) \geq l$ and $\eta(M|W') \geq l$, then $\eta(M|(W \cup W')) \geq l$.*

Proof. Let $N = M|(W \cup W')$. Since $N/W = (M|W')/(W \cap W')$, it follows from Corollary 2.5(C2) that $\eta(N/W) = \eta((M|W')/(W \cap W')) \geq \eta(M|W') \geq l$. Hence both $N/W \in \mathcal{S}_l$ and $M|W \in \mathcal{S}_l$. It then follows from Corollary 2.5(C3) that $N \in \mathcal{S}_l$. Thus $\eta(N) \geq l$. □

If N_1 and N_2 are two restrictions of M, we denote by $N_1 \cup N_2 = M|(E(N_1) \cup E(N_2))$, the restriction of M to the union of the ground sets of N_1 and N_2. This notation can be extended to any finite union of restrictions.

Lemma 3.3. *Let N be a restriction of M. Then M must have an η-maximal restriction L such that both $E(N) \subseteq E(L)$ and $\eta(L) \geq \eta(N)$.*

Proof. Suppose that $\eta(N) = l$ for some rational number $l \geq 1$. Let $\mathcal{F}_N$ be the collection of all restrictions N' of M such that $\eta(N') \geq l$. Define $L = \bigcup_{N' \in \mathcal{F}_N} N'$. As $N \in \mathcal{F}_N$, $E(N) \subseteq E(L)$. By Lemma 3.2, $\eta(L) \geq l$. By the definition of L, L must be η-maximal. □

Lemma 3.4. *For any restriction N of M, $\eta(N) \leq \gamma(M)$.*

Proof. By (2), $\eta(N) \leq d(N) \leq \gamma(M)$, and so it follows from the definition of $\gamma(M)$. □

Theorem 3.5. *Let M be a matroid with $\rho(M) > 0$. Then each of the following holds.*

(i) *There exist an integer $m > 0$, and an m-tuple $(l_1, l_2, \ldots, l_m)$ of positive rational numbers such that*

$$\eta(M) = l_1 < l_2 < \cdots < l_m = \gamma(M), \tag{5}$$

and a sequence of subsets

$$J_m \subset \cdots \subset J_2 \subset J_1 = E(M), \tag{6}$$

such that for each i with $1 \le i \le m$, $M|J_i$ is an η-maximal restriction of M with $\eta(M|J_i) = l_i$.

(ii) *The integer m and the sequences (5) and (6) are uniquely determined by M.*

(iii) *For every i with $1 \le i \le m$, J_i is a closed set in M.*

Proof. Let $\mathcal{R}(M)$ denote the collection of all η-maximal restrictions of M. By Lemma 3.3, $\mathcal{R}(M)$ is not empty. Since $E(M)$ is finite,

$$|\mathcal{R}(M)| \text{ is a finite number.} \tag{7}$$

Define

$$sp_\eta(M) = \{\eta(N) : N \in \mathcal{R}\}.$$

By (7), $|sp_\eta(M)|$ is finite. Since $M \in \mathcal{R}$, $|sp_\eta(M)| \ge 1$.

Let $m = |sp_\eta(M)|$. Denote

$$sp_\eta(M) = \{l_1, l_2, \ldots, l_m\}, \quad \text{such that } l_1 < l_2 < \cdots < l_m.$$

By Corollary 2.5(C3), and by the definition of $\gamma(M)$, we have

$$\eta(M) = l_1, \quad \text{and} \quad \gamma(M) = l_m. \tag{8}$$

For each $j \in \{1, 2, \ldots, m\}$, let N_j denote the η-maximal restriction of M with $\eta(N_j) = l_j$, and define

$$J_j = E(N_j). \tag{9}$$

By the definition of $\mathcal{S}_l$,

$$\mathcal{S}_{l_1} \supset \mathcal{S}_{l_2} \supset \cdots \supset \mathcal{S}_{l_m}. \tag{10}$$

Hence by (8)–(10),

$$E(M) = J_1 \supseteq J_2 \supseteq \cdots \supseteq J_m. \tag{11}$$

Since $\mathcal{R}$ and $sp_\eta(M)$ are uniquely determined by M, the integer m, the m-tuple $(l_1, l_2, \ldots, l_m)$ and the sequence (6) are all uniquely determined by M.

(iii) This follows from Lemma 3.1. □

For a matroid M, the m-tuple $(l_1, l_2, \ldots, l_m)$ and the sequence in (6) will be referred as the *η-spectrum* and the *η-decomposition* of M, respectively.

Corollary 3.6. *Let M be a matroid with η-spectrum (5) and η-decomposition (6) such that $m > 1$. Then each of the following holds.*

(i) *M/J_2 is a uniformly dense matroid with $\eta(M/J_2) = \gamma(M/J_2) = \eta(M)$.*

(ii) *For any integer k with $l_1 \le k < l_m$, $E(M)$ has a unique subset Z_k such that Z_k is η-maximal and $\eta(M|Z_k) > k$.*

Proof. (i) Since $m > 1$, $\eta(M|J_2) = l_2 > l_1 = \eta(M)$. It follows from Lemma 2.4 that $\eta(M/J_2) = \eta(M)$. To see that M/J_2 is uniformly dense, we argue by contradiction. Suppose that M/J_2 is not uniformly dense, and that $\gamma(M/J_2) > \eta(M/J_2)$. It follows from the definition of γ that there is a subset $J' \subset E(M/X_2)$ such that $d_{M/J_2}(J') = \gamma(M/J_2)$. By Lemma 3.3, M/J_2 has an η-maximal subset J'' (containing J') such that $\eta((M/J_2)|J'') = l' > \eta(M) = l_1$. If $l' \ge l_2$, then by Lemma 3.2, $\eta(M|(J_2 \cup J')) \ge l_2$, and so J_2 is not η-maximal, contrary to the conclusion of Theorem 3.5. Thus we may assume that $l_2 > l' > l_1$. Since J'' is η-maximal in M/J_2, by Lemma 2.4(i), $J_2 \cup J''$ is also η-maximal, and so by Theorem 3.5, the η-spectrum of M must contain l'. It follows that $(l_1, l_2, \ldots, l_m)$ cannot be the η-spectrum of M, contrary to the assumption of the corollary. This proves (i).

(ii) Let $j < m$ be the smallest integer such that $l_j > k$, and let $Z_k = J_{l_j}$. Then (ii) of this corollary follows from Theorem 3.5. □

The unique subset Z_k stated in Part (ii) of Corollary 3.6 will be called the *η-maximal subset at level k* of M.

Corollary 3.7. *Let M be a matroid with η-spectrum (5). Then M is uniformly dense if and only if $m = 1$.*

Proof. By definition, M is uniformly dense if and only if $\gamma(M) = \eta(M)$. Since $l_1 = \eta(M)$ and $l_m = \gamma(M)$, it follows that M is uniformly dense if and only if $m = 1$. □

4. Characterization of the removable elements with respect to having *k* disjoint bases

The main purpose of this section is to investigate the behavior of the set $E_k(M)$. We first observe that matroids M with $E_k(M) = \emptyset$ can be characterized in terms of the density of M.

Proposition 4.1. *Let $k > 0$ be an integer, and M be a matroid with $\tau(M) \ge k$. Then $E_k(M) = \emptyset$ if and only if $d(M) = k$.*

Proof. Since $\tau(M) \ge k$, M has disjoint spanning bases $B_1, B_2, \ldots, B_k$, and so

$$k\rho(M) = \sum_{i=1}^{k} |B_i| \le |E(M)| = d(M)\rho(M),$$

where equality holds if and only if $k = d(M)$. It follows from Theorem 2.2(iv) (with $s = k$ and $t = 1$) that $k = d(M)$ if and only if $E(M) = \bigcup_{i=1}^{k} B_i$, and so if and only if $E_k(M) = \emptyset$. □

Accordingly, when $\tau(M) \ge k$, $E_k(M) \ne \emptyset$ if and only if $d(M) > k$. We have the following characterization.

Theorem 4.2. *Let $k \ge 2$ be an integer. Let M be a graph with $\tau(M) \ge k$. Then each of the following holds.*

(i) *$E_k(M) = E(M)$ if and only if $\eta(M) > k$.*
(ii) *In general, if $\eta(M) = k$ and if $m > 1$, then $E_k(M) = J_2$, which is the η-maximal subset at level k of M.*

Proof. Since $\tau(M) \ge k$, it follows from (1) that $\eta(M) \ge k$.

(i) If $\eta(M) = k$, then by Theorem 3.5 or by Corollary 3.6, there exists a unique subset $J \subset E(M)$ (say, $J = J_2$ in the η-decomposition of M) such that M/J is uniformly dense with $\eta(M/J) = \gamma(M/J) = \eta(M) = k$. It follows from Theorem 2.2 that $d(E(M/J)) = k$, and so by Proposition 4.1, for any $e \in E(M) - J = E(M/J)$, $\tau((M-e)/J) = \tau(M/J - e) < k$. Thus by $\tau((M-e)|J) = \tau(M|J) \ge k$ and Proposition 2.3(C3), $\tau(M-e) < k$. This proves the necessity of (i).

We shall argue by contradiction to prove the sufficiency. Assume that the sufficiency of (i) fails, and that

$$M \text{ is a counterexample with } \rho(M) \text{ minimized.} \tag{12}$$

Then

$$\eta(M) > k \text{ but } E_k(M) \ne E(M). \tag{13}$$

Claim 1. *M does not have a restriction N with $r(N) < \rho(M)$ and $\eta(N) > k$.*

Suppose not, and that M has such a restriction N with $\eta(N) > k$. As $r(N) < \rho(M)$, it follows from (12) that $E_k(N) = E(N)$. By Lemma 2.4, $\eta(M/N) \ge \eta(M) > k$. Since $\eta(N) > k$, $r(N) > 0$, and so $r(M/N) < \rho(M)$. By (12), $E_k(M/N) = E(M/N)$. By (1), both $M/N, N \in \mathcal{T}_k$, and so by Lemma 2.7 $E_k(M) = E(M)$, contrary to (13). This proves Claim 1.

The next claim follows from Claim 1 and Lemma 2.4(iv).

Claim 2. *M is uniformly dense.*

By (12) and (13), we may assume that

$$\tau(M) \ge k \text{ and } \eta(M) > k, \text{ but } \exists e \in E(M),\ \tau(M-e) \le k-1. \tag{14}$$

Fix $e \in E(M)$ so that $\tau(M-e) \le k-1$ as in (14). It follows from (2) and $\tau(M-e) \le k-1$ that $\eta(M-e) < k$. On the other hand, by Claim 2, M is uniformly dense, and so by Theorem 2.2,

$$k < \eta(M) = d(M) = \frac{|E(M)|}{\rho(M)}.$$

This implies $|E(M)| \ge k\rho(M) + 1$. Since M has $k \ge 2$ disjoint bases, e cannot be a coloop of M, and so $r(M-e) = \rho(M)$. Hence

$$d(E-e) = \frac{|E(M-e)|}{r(M-e)} \ge k.$$

By Lemma 2.4(v), $E(M)$ has a subset $X \subseteq E(M)$ with $\rho(X) > 0$ such that $\eta(M|X) \ge k$. Hence $\tau(M|X) = \lfloor \eta(M|X) \rfloor \ge k$. By Corollary 2.5(C2), $\eta(M/X) \ge \eta(M) > k$. Since $\rho(X) > 0$, $r(M/X) < \rho(M)$.

By $e \in E(M/X)$, and (12), $\tau((M-e)/N) = \tau(M/N - e) \ge k$. As $\tau(N) \ge k$, it follows from Proposition 2.3(C3) that $\tau(M-e) \ge k$, contrary to (14). This proves the sufficiency of (i).

(ii) We assume that $\eta(M) = k$. If $d(M) = k$, then by Proposition 4.1, $E_k(M) = \emptyset$. On the other hand, by Theorem 2.2, M is uniformly dense and so by Corollary 3.7, the η-maximal subset of level k of M is an empty set. Thus if $d(M) = k$, then (ii) holds with $E_k(M) = \emptyset$.

Now assume that $d(M) > k$. By Lemma 2.4(v), $\gamma(M) \geq d(M) > k = \eta(M)$, and so M is not uniformly dense. By Corollary 3.7, if M has (5) as its η-spectrum and sequence (6) as its η-decomposition, then $m > 1$. Hence by Corollary 3.6(ii), the η-maximal subset of level k of M equals J_2. It follows from Part (i) of this theorem that $E_k(M|J_2) = J_2$. By Lemma 2.6,

$$J_2 = E_k(M|J_2) \subseteq E_k(M). \tag{15}$$

On the other hand, by Corollary 3.6(i), M/J_2 is uniformly dense with $\eta(M/J_2) = \eta(M) = k$, and so by Proposition 4.1, $E_k(M/J_2) = \emptyset$. By Theorem 3.5(iii), J_2 is closed in M, and so

$$E_k(M) \subseteq E(M) - E(M/J_2) = J_2. \tag{16}$$

Combining (15) and (16), we have $E_k(M) = J_2$, which proves Part (ii) of the theorem. □

Applying Theorem 4.2 to cycle matroids of connected graphs, we obtain the corresponding theorem for graphs.

Corollary 4.3. *Let $k \geq 2$ be an integer, and G be a connected graph with $\tau(G) \geq k$. Let (5) and (6) denote the η-spectrum and η-decomposition of $M(G)$, respectively. Then each of the following holds.*

(i) *$E_k(G) = E(G)$ if and only if $\eta(G) > k$.*
(ii) *In general, if $\eta(G) = k$ and if $m > 1$, then $E_k(G) = J_2$ equals the η-maximal subset at level k of $M(G)$.*

5. Polynomial algorithms identifying the excessive elements

We remark that there exists a polynomial algorithm which can identify the excessive element subset $E_k(M)$ for any given integer $k > 0$ and any matroid M.

Modifying an algorithm of Kruth (see p. 368 of [24]), Hobbs in [12] obtained an algorithm in $O(|E(M)|^3)(\rho(M)^4)$ time (referred as *Hobbs' Algorithm* below) such that for any matroid M, it computes $\eta(M)$ and $\gamma(M)$, and finds the η-maximal subset J of M such that $\eta(M|J) = \gamma(M)$. By Theorem 3.5, this η-maximal subset J of M equals J_m in (6).

For any matroid M, Hobbs' Algorithm outputs $i_m = \gamma(M)$ and J_m in (6). If $E(M) \neq J_m$ (which means $m > 1$), then by Lemma 2.4(i), we replace M by M/J_m, and run Hobbs' Algorithm to get $\gamma(M) = i_{m-1}$ and the η-maximal subset J' of M/J_m, and so $J_{m-1} = J' \cup J_m$. This process can be repeated m times to generate all subsets $J_1, J_2, \ldots, J_m$ in (6). In particular, by Theorem 4.2, it also computes $E_k(M)$.

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