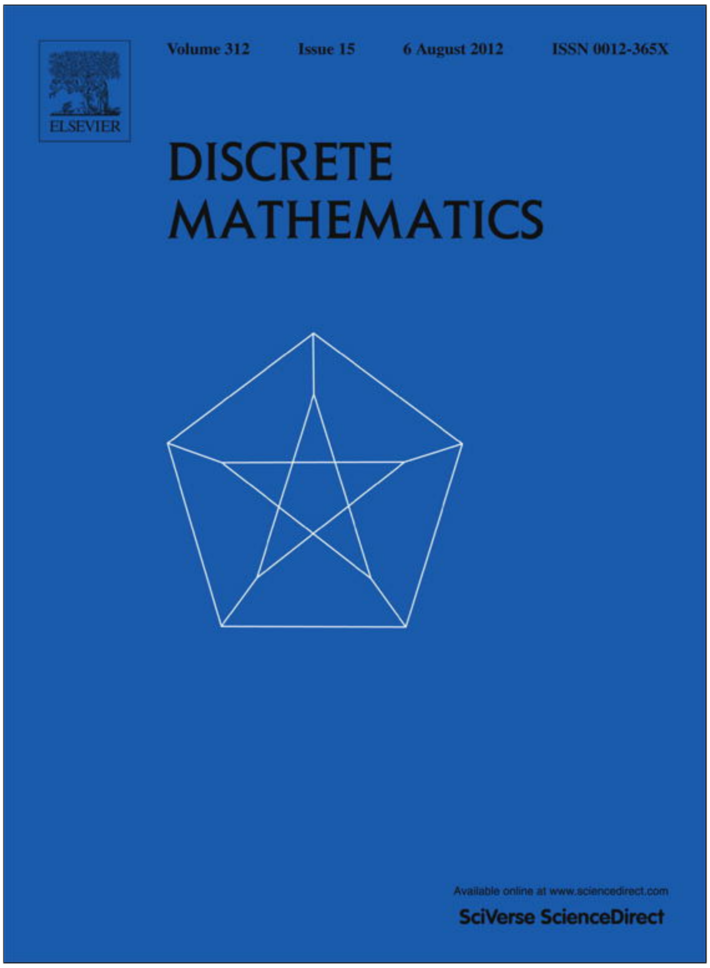
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A dual version of the Brooks group coloring theorem

Deying Li [a], Suohai Fan [b], Hong-Jian Lai [c,d,*], Senmei Yao [d]

[a] School of Information, Renmin University of China, Beijing 100872, PR China
[b] Department of Mathematics, Jinan University, Guangzhou 510632, PR China
[c] College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, PR China
[d] Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

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ABSTRACT

Let G be a 2-edge-connected undirected graph, A be an (additive) Abelian group, and $A^* = A - \{0\}$. A graph G is *A-connected* if G has an orientation $D(G)$ such that for every mapping $b\colon V(G) \mapsto A$ satisfying $\sum_{v\in V(G)} b(v) = 0$, there is a function $f\colon E(G) \mapsto A^*$ such that for each vertex $v \in V(G)$, the sum of f over the edges directed out from v minus the sum of f over the edges directed into v equals $b(v)$. For a 2-edge-connected graph G, define $\Lambda_g(G) = \min\{k$: for any Abelian group A with $|A| \geq k$, G is A-connected $\}$. Let P denote a path in G, let $\beta_G(P)$ be the minimum length of a circuit containing P, and let $\beta_i(G)$ be the maximum of $\beta_G(P)$ over paths of length i in G. We show that $\Lambda_g(G) \leq \beta_i(G) + 1$ for any integer $i > 0$ and for any 2-connected graph G. Partial solutions toward determining the graphs for which equality holds were obtained by Fan et al. in [G. Fan, H.-J. Lai, R. Xu, C.-Q. Zhang, C. Zhou, Nowhere-zero 3-flows in triangularly connected graphs, Journal of Combinatorial Theory, Series B 98 (6) (2008) 1325–1336], among others. In this paper, we completely determine all graphs G with $\Lambda_g(G) = \beta_2(G) + 1$.

1. Introduction

Graphs in this paper are finite and connected, with parallel edges permitted. We follow [1] for undefined terms in graphs. In contrast to [1], we call a 2-regular nontrivial connected graph a *circuit*, and a circuit with k edges is a *k-circuit*. For a graph G, let *girth*(G) be the minimum length of a circuit in G. Let the *circumference* of G, denoted by $c(G)$, be the maximum length of a circuit in G. All groups considered in this paper are (additive) Abelian groups with at least two elements. For undefined terms in group theory, see [5]. Let $\mathbb{Z}_k$ denote the cyclic group of order k. For groups A and B, $A \times B$ denotes the direct product of A and B (see page 26 in [5]).

Let G be a graph with an orientation D. For a vertex $v \in V(G)$, let $E_D^+(v)$ denote the set of edges directed away from v, and let $E_D^-(v)$ denote the set of edges directed in to v.

Let A be an Abelian group with identity 0, let $A^* = A - \{0\}$, let $F(G, A)$ be the set of all functions from $E(G)$ to A, and let $F^*(G, A)$ be the set of all functions from $E(G)$ to A^*. Given a function $f \in F(G, A)$, define $\partial f\colon V(G) \mapsto A$ by

$$\partial f(v) = \sum_{e\in E_D^+(v)} f(e) - \sum_{e\in E_D^-(v)} f(e),$$

where "$\sum$" refers to the addition in A. Define $F_0(G, A) = \{f \in F(G, A)\colon \partial f = 0\}$. Unless otherwise stated, we shall adopt the following convention: if $X \subseteq E(G)$ and $f\colon X \mapsto A$ is a function, then we regard f as a function $f\colon E(G) \mapsto A$ such that $f(e) = 0$ for all $e \in E(G) - X$.

* Corresponding author at: Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA.
E-mail address: hjlai@math.wvu.edu (H.-J. Lai).

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A mapping b: $V(G) \mapsto A$ is an *A-valued zero-sum mapping* on G if $\sum_{v\in V(G)} b(v) = 0$. The set of all A-valued zero-sum mappings on G is denoted by $Z(G, A)$. A function $f \in F(G, A)$ is called an *A-flow* of G if $\partial f(v) = 0$ for every vertex $v \in V(G)$. A graph G is A-connected if G has an orientation $D(G)$ such that for all $b \in Z(G, A)$, there exists f: $E(G) \mapsto A^*$ such that $\partial f(v) = b(v)$ for $v \in V(G)$. Let $\langle A\rangle$ denote the family of graphs that are A-connected. The *group connectivity* of a 2-edge-connected graph G is defined as

$$\Lambda_g(G) = \min\{k: G \text{ is } A\text{-connected for every Abelian group } A \text{ with } |A| \geq k\}.$$

Fix an orientation D of G. An oriented edge uv of D (assumed to be directed from u to v) is called an *arc* (u, v). For $f \in F(G, A)$, an *(A, f)-coloring* of G under the orientation D is a function c: $V(G) \mapsto A$ such that for $e = (u, v) \in D(G)$, $c(u) - c(v) \neq f(e)$. A graph G is *A-colorable* under an orientation D if and only if for every $f \in F(G, A)$, there exists an (A, f)-coloring. Define the *group chromatic number* of a graph G, denoted by $\chi_g(G)$, to be the minimum m such that G is A-colorable for any Abelian group A of order at least m under the orientation D. It was proved in [13,14] that for any finite graph G, the value $\chi_g(G)$ is well defined and finite. Note that if $f(e) = 0$ for any $e \in E(G)$, then an $(A, 0)$-coloring is a proper $|A|$-coloring. Recall that $\chi(G)$, the *chromatic number* of a graph G, is the smallest integer $|A|$ such that G has an $(A, 0)$-coloring. Thus it follows that $\chi(G) \leq \chi_g(G)$.

It is known that if G is a loopless plane graph without cut edges and with geometric dual G^*, then G has a mapping f: $E(G) \mapsto A^*$ with $\partial f = 0$ if and only if $\chi(G^*) = |A|$ (Tutte [16]). In addition, $\Lambda_g(G) = \chi_g(G^*)$ ([7], also see Theorem 3.6 of [2]).

Brooks' Theorem states that equality in the trivial bound $\chi(G) \leq \Delta(G)+1$ hold among connected graphs if and only if G is an odd circuit or a complete graph. The group coloring analogue (Theorem 4.2 of [13]) implies that if G is a 2-edge-connected plane graph, then $\Lambda_g(G) \leq \Delta(G^*) + 1$. The fact that edges incident with a vertex in G^* induce a circuit in G motivates us to consider the problem of using certain circuit lengths of G to describe best possible upper bounds for $\Lambda_g(G)$ for general 2-edge-connected graphs that may not be planar. An objective of this paper is to seek the best possible upper bounds on $\Lambda_g(G)$ with such a feature.

Let P denote a path in G, and let $\beta_G(P)$ be the minimum length of a circuit containing P. For a positive integer i, let $\beta_i(G)$ be the maximum of $\beta_G(P)$ over paths of length i in G. By this definition, we have

$$\text{girth}(G) \leq \beta_1(G) \leq \beta_2(G) \leq \cdots \leq \beta_i(G) \leq \beta_{i+1}(G) \leq \cdots \leq \beta_{c(G)}(G) = c(G). \quad (1)$$

Let H_1 and H_2 be two subgraphs of a graph G. We say that G is a *parallel connection* of H_1 and H_2, if $E(H_1) \cup E(H_2) = E(G)$, $|V(H_1) \cap V(H_2)| = 2$ and $|E(H_1) \cap E(H_2)| = 1$.

In Section 2, we shall show that for any positive integer i with $i \geq 1$,

$$\Lambda_g(G) \leq \beta_i(G) + 1. \quad (2)$$

Determining exactly when the equality $\Lambda_g(G) = \beta_1(G) + 1$ holds seems to be difficult. When $\beta_1(G) = 3$, Fan et al. [4] solved a special case of this problem by showing that if in G every pair of edges are connected by a sequence of mutually intersecting circuits of length at most 3, then $\Lambda_g(G) = \beta_1(G) + 1$ if and only if G can be constructed from odd wheels and K_3 by a finite number of parallel connections. Xu and Zhang [17] conjectured a weaker version of Tutte's 3-flow conjecture (see [6,18]): if G is 4-edge-connected and $\beta_1(G) = 3$, then there exists f: $E(G) \mapsto \mathbb{Z}_3^*$ with $\partial f = 0$. It was further conjectured by DeVos ([3,11]) that every 4-edge-connected graph G with $\beta_1(G) = 3$ satisfies $\Lambda_g(G) \leq 3$. This stronger conjecture was disproved in [11]. As of today, it is not known (see [11]) whether every 5-edge-connected graph G with $\beta_1(G) = 3$ satisfies $\Lambda_g(G) \leq 3$. See a recent survey [10] for more in the literature.

The main purpose of this paper is to prove the inequality (2) and to determine for $i = 2$ the graphs such that equality holds in (2). To describe the main result of this paper, we need to introduce some notation.

Let m and t be positive integers, with $t \geq 2$. We use tK_2 to denote the loopless connected graph with two vertices and t edges. Now we replace each edge of tK_2 by a path of length exactly m, and denote the resulting graph by $K_{2,t}^{1/m}$. Let $K_n^{1/k}$ be the graph obtained from the complete graph K_n by subdividing each edge into exactly k edges.

Theorem 1.1. *If G is a 2-connected graph, then*

$$\Lambda_g(G) \leq \beta_2(G) + 1, \quad (3)$$

where equality holds in (3) if and only if $G \in \{C_k: k \geq 2\} \cup \{K_{2,t}^{1/m}: m \geq 1, t \geq 3\} \cup \{K_4^{1/k}: k \geq 1\}$.

Corollary 1.2. *If G is a 2-connected graph, then*

$$\Lambda_g(G) \leq c(G) + 1, \quad (4)$$

where equality holds if and only if either $c(G)$ is odd and G is an odd circuit, or $c(G)$ is even and G is isomorphic to a $K_{2,t}^{1/(c(G)/2)}$, for some t.

Corollary 1.3. *If G is a 2-edge-connected graph, then (4) holds, with equality if and only if each of the following holds:*

(i) *G has at least one block B such that either $c(G)$ is odd and B is an odd circuit of length $c(G)$, or $c(G)$ is even and B is isomorphic to a $K_{2,t}^{1/(c(G)/2)}$, for some t.*
(ii) *Every block H of G is either a subgraph with $\Lambda_g(H) \leq c(G)$, or $c(G)$ is odd and H is a circuit of length $c(G)$, or $c(G)$ is even and H is isomorphic to a $K_{2,t}^{1/(c(G)/2)}$, for some integer $t \geq 2$.*

Jaeger et al. [7] showed that if G is 3-edge-connected, then $\Lambda_g(G) \leq 6$, which extends Seymour's famous 6-flow theorem from [15]. Thus it is clear that when $\beta_2(G) \geq 6$, all the extremal graphs in Theorem 1.1 will have 2-edge cuts. In Section 2, we investigate some preliminary properties of $\beta_2(G)$, which lead to a proof for (3), and present the extremal graphs in Theorem 1.1, as well as the proofs for Corollaries 1.2 and 1.3, assuming the validity of Theorem 1.1. In Section 3, we prove Theorem 1.1 by completing the characterization of the extremal graphs in Theorem 1.1. We make some remarks on the applications of Theorem 1.1 in the last section.

2. Elementary properties and the extremal examples

In this section, we present some useful properties of $\beta_2(G)$ and display the extremal graphs for Theorem 1.1.

Theorem 2.1 (*Proposition 2.2 of [7]*)**.** *Let G be a connected graph and A be an Abelian group. The following are equivalent.*

(i) $G \in \langle A \rangle$.
(ii) *For all $\bar{f} \in F(G, A)$, there exists $f \in F_0(G, A)$ such that for all $e \in E(G)$, $f(e) \neq \bar{f}(e)$.*
(iii) *For all $b \in Z(G, A)$, and for all $\bar{f} \in F(G, A)$, there exists $f \in F(G, A)$ such that $\partial f = b$ and for all $e \in E(G)$, $f(e) \neq \bar{f}(e)$.*

Let G be a graph and $X \subseteq E(G)$. The *contraction* G/X is the graph obtained from G by identifying the two ends of every edge $e \in X$ and deleting the resulting loops. Note that even when G is a simple graph, the contraction G/X may have multiple edges. For convenience, we define $G/\emptyset = G$, and write G/e for $G/\{e\}$, where $e \in E(G)$. If H is a subgraph of G, then we write G/H for $G/E(H)$.

Proposition 2.2 (*Proposition 3.2 of [8]*)**.** *If A is an Abelian group with $|A| \geq 3$, then $\langle A \rangle$ satisfies each of the following:*

(C1) $K_1 \in \langle A \rangle$,
(C2) *if $G \in \langle A \rangle$ and $e \in E(G)$, then $G/e \in \langle A \rangle$,*
(C3) *if H is a subgraph of G and if both $H \in \langle A \rangle$ and $G/H \in \langle A \rangle$, then $G \in \langle A \rangle$.*

Lemma 2.3 (*[7,8]*)**.** *Letting C_n denote the circuit with n vertices, we have $C_n \in \langle A \rangle$ if and only if $|A| \geq n + 1$. (Equivalently, $\Lambda_g(C_n) = n + 1$).*

Part (ii) of the next lemma follows immediately from definitions.

Lemma 2.4. *Let G be a connected graph, and let A be an Abelian group.*

(i) *(Lemma 2.1 of [9]) Let T be a connected spanning subgraph of G. If for each edge $e \in E(T)$, G has a subgraph $H_e \in \langle A \rangle$ with $e \in E(H_e)$, then $G \in \langle A \rangle$.*
(ii) *$G \in \langle A \rangle$ if and only if every block of G is A-connected.*

Let G be a graph and H be a subgraph of G. Following Seymour [15], we define the *k-closure* of H in G, denoted $cl_k(H)$, to be $H \cup C^1 \cup C^2 \cup \cdots$, where $C^1, C^2, \ldots$ are circuits of G such that $|E(C^i) - (E(H) \cup_{j=1}^{i-1} E(C^j))| \leq k$.

Corollary 2.5. *Let G be a graph and H be a subgraph of G with $cl_k(H) = G$. Let A be an Abelian group with $|A| \geq k + 1$. If H is A-connected, then G is also A-connected. In particular, if $\Lambda_g(H) \leq k + 1$ and if $cl_k(H) = G$, then $\Lambda_g(G) \leq k + 1$.*

Proof. Suppose that $H \cup C^1 \cup \cdots \cup C^m = G$. We argue by induction on m to show that for any A with $|A| \geq \Lambda_g(H)$, $G \in \langle A \rangle$. Since $|A| \geq \Lambda_g(H)$, this holds if $m = 0$. Now assume that $m \geq 1$. Let $H' = H \cup C^1 \cup \cdots \cup C^{m-1}$. By the induction hypothesis, $H' \in \langle A \rangle$. By the definition of k-closure, every circuit of $G/H' = C^m/(C^m \cap H')$ has length at most k, where $k < |A|$. By Lemmas 2.3 and 2.4, $G/H' \in \langle A \rangle$. It follows by Proposition 2.2(C3) that $G \in \langle A \rangle$. □

Lemma 2.6. *For any graph G with $\kappa'(G) \geq 2$,*

$$\Lambda_g(G) \leq \max\{girth(G) + 1, \beta_1(G)\} \leq \max\{girth(G) + 1, \beta_2(G)\} \leq \beta_2(G) + 1. \tag{5}$$

Proof. By (1), $\text{girth}(G) \leq \beta_1(G)$. If $\beta_1(G) = \text{girth}(G)$, then for every edge $e \in E(G)$, G has a circuit C_e with length $\text{girth}(G)$ and with $e \in E(C_e)$. By Lemma 2.3, for any Abelian group A with $|A| \geq \text{girth}(G) + 1$, $C_e \in \langle A \rangle$. By Lemma 2.4, $G \in \langle A \rangle$, and so $\Lambda_g(G) \leq \text{girth}(G) + 1 = \beta_1(G) + 1$. Hence we may assume $\beta_1(G) > \text{girth}(G)$. Let C^0 be a circuit in G with $|E(C^0)| = \text{girth}(G)$. By Lemma 2.3, $\Lambda_g(C^0) = \text{girth}(G) + 1 \leq \beta_1(G)$. By the definition of $\beta_1(G)$ and by $\kappa'(G) \geq 2$, $cl_{\beta_1(G)-1}(C^0) = G$. By Corollary 2.5, $\Lambda_g(G) \leq \beta_1(G)$. Hence we proved the first inequality of (5). The second inequality of (5) follows from (1). □

Thus (2) and (3), now follow from (1) and (5). Let $\mathcal{E}$ denote the set of all 2-connected graphs satisfying equality in (3), and define

$$\mathcal{E}_k = \{G \in \mathcal{E} : \text{girth}(G) = \beta_2(G) = k\}. \tag{6}$$

By Lemma 2.3, $C_k \in \mathcal{E}_k$. We next show that two other classes of graphs are also in $\mathcal{E}_k$.

Lemma 2.7. *Let $t \geq 2$ and $m \geq 1$ be integers. If $G \cong K_{2,t}^{1/m}$, then $\Lambda_g(G) = \beta_2(G) + 1$.*

Proof. The lemma holds trivially for $m = 1$, and so we assume that $m \geq 2$. Let the two (nonadjacent) vertices of degree t in G be w_1 and w_2; and let the vertices of degree 2 in $V(G) - \{w_1, w_2\}$ be v_j^i, for $1 \leq j \leq m-1$ and $1 \leq i \leq t$, such that for each i with $1 \leq i \leq t$,

$$w_1, v_1^i, v_2^i, \ldots, v_{m-1}^i, w_2$$

is a directed path under a fixed orientation D of G. Note that in this case, $\beta_2(G) = 2m$, and by (5), $\Lambda_g(G) \leq 2m + 1$. To prove that $\Lambda_g(G) = 2m + 1$, it suffices to show that G is not $\mathbb{Z}_{2m}$-connected.

We shall apply the equivalence between Theorem 2.1(i) and (iii) to prove that G is not $\mathbb{Z}_{2m}$-connected. Let $A = \mathbb{Z}_{2m}$, and let $A_e = \{\overline{2}, \overline{4}, \ldots, \overline{2m-2}\} \subset \mathbb{Z}_{2m}$. We shall assume that G is A-connected to show that either of the the following two cases will lead to a contradiction.

Case 1. $t = 2k$. Choose $b \colon V(G) \mapsto A$ to be the mapping given by

$$b(z) = \begin{cases} \overline{0} & \text{if } z \notin \{w_1, w_2\} \\ \overline{1} & \text{if } z = w_1 \\ -\overline{1} & \text{if } z = w_2. \end{cases}$$

Note that $b \in Z(G, A)$. Choose $\overline{f} \colon E(G) \mapsto A$ by, for each i with $1 \leq i \leq t$,

$$\overline{f}(e) = \begin{cases} \overline{1} & \text{if } e = (w_1, v_1^i) \\ \overline{3} & \text{if } e = (v_1^i, v_2^i) \\ \vdots & \vdots \\ \overline{2j+1} & \text{if } e = (v_j^i, v_{j+1}^i) \\ \vdots & \vdots \\ \overline{2m-1} & \text{if } e = (v_{m-1}^i, w_2). \end{cases}$$

Since G is A-connected, by Theorem 2.1(iii) there must be a function $f \in F(G, A)$ such that $\partial f = b$ and such that $f(e) \neq \overline{f}(e)$ for $e \in E(G)$. For each i with $1 \leq i \leq t$, let $x_i = f(w_1, v_1^i)$. Since $b(z) = 0$ for $z = v_j^i$, and since the path $w_1, v_1^i, v_2^i, \ldots, v_{m-1}^i, w_2$ is a directed path, $f(v_j^i, v_{j+1}^i) = x_i$ for $1 \leq j \leq m-2$ and $f(v_{m-1}^i, w_2) = x_i$. By the choice of $\overline{f}$, since $f - \overline{f} \in F^*(G, A)$, we must have $x_i \in A_e$. It follows by $\partial f(w_1) = b(w_1)$ that $1 \equiv \sum_{i=1}^{t} x_i \pmod{2m}$. This implies that the sum of certain even numbers can be equal to an odd number, leading to a contradiction.

Case 2. $t = 2k + 1$. Choose $b \colon V(G) \mapsto A$ to be the mapping given by $b(z) = 0$, for all $z \in V(G)$, so $b \in Z(G, A)$. Choose $\overline{f} \colon E(G) \mapsto A$ by defining, for each i with $2 \leq i \leq t$,

$$\overline{f}(e) = \begin{cases} \overline{1} & \text{if } e = (w_1, v_1^i) \\ \overline{3} & \text{if } e = (v_1^i, v_2^i) \\ \vdots & \vdots \\ \overline{2j+1} & \text{if } e = (v_j^i, v_{j+1}^i) \\ \vdots & \vdots \\ \overline{2m-1} & \text{if } e = (v_{m-1}^i, w_2). \end{cases}$$

and,

$$\overline{f}(e) = \begin{cases} \overline{0} & \text{if } e = (w_1, v_1^1) \\ \overline{2} & \text{if } e = (v_1^1, v_2^1) \\ \vdots & \vdots \\ \overline{2j} & \text{if } e = (v_j^1, v_{j+1}^1) \\ \vdots & \vdots \\ \overline{2m-2} & \text{if } e = (v_{m-1}^1, w_2). \end{cases}$$

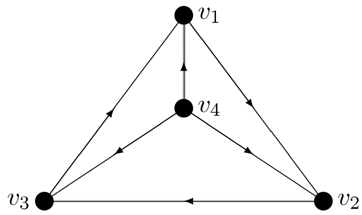

Fig. 2.1. Oriented $K_4^{1/k}$, each line representing a path of k edges.

Since G is A-connected, by Theorem 2.1(iii) there must be a function $f \in F(G, A)$ such that $\partial f = b$ and such that $f(e) \neq \overline{f}(e)$ for $e \in E(G)$. For each i with $1 \leq i \leq t$, let $x_i = f(w_1, v_1^i)$. Since $b(z) = 0$ for $z = v_j^i$, and since the path $w_1, v_1^i, v_2^i, \ldots, v_{m-1}^i, w_2$ is a directed path, $f(v_j^i, v_{j+1}^i) = x_i$ for $1 \leq j \leq m-2$ and $f(v_{m-1}^i, w_2) = x_i$. By the choice of $\overline{f}$, since $f - \overline{f} \in F^*(G, A)$, we must have $x_i \in A_e$ for $i > 1$, and $x_1 \in A - A_e$. It follows by $\partial f(w_1) = b(w_1)$ that $0 \equiv \sum_{i=1}^{t} x_i \pmod{2m}$. This implies that the sum of certain even numbers plus one odd number can be equal to an even number, leading to a contradiction.

These contradictions establish the validity of the lemma. □

Lemma 2.8. *If k is a positive integer, then $\Lambda_g(K_4^{1/k}) = \beta_2(K_4^{1/k}) + 1 = 3k + 1$.*

Proof. By the definition of $K_4^{1/k}$, $\beta_2(K_4^{1/k}) = 3k$. By (5), it suffices to prove that $K_4^{1/k} \notin \langle \mathbb{Z}_3 \times \mathbb{Z}_k \rangle$. Denote the four vertices of degree 3 in $K_4^{1/k}$ by v_1, v_2, v_3, v_4 and orient the edges as shown in Fig. 2.1.

Let $P^{(v_i,v_j)}$ denote the directed (v_i, v_j)-path whose internal vertices have degree 2, and label these paths by

$$P_1 = P^{(v_1,v_4)}, \qquad P_2 = P^{(v_2,v_4)}, \qquad P_3 = P^{(v_3,v_4)}, P_4 = P^{(v_2,v_3)}, \qquad P_5 = P^{(v_3,v_1)}, \quad \text{and} \quad P_6 = P^{(v_1,v_2)}.$$

Let $\overline{f} \in F(K_4^{1/k}, \mathbb{Z}_3 \times \mathbb{Z}_k)$ be a function such that for each $P^{(v_i,v_j)}$, $\overline{f}\colon E(P^{(v_i,v_j)}) \to \{\overline{0}\} \times \mathbb{Z}_k$ is surjective. We argue by contradiction and assume that there exists an A-flow f such that $f(e) \neq \overline{f}(e)$ for any $e \in E(K_4^{1/k})$. Since $\partial f = 0$, f must have the same value on every edge in $E(P^{(v_i,v_j)})$. For $1 \leq j \leq 6$, let (x_j, y_j) denote the common value of f on the edges of P_j, where $x_i \in \mathbb{Z}_3$, $y_i \in \mathbb{Z}_k$. Then we have $x_i \neq \overline{0}$ for $1 \leq i \leq 6$ and $x_1 + x_2 + x_3 = \overline{0}$. Hence $x_1 = x_2 = x_3 = a \in \mathbb{Z}_3$, where either $a = \overline{1}$ or $a = -\overline{1}$. On the other hand, $x_5 = x_4 + a$, $x_6 = x_4 - a$, so $\overline{0} \in \mathbb{Z}_3 = \{x_4, x_5, x_6\}$. The contradiction completes the proof. □

The above results show that these three classes of graphs are extremal cases of Theorem 1.1 when equality in (3) holds. We shall prove that they are the only extremal graphs, mainly in the next section.

Lemma 2.9. *Let G be a 2-connected graph. Each of the following holds.*

(i) *If $\Delta(G) = 2$, then $\Lambda_g(G) = \beta_2(G) + 1$ if and only if $G \cong C_m$ for some integer $m \geq 2$.*
(ii) *If $\Delta(G) \geq 3$ and if G is not simple, then $\Lambda_g(G) = \beta_2(G) + 1$ if and only if $G = K_{2,s}^1$ for some integer $s \geq 3$.*
(iii) *Let G be a graph with girth$(G) = k \geq 3$ and let C^1, C^2 be two distinct k-circuits in G. If C^1 and C^2 have at least one common edge, then the intersection of C^1 and C^2 must be a path of length at most $k/2$.*

Proof. (i) follows from Lemma 2.3. Suppose that G has parallel edges. By (5), $\beta_2(G) = \text{girth}(G) = 2$. Thus (ii) follows from Lemma 2.7. Part (iii) follows from the assumption that girth$(G) = k$. □

We now assume that validity of Theorem 1.1 to prove Corollaries 1.2 and 1.3.

Proof of Corollary 1.2. By (3), $\Lambda_g(G) \leq \beta_2(G) + 1 \leq c(G) + 1$. By (5), when the equality holds in (4), we must have girth$(G) = \beta_2(G) = c(G)$. As $g(K_4^{1/k}) < c(K_4^{1/k})$, Corollary 1.2 follows from Theorem 1.1. □

Proof of Corollary 1.3. Let $H_1, H_2, \ldots, H_s$ be the blocks of G. If $\Lambda_g(G) = c(G) + 1$, then by Lemma 2.4, some H_i has its group connectivity number equal to $c(G) + 1$, in which case (4) implies $c(H_i) = c(G)$. Without loss of generality, and by Theorem 1.1, we may assume that $\Lambda_g(H_i) = c(G) + 1$ for $H_i \in \{H_1, \ldots, H_{s'}\}$ and $\Lambda_g(H_i) \leq c(G)$ for $H_i \in \{H_{s'+1}, \ldots, H_s\}$. Thus Corollary 1.3 follows from Corollary 1.2. □

3. Characterization of the extremal graphs

By Lemma 2.9, it suffices to characterize the extremal graphs for 2-connected simple graphs G with girth$(G) \geq 3$ and $\Delta(G) \geq 3$. Moreover, the intersection of any two circuits in G has at most $\lfloor k/2 \rfloor$ edges.

Define $C^2(k, l)$, where $1 \leq l \leq k/2$, to be the union of two k-circuits whose intersection is a path of length l; and $C^3(k, l_1, l_2, l_3)$, where $1 \leq l_1, l_2, l_3 \leq k/2$ and $l_1 + l_2 + l_3 \leq k$, to be the union of three k-circuits among which the intersection of any two circuits is a path of length l_1, l_2, l_3, respectively. See Fig. 3.1 for examples of these graphs.

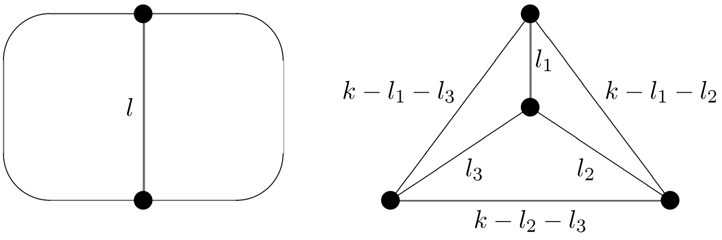

Fig. 3.1. $C^2(k, l)$ and $C^3(k, l_1, l_2, l_3)$.

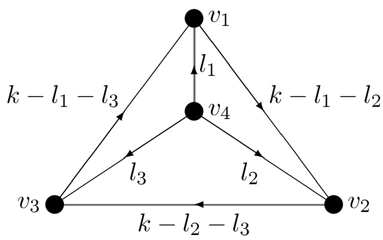

Fig. 3.2. Oriented $C^3(k, l_1, l_2, l_3)$.

Lemma 3.1. *If l_1, l_2, l_3 are not identically equal, then $\Lambda_g(C^3(k, l_1, l_2, l_3)) \le k$.*

Proof. Let A be a group of order at least k. Without loss of generality, we assume that $l_1 < l_3$. Let $H = C^3(k, l_1, l_2, l_3)$ be annotated and oriented as in Fig. 3.2. We shall adopt the same notation as in the proof of Lemma 2.8 and denote $P^{(v_i,v_j)}$ to be the undirected (v_i, v_j)-path of which all the internal vertices have degree 2. Let $\bar{f} \in F(H, A)$. We shall construct an A-flow f such that $f(e) \neq \bar{f}(e)$ for any e in H. Thus by Theorem 2.1, H is A-connected for any A with $|A| \ge k$. Denote $\bar{f}(P^{(v_1,v_4)}) = \{\bar{a}_1, \dots, \bar{a}_{l_1}\}, \bar{f}(P^{(v_2,v_4)}) = \{\bar{b}_1, \dots, \bar{b}_{l_2}\}, \bar{f}(P^{(v_3,v_4)}) = \{\bar{c}_1, \dots, \bar{c}_{l_3}\}, \bar{f}(P^{(v_1,v_2)}) = \{\bar{x}_1, \dots, \bar{x}_{k-l_1-l_2}\}$, $\bar{f}(P^{(v_2,v_3)}) = \{\bar{y}_1, \dots, \bar{y}_{k-l_2-l_3}\}$, and $\bar{f}(P^{(v_3,v_1)}) = \{\bar{z}_1, \dots, \bar{z}_{k-l_1-l_3}\}$. □

Claim 1. *There exist $x, y, z \in A$ satisfying each of the following:*

(i) $y \in A - \{\bar{y}_1, \dots, \bar{y}_{k-l_2-l_3}\}$;
(ii) $x \in A - \{\bar{x}_1, \dots, \bar{x}_{k-l_1-l_2}, y - \bar{b}_1, \dots, y - \bar{b}_{l_2}\}$;
(iii) $z \in A - \{\bar{z}_1, \dots, \bar{z}_{k-l_1-l_3}, y + \bar{c}_1, \dots, y + \bar{c}_{l_3}, x - \bar{a}_1, \dots, x - \bar{a}_{l_1}\}$.

Since $|A| \ge k \ge l_1 + l_2, +l_3$, $|A - \{\bar{y}_1, \dots, \bar{y}_{k-l_2-l_3}\}| \ge k - l_1 \ge l_2 + l_3$. An element y satisfying (i) has at least $l_2 + l_3$ choices. Since $l_1 < l_3$, we have $l_2 + l_3 > l_1 + l_2$. If

$$|\{\bar{x}_1, \dots, \bar{x}_{k-l_1-l_2}\}| < k - l_1 - l_2,$$

then there exists $y \in A - \{\bar{y}_1, \dots, \bar{y}_{k-l_2-l_3}\}$ such that

$$|\{\bar{x}_1, \dots, \bar{x}_{k-l_1-l_2}, y - \bar{b}_1, \dots, y - \bar{b}_{l_2}\}| < k - l_1, \tag{7}$$

and so an x satisfying (ii) can also be chosen. If $|\{\bar{x}_1, \dots, \bar{x}_{k-l_1-l_2}\}| = k - l_1 - l_2$ and $y_1 - \bar{b}_1, \dots, y_{l_2+l_3} - \bar{b}_1$ are $(l_2 + l_3)$ distinct elements, then $y_i - \bar{b}_1 \in \{\bar{x}_1, \dots, \bar{x}_{k-l_1-l_2}\}$ for some i, and so (7) holds as well. Hence we can find x and y satisfying both (i) and (ii) in either case.

With a similar argument, for a given y, either $|\{\bar{z}_1, \cdots, \bar{z}_{k-l_1-l_3}, y + \bar{c}_1, \dots, y + \bar{c}_{l_3}\}| < k - l_1$ or we can choose x such that $x - \bar{a}_1 \in \{\bar{z}_1, \dots, \bar{z}_{k-l_1-l_3}, y + \bar{c}_1, \dots, y + \bar{c}_{l_3}\}$. In either case, $|\{\bar{z}_1, \dots, \bar{z}_{k-l_1-l_3}, y + \bar{c}_1, \dots, y + \bar{c}_{l_3}, x - \bar{a}_1, \dots, x - \bar{a}_{l_1}\}| < k$ and so there must be at least a z satisfying (iii). This proves Claim 1.

By Claim 1, there exist $x, y, z \in A$ satisfying Claim 1(i)–(iii). Set $a = x - z$, $b = y - x$, and $c = z - y$. We define $f : E(H) \mapsto A$ such that $f(P^{(v_1,v_4)}) = \{a\}, f(P^{(v_2,v_4)}) = \{b\}, f(P^{(v_3,v_4)}) = \{c\}, f(P^{(v_1,v_2)}) = \{x\}, f(P^{(v_2,v_3)}) = \{y\}$, and $f(P^{(v_3,v_1)}) = \{z\}$. Note that f defines an A-flow on H. Moreover, by Claim 1(ii) and (iii),

$$a \notin \{\bar{a}_1, \dots, \bar{a}_{l_1}\}, \qquad b \notin \{\bar{b}_1, \dots, \bar{b}_{l_2}\}, \qquad c \notin \{\bar{c}_1, \dots, \bar{c}_{l_3}\},$$

and

$$x \notin \{\bar{x}_1, \dots, \bar{x}_{k-l_1-l_2}\}, \qquad y \notin \{\bar{y}_1, \dots, \bar{y}_{k-l_2-l_3}\}, \quad \text{and} \quad z \notin \{\bar{z}_1, \dots, \bar{z}_{k-l_1-l_3}\}.$$

Hence $f(e) \neq \bar{f}(e)$ for any e in H. This completes the proof. □

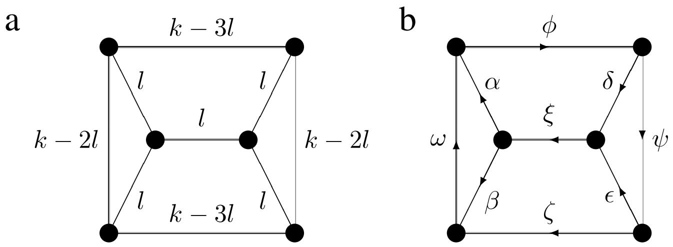

Fig. 3.3.

Lemma 3.2. *Let $K_3 \times K_2$ denote the Cartesian product (see Page* 30 *of* [1]*) of the complete graphs K_3 and K_2. For integers k, l with $k \geq 3l > 0$, let H denote the subdivided $K_3 \times K_2$ as depicted in Fig. 3.3 (a), where the integer i_e on each edge e of $K_3 \times K_2$ indicates that the edge e is subdivided into a path of i_e edges. If $k > 3l$, then H is A-connected for any A with $|A| \geq k$.*

Proof. Let A be an Abelian group of order at least k, and assume that H is oriented as in Fig. 3.3(b). Let $\overline{f} \in F(H, A)$. We shall adopt the convention so that the path labeled with $\gamma \in \{\alpha, \beta, \xi, \delta, \epsilon, \phi, \psi, \zeta, \omega\}$ in Fig. 3.3(b) is denoted as P_γ, and the values that $\overline{f}$ has assigned on the edges in P_γ are denoted as $\overline{\gamma}_1, \overline{\gamma}_2, \ldots$. We will show $\Lambda_g(H) \leq k$ by applying the equivalence between Theorem 2.1(iii) and Theorem 2.1(i) again. □

Claim 2. *There exist $x, y, z, w \in A$ satisfying each of the following:*

(i) $x \in A - \{\overline{\phi}_1, \ldots, \overline{\phi}_{k-3l}\}$,
(ii) $y \in A - \{\overline{\psi}_1, \ldots, \overline{\psi}_{k-2l}, x - \overline{\delta}_1, \ldots, x - \overline{\delta}_l\}$,
(iii) $z \in A - \{\overline{\zeta}_1, \ldots, \overline{\zeta}_{k-3l}, y - \overline{\epsilon}_1, \ldots, y - \overline{\epsilon}_l, x - \overline{\xi}_1, \ldots, x - \overline{\xi}_l\}$,
(iv) $w \in A - \{\overline{\omega}_1, \ldots, \overline{\omega}_{k-2l}, x - \overline{\alpha}_1, \ldots, x - \overline{\alpha}_l, z + \overline{\beta}_1, \ldots, z + \overline{\beta}_l\}$.

The following observations are straightforward.

$$\forall x \in A, \quad |\{\overline{\psi}_1, \ldots, \overline{\psi}_{k-2l}, x - \overline{\delta}_1, \ldots, x - \overline{\delta}_l\}| \leq k - l < k,$$

and

$$\forall x, y \in A, \quad |\{\overline{\zeta}_1, \ldots, \overline{\zeta}_{k-3l}, y - \overline{\epsilon}_1, \ldots, y - \overline{\epsilon}_l, x - \overline{\xi}_1, \ldots, x - \overline{\xi}_l\}| \leq k - l < k.$$

Denote $A - \{\overline{\phi}_1, \ldots, \overline{\phi}_{k-3l}\} = \{x_1, \ldots, x_t\}$, where $t \geq 3l$. If $|\{\overline{\omega}_1, \ldots, \overline{\omega}_{k-2l}\}| < k - 2l$, pick any $x \in \{x_1, \ldots, x_t\}$. Otherwise $|\{\overline{\omega}_1, \ldots, \overline{\omega}_{k-2l}\}| = k - 2l > k - 3l$, and so there exists an i such that $x_i - \overline{\alpha}_1 \in \{\overline{\omega}_1, \ldots, \overline{\omega}_{k-2l}\}$. Hence we can pick $x = x_i$. Thus in either case, there always exists an x so that $|\{\overline{\omega}_1, \ldots, \overline{\omega}_{k-2l}, x - \overline{\alpha}_1, \ldots, x - \overline{\alpha}_l\}| < k - l$.

After x has been chosen, pick any $y \in A - \{\overline{\psi}_1, \ldots, \overline{\psi}_{k-2l}, x - \overline{\delta}_1, \ldots, x - \overline{\delta}_l\}$; and pick any $z \in A - \{\overline{\zeta}_1, \ldots, \overline{\zeta}_{k-3l}, y - \overline{\epsilon}_1, \ldots, y - \overline{\epsilon}_l, x - \overline{\xi}_1, \ldots, x - \overline{\xi}_l\}$. By the choice of x, $|\{\overline{\omega}_1, \ldots, \overline{\omega}_{k-2l}, x - \overline{\alpha}_1, \ldots, x - \overline{\alpha}_l, z + \overline{\beta}_1, \ldots, z + \overline{\beta}_l\}| < k$. Hence there exists $w \in A - \{\overline{\omega}_1, \ldots, \overline{\omega}_{k-2l}, x - \overline{\alpha}_1, \ldots, x - \overline{\alpha}_l, z + \overline{\beta}_1, \ldots, z + \overline{\beta}_l\}$. This proves Claim 2.

By Claim 2, there exist $x, y, z, w \in A$ satisfying Claim 2(i)–(iv). Set $a = x - w$, $b = w - z$, $c = x - z$, $d = x - y$, and $e = y - z$. Define $f: E(H) \mapsto A$ in such a way that f takes a constant value $f(\gamma)$ on every edge of P_γ, for all $\gamma \in \{\alpha, \beta, \xi, \delta, \epsilon, \phi, \psi, \zeta, \omega\}$, as follows:

$$f = \begin{pmatrix} \alpha & \beta & \xi & \delta & \epsilon & \phi & \psi & \zeta & \omega \\ a & b & c & d & e & x & y & z & w \end{pmatrix}.$$

For notational convenience, we also view f as a bijection from $\{\alpha, \beta, \xi, \delta, \epsilon, \phi, \psi, \zeta, \omega\}$ onto $\{a, b, c, d, e, x, y, z, w\}$.

As a mapping $f: E(H) \mapsto A$ under the indicated orientation in Fig. 3.3 (b), f defines an A-flow on H. Moreover, by Claim 2(i)–(iv), for all $\gamma \in \{\alpha, \beta, \xi, \delta, \epsilon\}$, $f(\gamma) \notin \{\overline{\gamma}_i : i = 1, 2, \ldots, l\}$ and $y \notin \{\overline{\psi}_i : 1 \leq i \leq k - 2l\}$, $w \notin \{\overline{\omega}_i : 1 \leq i \leq k - 2l\}$, $z \notin \{\overline{\zeta}_i : 1 \leq i \leq k - 3l\}$, and $x \notin \{\overline{\phi}_i : 1 \leq i \leq k - 3l\}$. Hence $f(e) \neq \overline{f}(e)$, for any $e \in E(H)$. This completes the proof. □

From now on in this section, we assume that

$$G \in \mathcal{E}_k \quad \text{and} \quad \Delta(G) \geq 3. \tag{8}$$

By (6), $\text{girth}(G) = \beta_2(G) = k$ and $\Lambda_g(G) = \beta_2(G) + 1$. We shall prove that either $G = K_{2,\Delta(G)}^{1/(k/2)}$ and k is even, or $G = K_4^{1/(k/3)}$ and $k \equiv 0 \pmod 3$.

Let v be a vertex of degree at least 3 in G and let e_1, e_2, e_3 be three edges incident with v. By the definition of β_2, there exists a k-circuit C^{ij} such that $e_i, e_j \in C^{ij}$ for any $1 \leq i < j \leq 3$. Let U denote the union of the C^{ij}'s. Then U is either $K_{2,3}^{1/(k/2)}$ or $C^3(k, l_1, l_2, l_3)$ for some l_1, l_2, l_3.

Lemma 3.3. *If G satisfies (8) and contains $C^3(k, l, l, l)$, then $l = k/3$.*

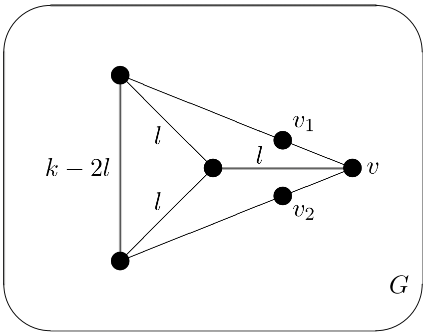

Fig. 3.4.

Proof. By the definition of $C^3(k,l,l,l)$, we have $1 \le l \le k/3$. We argue by contradiction and assume that $l < k/3$ and that G contains $C^3(k,l,l,l)$ as depicted and annotated in Fig. 3.4, where v_1, v_2 are two neighbors of v. Let C^1 and C^2 denote the two k-circuits containing v in this subgraph. Now consider the adjacent edges v_1v, v_2v, and let C be a k-circuit in G containing these two edges. Note that $C^1 \cup C^2 \cup C$ is a subgraph of G that is not $K_{2,3}^{1/(k/2)}$. Since C^1 and C^2 intersect in a path of length $l < k/2$, by Lemma 3.1 it must also be a $C^3(k,l,l,l)$. Hence G contains a subgraph H as depicted in Fig. 3.3 (a). Since $\beta_2(H) = k$, it follows by Lemma 3.2 that H is A-connected for any A with $|A| \ge k$. By the definition of $\beta_2(G)$, if $H \ne G$, then any edge $e \in E(H)$ adjacent to an edge in $E(G) - E(H)$ must be in a circuit of length at most k, and so by the 2-edge-connectedness of G, the closure $cl_{\beta_2(G)-1}(H) = G$. It follows by Corollary 2.5 that $\Lambda_g(G) \le k$, contrary to the assumption that $\Lambda_g(G) = \beta_2(G) + 1$. Hence we must have $k = 3l$. □

Lemma 3.4. *If G satisfies (8) and contains $C^2(k,l)$ with $1 \le l < k/2$, then $l = k/3$ and G contains $C^3(k,l,l,l) = K_4^{1/l}$.*

Proof. Let C^1 and C^2 be two k-circuits in G that intersect in a path of length l. Let v be an endpoint of the intersection path, and let v' and v'' be the two neighbors of v, with $v' \in C^1 - C^2$ and $v'' \in C^2 - C^1$. Let C be a k-circuit containing vv' and vv''. Since $l < k/2$ and $C \ne C^1 \triangle C^2$, $C^1 \cup C^2 \cup C$ is isomorphic $C^3(k,l,m,n)$ for some m, n. By Lemma 3.1, $l = m = n$, so G contains a $C^3(k,l,l,l)$. By Lemma 3.3, $l = k/3$. □

Lemma 3.5. *If G satisfies (8), then each of the following holds:*

(i) *If $k \equiv 0 \pmod 3$, and if C^1 and C^2 are two k-circuits in G which intersect in a path of length $k/3$, then any internal vertex in this path has degree 2 in G.*

(ii) *Suppose that k is even and that any two circuits of G intersect in a path of length either 0 or $k/2$. If G contains $H = K_{2,3}^{1/(k/2)}$ as a subgraph, and u and v are the two degree 3 vertices, then $d_G(w) = 2$ for all $w \in V(H) - \{u, v\}$.*

Proof. (i) Let $P = C^1 \cap C^2$, and let u and v be the endpoints of P. If P has an internal vertex w with $d_G(w) > 2$, then there exists $e, e' \in E(G)$ incident with w such that $e \in E(C^1) \cap E(C^2)$ and $e' \notin E(C^1) \cap E(C^2)$. By definition of β_2, G has a k-circuit C containing both e and e'. Since $e \in E(C^1) \cap E(C^2)$, by Lemmas 2.9 and 3.4, C and C^i intersect in a path of length at least $k/3$ for $i = 1, 2$. However, since w is an internal vertex of P, this is not possible. The contradiction proves (i).

(ii) Denote the three paths in H by P^i, $1 \le i \le 3$, respectively. Let w be an internal vertex of P^i. Arguing similarly as in (i), we conclude that $d_G(w) = 2$. □

Corollary 3.6. *If G satisfies (8) and contains a subgraph $H = K_n^{1/(k/3)}$ for some $n \ge 4$ and $k \equiv 0 \pmod 3$, then for any $v \in V(H)$ with $d_H(v) = 2$, $d_G(v) = 2$.*

Proof. This follows from Lemma 3.5(i). □

Lemma 3.7. *If l is a positive integer, then $\Lambda_g(K_5^{1/l}) \le 3l$.*

The proof of Lemma 3.7 use Proposition 3.8 below. Let H be a graph and let $v \in V(H)$ with $d = d(v) \ge 4$. Denote $N(v) = \{v_1, \ldots, v_d\}$ and denote $e_i = vv_i$. Define $H_{ij} = H - \{e_i, e_j\} + v_iv_j$.

Proposition 3.8 (*Lemma 3.1 (i) of [8]*). *If $H_{ij} \in \langle A \rangle$ for some $i \ne j$, then $H \in \langle A \rangle$.*

Proof of Lemma 3.7. Let A be an Abelian group of order at least 3l. Let $H = K_5^{1/l}$ and let v be a vertex of degree 4 in H. Denote the 4 neighbors of v by v_1, v_2, v_3, v_4. Let $H_{12} = H - \{vv_1, vv_2\} + v_1v_2$. Let C be the $3l$-circuit containing vv_1, vv_2 and $C' = C - \{vv_1, vv_2\} + v_1v_2$. Now C' has length $3l - 1$ and thus is A-connected. Also $cl_{2l}(C') = H_{12}$ and thus $H_{12} \in \langle A \rangle$. By Proposition 3.8, $H \in \langle A \rangle$. □

Lemma 3.9. *If G satisfies (8) and contains a $K_4^{1/(k/3)}$, then $G = K_4^{1/(k/3)}$.*

Proof. Let $H = K_4^{1/(k/3)}$ be a subgraph of G. Denote the four vertices of degree 3 in H by v_1, v_2, v_3, v_4, and refer to all other vertices in H as internal vertices of H. For $1 \le i < j \le 4$, let P^{ij} denote the (v_i, v_j)-path in H of length $k/3$, and for $1 \le i < j < s \le 4$ let C^{ijs} denote the k-circuit in H containing v_i, v_j, v_s.

Assume that $G \ne H$. By Corollary 3.6, every internal vertex of H has degree 2 in G. Since $G \ne H$ and $\kappa'(G) \ge 2$, we may assume that v_4 is incident with an edge $e \notin E(H)$. Let e_i be the edge incident with v_4 in P^{i4} for $1 \le i \le 3$. Now e is adjacent with e_i, for $1 \le i \le 3$. Let C^i be a k-circuit in G containing e, e_i.

We claim that C^1 must intersect C^{124} in a path of $k/3$. Otherwise, by Lemma 3.4, C^1 and C^{124} intersect in a path of length $k/2$. This marks some internal vertex in P^{12} incident with an edge that is not in H. This leads to a contradiction. Hence C^1 intersects C^{124} exactly in P^{14}. Similarly, $C^1 \cap C^{134} = P^{14}$, and so $C^1 \cap H = P^{14}$. With similar arguments, $C^2 \cap H = P^{24}$ and $C^3 \cap H = P^{34}$.

Note that $C^1 \ne C^2$ and $C^1 \cap C^2 = e$. Apply Lemma 3.1 to $C^1 \cup C^2 \cup C^{124}$ to see that C^1 and C^2 must intersect in a path of $k/3$ containing e. Let u be the vertex in the path of $C^1 - P^{14}$ such that the two paths from v_1 to u and from v_4 to u have the same length in C^1. Let P denote the path in C^1 from u to v_4 containing e, so $C^1 \cap C^2 = P$. Similarly $C^1 \cap C^3 = P = C^2 \cap C^3$.

Let $L = H \cup C^1 \cup C^2 \cup C^3$. It follows by Lemma 3.7 that $L = K_5^{1/(k/3)}$ and that $\Lambda_g(L) \le k$. By repeating applications of Lemma 2.3, $\Lambda_g(G/L) \le k$. By Proposition 2.2(C3), $\Lambda_g(G) \le k$, contrary to (8). This contradiction proves that $G = H$. □

Lemma 3.10. *Suppose G satisfies (8). If any two circuits of G intersect in a path of length either 0 or $k/2$, then $G = K_{2,\Delta(G)}^{1/(k/2)}$.*

Proof. Let $v \in V(G)$ and $d(v) = \Delta(G)$. Denote the edges incident with v by $e_1, \ldots, e_{\Delta(G)}$. Let C^{12} be a k-circuit containing e_1 and e_2, and denote the vertex in C^{12} by u such that either of the two (u, v)-paths in C^{12} has the length $k/2$. Let P^1 denote the (u, v)-path in C^{12} containing e_1. Let C^{1m} be the k-circuit containing e_1 and e_m, where $m \ge 3$. By assumption, $|C^{12} \cap C^{1m}| = k/2$. Hence $C^{12} \cap C^{1m} = P^1$. Let $H = C^{12} \cup \cdots \cup C^{\Delta(G)}$. Note that $H = K_{2,\Delta(G)}^{1/(k/2)}$ with u and v as the two common ends of all the paths. By Lemma 3.5(ii), the internal vertices of the paths have degree 2 in G. Moreover, $d_H(u) = d_H(v) = d_G(v) = \Delta(G) \ge d_G(u)$. Hence $G = H$ as $\kappa'(G) \ge 2$. □

Proof of Theorem 1.1. By (5), $\Lambda_g(G) \le \beta_2(G) + 1$. By Lemmas 2.3, 2.7 and 2.8, for $G \in \{C_k: k \ge 2\} \cup \{K_{2,t}^{1/m}: m \ge 1, t \ge 3\} \cup \{K_4^{1/k}: k \ge 1\}$, we have $\Lambda_g(G) = \beta_2(G) + 1$.

Conversely, suppose that $\Lambda_g(G) = \beta_2(G) + 1$. If $\Delta(G) = 2$, then Theorem 1.1 follows from Lemma 2.9. Assume that $\Delta(G) \ge 3$. By Lemmas 3.3, 3.4, 3.9 and 3.10, either k is even and $G = K_{2,\Delta(G)}^{1/(k/2)}$, or $k \equiv 0 \pmod 3$ and $G = K_4^{1/(k/3)}$. This completes the proof of Theorem 1.1. □

4. Applications

We have seen that Theorem 1.1 can be applied to obtain Corollaries 1.2 and 1.3. In this section, we shall present additional evidence that Theorem 1.1 can be applied to study the group connectivity of certain families of graphs. For subgraphs H_1 and H_2 of a graph G, define

$$H_1 \triangle H_2 = H_1 \cup H_2 - E(H_1) \cap E(H_2).$$

Lemma 4.1. *If a graph G has $\Lambda_g(G) \ge m + 1$ with $|V(G)| + |E(G)|$ minimized, then G contains no nontrivial subgraph H such that $\Lambda_g(H) \le m$.*

Proof. If G has a nontrivial subgraph H with $\Lambda_g(H) \le m$, then $|V(G/H)| + |E(G/H)| < |V(G)| + |E(G)|$, we have $\Lambda_g(G/H) \le m$. Since $\Lambda_g(H) \le m$, by Proposition 2.2(C3), $\Lambda_g(G) \le m$, contrary to the assumption of the lemma. □

Corollary 4.2 (*Theorem 3.1, [12]*). *If G is a 2-edge-connected loopless graph with diameter at most 2, then $\Lambda_g(G) \le 6$, where equality holds if and only if G is a 5-circuit.*

Proof. By contradiction, assume that G is a counterexample with $|V(G)| + |E(G)|$ minimized. The diameter of G is at most 2, but $\Lambda_g(G) \ge 7$. By the definition of $\beta_2(G)$, G has a 2-path P_2 with $\beta_2(P_2) = \beta_2(G)$, and hence G has a circuit C containing P_2 with $|E(C)| = \beta_2(P_2)$. Let $V(P_2) = \{v_1, v_2, v_3\}$ and $V(C) = \{v_1, v_2, \ldots, v_m\}$. By Theorem 1.1, $m \ge 6$.

Since the diameter of the graph G is at most 2, G has a (v_1, v_4)-path P' with $|E(P')| \le 2$. Assume $P' = v_1v'v_4$. Since $m \ge 6$, P' is not a path on C. Let P denote a (v_1, v_4)-path on C with $|E(P)| = 3$. Since P' and P are both (v_1, v_4)-paths, $P' \Delta P$ contains a circuit C' whose length is at most $|E(P')| + |E(P)| = 5$. By Lemma 2.3, G has a nontrivial subgraph C' with $\Lambda_g(C') \le 6$, contrary to Lemma 4.1 with $m = 6$. □

A argument similar to the proof above can also be employed to prove the following.

Corollary 4.3. *If G is a 2-edge-connected loopless graph with diameter at most m, where $m \ge 3$, then $\Lambda_g(G) \le 2m + 2$.*

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