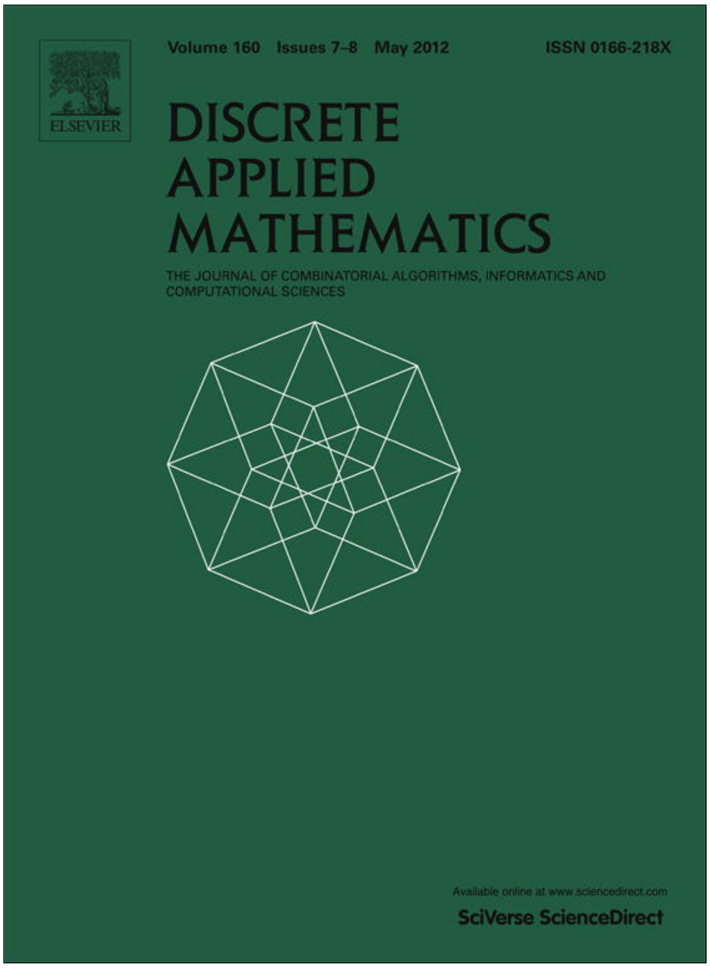
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On dynamic coloring for planar graphs and graphs of higher genus

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ABSTRACT

For integers $k, r > 0$, a (k, r)-*coloring* of a graph G is a proper coloring on the vertices of G by k colors such that every vertex v of degree $d(v)$ is adjacent to vertices with at least $\min\{d(v), r\}$ different colors. The *dynamic chromatic number*, denoted by $\chi_2(G)$, is the smallest integer k for which a graph G has a $(k, 2)$-coloring. A list assignment L of G is a function that assigns to every vertex v of G a set $L(v)$ of positive integers. For a given list assignment L of G, an (L, r)-coloring of G is a proper coloring c of the vertices such that every vertex v of degree $d(v)$ is adjacent to vertices with at least $\min\{d(v), r\}$ different colors and $c(v) \in L(v)$. The *dynamic choice number* of G, $ch_2(G)$, is the least integer k such that every list assignment L with $|L(v)| = k, \forall v \in V(G)$, permits an $(L, 2)$-coloring. It is known that for any graph G, $\chi_r(G) \leq ch_r(G)$. Using Euler distributions in this paper, we prove the following results, where (2) and (3) are best possible.

(1) If G is planar, then $ch_2(G) \leq 6$. Moreover, $ch_2(G) \leq 5$ when $\Delta(G) \leq 4$.
(2) If G is planar, then $\chi_2(G) \leq 5$.
(3) If G is a graph with genus $g(G) \geq 1$, then $ch_2(G) \leq \frac{1}{2}(7 + \sqrt{1 + 48g(G)})$.

1. Introduction

Graphs in this paper are simple and finite. For undefined terminologies and notations see [5,18]. Thus for a graph G, $\Delta(G)$, $\delta(G)$ and $\chi(G)$ denote the maximum degree, minimum degree and chromatic number of G respectively. For $v \in V(G)$, let $N_G(v)$ denote the set of vertices adjacent to v in G, and $d_G(v) = |N_G(v)|$. Vertices in $N_G(v)$ are *neighbors* of v. For an integer $g \geq 0$, let S_g be the orientable surface obtained from the sphere by adding g handles, and let N_g be the non-orientable surface obtained from the sphere by adding g *Möbius* strips (cross-caps). Given an embedding of G on a closed surface, the *genus* $g(G)$ of a graph G is the minimum number g such that G can be embedded on the surface S_g or N_g.

Let G be a graph, $k > 0$ be an integer, $\bar{k} = \{1, 2, \ldots, k\}$, and $c : V(G) \mapsto \bar{k}$ be a map. For $S \subseteq V(G)$, define $c(S) = \{c(u) \mid u \in S\}$. For integers $k > 0$ and $r > 0$, a *(k,r)-coloring* of a graph G is a map $c : V(G) \mapsto \bar{k}$ satisfying both the following.

(C1) $c(u) \neq c(v)$, for every edge $uv \in E(G)$;
(C2) $|c(N_G(v))| \geq \min\{d_G(v), r\}$, for every $v \in V(G)$.

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For a fixed integer $r > 0$, the r-hued chromatic number of G, denoted by $\chi_r(G)$, is the smallest k such that G has a (k, r)-coloring. The concept was first introduced in [13,11], where $\chi_2(G)$ is called *the dynamic chromatic number* of G. Later in [10], a referee suggested the name of conditional chromatic number of G. Recently, we received several comments on the name of conditional coloring, suggesting that does not reveal the nature of the coloring. Therefore, we decided to use the name *r-hued chromatic number* to reflect the use of many colors near a vertex.

By the definition of $\chi_r(G)$, it follows immediately that $\chi(G) = \chi_1(G)$, and so r-hued coloring is a generalization of the classical graph coloring. Let G^2 be the graph defined as the following, $V(G^2) = V(G)$, $E(G^2) = \{uv \mid d_G(u, v) \le 2\}$, then $\chi_{\Delta(G)}(G) = \chi(G^2)$. For any integers $i > j > 0$, any (k, i)-coloring of G is also a (k, j)-coloring of G, and so

$$\chi(G) \le \chi_2(G) \le \cdots \le \chi_{r-1}(G) \le \chi_r(G) \le \cdots \le \chi_{\Delta(G)}(G) = \chi_{\Delta(G)+1}(G) = \cdots. \tag{1}$$

A list assignment L of G is a function that assigns to every vertex v of G a set $L(v)$ of positive integers. An L-coloring is a proper coloring c such that $c(v) \in L(v)$, for every $v \in V(G)$. Such coloring is also called list coloring. G is said to be *k-choosable* if, for every list assignment L with $|L(v)| = k$, for all $v \in V(G)$, there exists an L-coloring of G. The *choice number* (or *list chromatic number*) $ch(G)$ of G, is the least integer k such that G is k-choosable.

There is also a similar generalization for the list coloring. For a given list assignment L of G and a given positive integer r, an *r-hued L-coloring c* of G is an L-coloring of G such that $|c(N_G(v))| \ge \min\{d_G(v), r\}$, for every vertex $v \in V(G)$. We call such coloring an (L, r)-coloring. The *r-hued choice number* (or *list chromatic number*) of G, $ch_r(G)$, is the least integer k such that G admits an (L, r)-coloring, for any list assignment L with $|L(v)| = k$, for every vertex $v \in V(G)$. Similarly, $ch(G) = ch_1(G)$ and $ch_{\Delta(G)}(G) = ch(G^2)$. As for any integers $i > j > 0$, any (L, i)-coloring of G is also an (L, j)-coloring of G, it follows

$$ch(G) \le ch_2(G) \le \cdots \le ch_{r-1}(G) \le ch_r(G) \le \cdots \le ch_{\Delta(G)}(G) = ch_{\Delta(G)+1}(G) = \cdots. \tag{2}$$

For any positive integers k and r, let $L(v) = \bar{k}$, for every vertex v of a graph G. Then every (k, r)-coloring of G is also an (L, r)-coloring of G, and so

$$\chi_r(G) \le ch_r(G). \tag{3}$$

Some recent results are published for the case $r = 2$. In [11], an analogue of Brooks' Theorem for χ_2 is proved. Akbari et al. [1] proved that $ch_2(G) \le \Delta(G) + 1$ if G has no component isomorphic to C_5 and if $\Delta(G) \ge 3$. Later in [7], Esperet disproved a conjecture $ch_2(G) = \max\{ch(G), \chi_2(G)\}$ made in [1]. In [2], Alishahi obtained that $\chi_2(G) \le \chi(G) + 14.06 \ln k + 1$, for any k-regular graph.

The research for general r is also of interest. In [10], it is shown that for $r \ge 2$, $\chi_r(G) \le \Delta(G) + r^2 - r + 1$ if $\Delta(G) \le r$. A *Moore graph* is a regular graph with diameter d and girth $2d + 1$. Ding et al. [6] proved that $\chi_r(G) \le (\Delta(G))^2 + 1$, where equality holds if and only if G is a Moore graph. This is also improved in [12] as $\chi_r(G) \le r(\Delta(G)) + 1$.

The r-hued coloring for graphs G embedded on surfaces is of particular interest. The famous Four Color Theorem [3,4,17] and the Heawood formula [9] provide complete answers to the case when $r = 1$. Heawood [9] proved that if G is a connected graph with a 2-cell embedding on $S_{g(G)}$, then $\chi(G) \le \frac{1}{2}\left(7 + \sqrt{1 + 48g(G)}\right)$. The main results of this paper are given below.

Theorem 1.1. *If G is a planar graph, then the following hold.*

(i) *If* $\Delta(G) \le 4$, *then* $ch_2(G) \le 5$;
(ii) $ch_2(G) \le 6$;
(iii) $\chi_2(G) \le 5$.

Theorem 1.2. *If G is a graph with genus* $g(G) \ge 1$, *then* $ch_2(G) \le \frac{1}{2}\left(7 + \sqrt{1 + 48g(G)}\right)$.

In Section 2, we present some of the mechanisms to be used in the proofs for the main results. Our main tool is the edge-distribution of a plane graph, which allows us to apply induction in our arguments. The proofs for the two main theorems are presented in the last two sections, respectively.

2. Preliminaries

A *plane graph* is a planar graph that is embedded in the plane. Let G be a connected plane graph, and let F be a face of G. Then the boundary of F is the boundary of the open set in the usual topological sense, and it contains the vertices and edges that are incident with F. The degree of F is the number of edges incident with F. We call the face with degree k a *k-face*.

For a given edge $e = v_1v_2$ of G, let d_1, d_2 denote the degrees of the two endpoints v_1 and v_2 of e, and d_1^*, d_2^* denote the degrees of the two faces adjacent at e, respectively. The *edge contribution* of e is defined to be $\Phi(e) = \frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_1^*} + \frac{1}{d_2^*} - 1$. The next result is known as a Lebesgue's formulae.

Lemma 2.1 (*P. 55 in [14]*). *Let G be a plane graph, then* $\sum_{e \in E(G)} \Phi(e) = 2$.

Throughout this paper, for an edge e of a plane graph G, we shall represent the *edge configuration* of e as the 4-tuple (x_1, x_2, x_3, x_4) such that $x_1 \le x_2 \le x_3 \le x_4$, where $\{x_1, x_2, x_3, x_4\} = \{d_1, d_2, d_1^*, d_2^*\}$ as multisets. For convenience, we use (x_1, x_2, x_3, S) with S being a set of integers, to mean that in this configuration, x_4 can be any integer in S. If S is given by an interval (such in Lemma 2.2), then S is the set of the integers inside the interval.

Lemma 2.2. *Let G be a plane graph with $\delta(G) \ge 3$. Then there must be an edge with its configuration falling into one of the following categories.*

(i) $(3, 3, 3, [3, \infty))$;
(ii) $(3, 3, 4, [4, 11])$;
(iii) $(3, 3, 5, [5, 7])$;
(iv) $(3, 4, 4, [4, 5])$;

Proof. We may assume that G is connected. By Lemma 2.1, $\sum_{e \in E(G)} \Phi(e) = 2 > 0$, and so G has an edge e with $\Phi(e) > 0$. We denote the configuration of e by (x_1, x_2, x_3, x_4). Then $\sum_{i=1}^{4} \frac{1}{x_i} > 1$.

Since $\delta(G) \ge 3$, we have $x_i \ge 3$, for each $i \in \{1, 2, 3, 4\}$. As $x_1 \le x_2 \le x_3 \le x_4$, $4 \cdot \frac{1}{x_1} > 1$, and so $x_1 < 4$. This implies that $x_1 = 3$. Thus $\sum_{i=2}^{4} \frac{1}{x_i} > 1 - \frac{1}{3} = \frac{2}{3}$. As $3 \cdot \frac{1}{5} < \frac{2}{3}$, thus $x_2 < 5$, it follows that $x_2 = 3$ or $x_2 = 4$.

If $x_2 = 3$, then $\frac{1}{x_3} + \frac{1}{x_4} > \frac{1}{3}$, hence $x_3 < 6$. It is routine to verify that if $x_3 = 3$, then x_4 can be any number no less than 3; if $x_3 = 4$, then $4 \le x_4 \le 11$; and if $x_3 = 5$, then $5 \le x_4 \le 7$.

If $x_2 = 4$, then $\frac{1}{x_3} + \frac{1}{x_4} > \frac{5}{12}$, and so $x_3 < 5$. Hence $x_3 = 4$ and $x_4 \le 5$. This completes the proof of the lemma. □

By Lemma 2.2, the following properties on the local structure of a plane graph can be obtained.

Lemma 2.3. *Let G be a plane graph with $\delta(G) \ge 3$. Then there must be an edge $e = v_1v_2$ which meets at least one of the following conditions.*

(i) *$d(v_1) \le 4$ and e lies in the boundary of a 3-face;*
(ii) *$d(v_1) = 3$ and e lies in the boundary of a 4-face;*
(iii) *$d(v_1) = d(v_2) = 3$ and e is the common boundary of a 5-face and another l-face where $5 \le l \le 7$;*
(iv) *$d(v_1) = 5$, $5 \le d(v_2) \le 7$ and e is the common boundary of two 3-faces.*

Proof. By Lemma 2.2, G has an edge $e = v_1v_2$ satisfying the conclusion of Lemma 2.2. The conclusions of this lemma will follow by analyzing the four cases listed in Lemma 2.2. □

Lemma 2.4. *Let G be a smallest counterexample to Theorem 1.1. Then G must be connected and $\delta(G) \ge 3$.*

Proof. We argue by contradiction and assume that

$$G \text{ is a counterexample with } |V(G)| \text{ minimized.} \quad (4)$$

Then for some list assignment $\{L(v) : v \in V(G)\}$, G has no $(L, 2)$-coloring. Furthermore, for one such list assignment L and any $v \in V(G)$, $|L(v)| = 5$ if (i) does not hold for G; $|L(v)| = 6$ if (ii) does not hold for G; $L(v) = \{1, 2, 3, 4, 5\}$ if (iii) does not hold for G. By (4), G must be connected with $|V(G)| \ge 6$.

If $\delta(G) = 1$, then let v be a vertex of degree 1 in G and w be the only neighbor of v. Denote $G' = G - v$. By (4), G' has an $(L, 2)$-coloring c. Extending c by coloring v with $c(v) \in L(v) \setminus c(\{w, w'\})$, where w' is another neighbor of w. Then c can be extended to an $(L, 2)$-coloring for G, contrary to (4).

Now suppose that $\delta(G) \ge 2$ and v is a vertex of degree 2. Denote the neighbors of v as x, y. Let x', y' be neighbors of x, y in $G - v$, respectively. By (4), $G' = G - v + xy$ has an $(L, 2)$-coloring c with $c(x) \ne c(y)$. Extending c by coloring v with $c(v) \in L(v) \setminus c(\{x, y\} \cup \{x', y'\})$. Then the extended c is an $(L, 2)$-coloring of G, contrary to (4). So we must have $\delta(G) \ge 3$. □

3. Proof of Theorem 1.1

Arguing by contradiction, we assume that

$$G \text{ is a counterexample to Theorem 1.1 with } |V(G)| \text{ minimized.} \quad (5)$$

Then for some list assignment $\{L(v) : v \in V(G)\}$, G has no $(L, 2)$-coloring. Equivalently, we may assume that for every $v \in V(G)$,

$$|L(v)| = 5, \quad \text{if (i) does not hold for G;} \quad (6)$$

$$|L(v)| = 6, \quad \text{if (ii) does not hold for G ;} \quad (7)$$

$$L(v) = \bar{5}, \quad \text{if (iii) does not hold for G.} \quad (8)$$

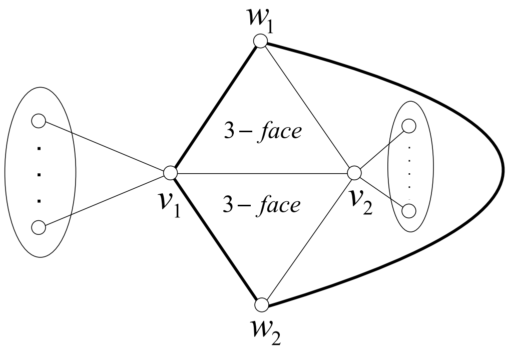

Fig. 1. Graph for Subcase 4.2.

By Lemma 2.4, G must be connected with $\delta(G) \geq 3$. In the arguments below, we start with a plane graph G' with $|V(G')| < |V(G)|$. Then by (5), G' has an $(L, 2)$-coloring c. To obtain a contradiction, we extend the $(L, 2)$-coloring c on G' to one on G. In the following arguments, for all unmentioned vertices w in G', $c(w)$ will not be changed in the extension. Throughout this section, let $e = v_1v_2$ denote an edge satisfying one of (i)–(iv) in Lemma 2.3. By Lemma 2.3, one of the following four cases must occur.

Case 1. $d(v_1) \leq 4$ and e lies in the boundary of a 3-face.

Let $G' = G - v_1$. By (5), G' has an $(L, 2)$-coloring c. Extending c by coloring v_1 with $c(v_1) \in L(v_1) \setminus c(N(v_1))$. As $\delta(G') \geq 2$, v_1 has a pair of adjacent vertices in the 3-face, and so the neighborhood of every vertex has at least 2 different colors. Hence c is an $(L, 2)$-coloring of G, contrary to (5).

Case 2. $d(v_1) = 3$ and e lies in the boundary of a 4-face.

Let $F_1 = v_1v_2x_1x_2$ denote the boundary of this 4-face. Let $G' = G - v_1 + x_2v_2$. By (5), G' has an $(L, 2)$-coloring c. Extending c by coloring v_1 with $c(v_1) \in L(v_1) \setminus c(N(v_1) \cup \{x_1\})$. As c is an $(L, 2)$-coloring of G', $c(x_2) \neq c(v_2)$. The choice of $c(v_1)$ makes c satisfy both (C1) and (C2). And so c is an $(L, 2)$-coloring of G, contrary to (5).

Case 3. $d(v_1) = d(v_2) = 3$ and e is the common boundary of a 5-face and an l-face where $5 \leq l \leq 7$.

Let F_1 denote the 5-face, and F_2 the l-face. For $i = 1, 2$, let x_i be the neighbor of v_i on the boundary of F_1, y_i be the neighbor of v_i on the boundary of F_2. Thus $N(v_1) = \{x_1, y_1, v_2\}$ and $N(v_2) = \{x_2, y_2, v_1\}$. Let $G' = G - v_1 - v_2$. By (5), G' has an $(L, 2)$-coloring c. Extending c by coloring v_1 with $c(v_1)$ from $L(v_1) \setminus c(\{x_1, y_1, x_2\})$ and $c(v_2)$ from $L(v_2) \setminus c(\{x_2, y_2, x_1, v_1\})$ respectively. As c is an $(L, 2)$-coloring of G', and by the choice of $c(v_1)$ and $c(v_2)$, the extended c satisfies both (C1) and (C2), and so c is an $(L, 2)$-coloring of G, contrary to (5).

Case 4. $d(v_1) = 5, 5 \leq d(v_2) \leq 7$ and e is the common boundary of two 3-faces. (This case is not applicable for Theorem 1.1(i).)

Suppose that Theorem 1.1(ii) does not hold. By (7), $|L(v)| = 6$, for all $v \in V(G)$. Let $G' = G - v_1$. By (5), G' has an $(L, 2)g$-coloring c. Since $d(v_1) = 5$ in G, $L(v_1) \setminus c(N_G(v_1)) \neq \emptyset$. Extending c by coloring v_1 with $c(v_1) \in L(v_1) \setminus c(N(v_1))$. Since e lies in a 3-face, $N_G(v_1)$ contains an edge, and so $|c(N(v_1))| \geq 2$. By the definition of $c(v_1)$ and by the assumption that c is an $(L, 2)$-coloring of G', the extended c is an $(L, 2)$-coloring of G, contrary to (5).

Suppose that Theorem 1.1(iii) does not hold. By (8), $L(v) = \bar{5}$, for all $v \in V(G)$. Denote the two faces as $F_1 = v_1v_2w_1$ and $F_2 = v_1v_2w_2$, respectively. Two subcases are discussed below.

Subcase 4.1. $w_1w_2 \notin E(G)$.

We obtain G' from $G - v_1$ by identifying w_1 with w_2 (denoting the new vertex by w). Let $L(w) = \bar{5}$. As w_1 and w_2 are in the same face of $G - v_1$, G' is again planar. By (5), G' has an $(L, 2)$-coloring c, which can also be viewed as an $(L, 2)$-coloring of $G - v_1$ with w_1, w_2 receiving the same color. Since w_1 and w_2 are identified in G', $|c(N_G(v_1))| \leq d_G(v_1) - 1 = 4$, and so $L(v_1) \setminus c(N(v_1)) \neq \emptyset$. Extending c by coloring v_1 with $c(v_1) \in L(v_1) \setminus c(N(v_1))$. By the definition of $c(v_1)$ and by the assumption that c is an $(L, 2)$-coloring of $G - v_1$, the extended c is an $(L, 2)$-coloring of G, contrary to (5).

Subcase 4.2. $w_1w_2 \in E(G)$.

For a plane graph G with a cycle C, let $Ext[C]$ (resp. $Int[C]$) be the subgraph obtained from G by deleting all vertices inside (resp. outside) the cycle C. If $V(Ext[C]) - V(C) \neq \emptyset$ and $V(Int[C]) - V(C) \neq \emptyset$, then C is called a *separating cycle* of G.

Note that the two faces F_1 and F_2 must be contained in one of the 3-cycles, $v_1w_1w_2$ or $v_2w_1w_2$. Without loss of generality, assume that $C = v_1w_1w_2$ that contains both F_i with $i = 1, 2$, see Fig. 1. Since both $d_G(v_i) \geq 5$ with $i = 1, 2$, C must be a separating cycle of G, and so each of $Ext[C]$ and $Int[C]$ has fewer vertices than G.

By (5), each of $Ext[C]$ and $Int[C]$ has an $(L, 2)$-coloring, denoted as c_1 and c_2, respectively. Since $G[v_1, w_1, w_2] \cong K_3$, we may assume that $c_1(v_1) = c_2(v_1)$, $c_1(w_1) = c_2(w_1)$, $c_1(w_2) = c_2(w_2)$.

Since $V(G) = V(Ext[C]) \cup V(Int[C])$ and $V(Ext[C]) \cap V(Int[C]) = \{v_1, w_1, w_2\}$, and since c_1 and c_2 agree on $\{v_1, w_1, w_2\}$, one can construct an $(L, 2)$-coloring c of G by combining c_1 and c_2:

$$c(v) = \begin{cases} c_1(v), & \text{if } z \in V(Ext[C]); \\ c_2(v), & \text{if } z \in V(Inc[C]). \end{cases}$$

As c_1 and c_2 are $(L, 2)$-colorings of $Ext[C]$ and $Int[C]$, respectively, and as $G[v_1, w_1, w_2] \cong K_3$, c is an $(L, 2)$-coloring for G, contrary to (5). This completes the proof of Theorem 1.1. □

As shown in [11], C_5 is planar with $\chi_2(C_5) = 5$. It follows by (3) that Theorem 1.1(i) and (iii) are best possible. We conjecture that C_5 is the only connected planar graph G with $\chi_2(G) = 5$.

When $r > 2$, the r-hued chromatic number $\chi_r(G)$ of a planar graph G may be larger than 5. For example, the wheel W_6 with six vertices has $\chi_3(W_6) = 6$, because any pair of vertices of degree 3 that are not adjacent are adjacent to a common vertex of degree 3, and the unique vertex of degree 5 is adjacent to all other vertices. In fact Lai et al. [10] showed that $\chi_r(T) = \min\{r, \Delta(T)\} + 1$ if T is a tree with $|V(T)| \geq 3$. Hence $\chi_5(T) > 5$ if $\Delta(T) \geq 5$.

4. Proof of Theorem 1.2

An embedding of a graph G on an orientable surface (resp. non-orientable surface) Σ is *minimal* if G cannot be embedded on any orientable (resp. non-orientable) surface Σ' where $g(\Sigma') < g(\Sigma)$. A graph G is said to have orientable (resp. non-orientable) genus g if G is minimally embedded on a surface with orientable (resp. non-orientable) genus g. An embedding of a graph is said to be 2-*cell* if every face of the embedding is homomorphic to an open unit disk. The *Euler characteristic* of a graph G is defined as follows.

$$\Phi(G) = \begin{cases} 2 - 2g, & \text{if } G \text{ has the orientable genus } g; \\ 2 - g, & \text{if } G \text{ has the non-orientable genus } g. \end{cases} \tag{9}$$

If G is a connected graph with a 2-cell embedding on a closed surface, then Euler formula indicates that

$$|V(G)| - |E(G)| + |F(G)| = \Phi(G).$$

The following results are needed in our proofs.

Theorem 4.1 ([19]). *If a connected graph G is minimally embedded on an orientable surface, then the embedding is 2-cell.*

Theorem 4.2 ([15]). *If G is a connected graph, which is not a tree, then G has a minimal non-orientable embedding which is 2-cell.*

Throughout this section, we assume that G is 2-cell embedded on a closed surface. Recall the edge contribution of an edge e is $\Phi(e) = \frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_1^*} + \frac{1}{d_2^*} - 1$. For convenience, let $\Phi'(e) = -\Phi(e)$.

Lemma 4.3 below follows from Theorems 4.1 and 4.2, with a similar argument in Chapter 4 of [14], where the case $g = 0$ is considered.

Lemma 4.3. *If a connected graph G is minimally embedded on a closed surface then*

$$\sum_{e \in E(G)} \Phi(e) = \Phi(G).$$

Proof of Theorem 1.2. Let $g(G)$ denote the genus of G and $h(G) = \frac{1}{2}\left(7 + \sqrt{1 + 48g(G)}\right)$. By contradiction, we assume that

$$G \text{ is a counterexample to Theorem 1.2} |V(G)| \text{ minimized.} \tag{10}$$

Then $g(G) \geq 1$, $ch_2(G) > h(G)$, and G has an assignment $\{L(v) : v \in V(G)\}$ with $|L(v)| = h(G)$, $\forall v \in V(G)$, such that G has no $(L, 2)$-coloring. By (10), G must be connected. We establish each of the following claims. The first claim is an observation following immediately from the definition of $(L, 2)$-colorings.

Claim 1. $|V(G)| \geq h(G) + 1$.

Claim 2. $\delta(G) \geq h(G) - 2$.

We prove $\delta(G) \geq 3$ first. Let v be a vertex with $d_G(v) = \delta(G)$. If $d_G(v) = 1$, let $N_G(v) = \{w\}$, $w' \in N_G(w) - \{v\}$ and $G' = G - v$. By (10), $ch_2(G') \leq h(G')$. By the definition of genus, $g(G') \leq g(G)$, and so $ch_2(G') \leq h(G') \leq h(G)$. Thus any $(L, 2)$-coloring c of G' can be extended to an $(L, 2)$-coloring of G by coloring v with $c(v) \in L(v) \setminus c(\{w, w'\})$, contrary to (10).

If $d_G(v) = 2$, denote $N_G(v) = \{x, y\}$, and let x' (resp. y') be a neighbor of x (resp. y) other than v. Let $G' = G - v + xy$. As G is 2-cell embedded on a surface with x and y on the same face of $G - v$, by the dentition of genus, $g(G') \leq g(G)$. Hence $ch_2(G') \leq h(G') \leq h(G)$. By (10), G' has an $(L, 2)$-coloring c. As $g(G) \geq 1$, $h(G) > 5$. Hence we can extend c by coloring v with $c(v) \in L(v) \setminus c(\{x, y, x', y'\})$. As c is an $(L, 2)$-coloring of G' and by the choice of $c(v)$, c is an $(L, 2)$-coloring of G, contrary to (10).

Hence $\delta(G) \geq 3$. We argue by contradiction to prove Claim 2. Assume that G has a vertex v with $d_G(v) \leq h(G) - 3$. As $\delta(G) \geq 3$, $\exists x, y \in N_G(v)$ with $x \neq y$. Let $G' = G - v + xy$. With the same argument above, $g(G') \leq g(G)$. Hence $ch_2(G') \leq h(G') \leq h(G)$. By (10), G' has an $(L, 2)$-coloring c. Let x', y' be a neighbor of x, y in $G - v$, respectively. Extending c by coloring v with $c(v) \in L(v) \setminus c(N(v) \cup \{x', y'\})$. Since x, y are adjacent in G', $c(x) \neq c(y)$. Since $\delta(G) \geq 3$, $\delta(G') \geq 2$, and so the extended c violates (10). This proves Claim 2.

Claim 3. *Let $e = v_1v_2$ be an edge in G. Then either $d_1 \geq h(G)$ or $d_2 \geq h(G)$.*

We assume otherwise that $d_i = d_G(v_i) \leq h(G) - 1$, $i = 1, 2$. Denote $G' = G - v_1 - v_2$. By (10), G' has an $(L, 2)$-coloring c. Denote $N_1 = N_G(v_1) \setminus \{v_2\}$, $N_2 = N_G(v_2) \setminus \{v_1\}$. Then $\max\{|N_1|, |N_2|\} \leq h(G) - 2$. If $\min\{|c(N_1)|, |c(N_2)|\} \geq 2$, then extend c by coloring v_1 with $c(v_1) \in L(v_1) \setminus c(N_1)$ and v_2 with $c(v_2) \in L(v_2) \setminus c(\{N_2 \cup v_1\})$. As c is an $(L, 2)$-coloring of G' and by the choices of $c(v_1)$ and $c(v_2)$, c is an $(L, 2)$-coloring of G, contrary to (10).

Thus we assume that $|c(N_2)| = 1$. Then pick $v_1' \in N_G(v_1) - \{v_2\}$. Extending c by coloring v_1 with $c(v_1) \in L(v_1) \setminus c(N_1 \cup N_2)$ and v_2 with $c(v_2) \in L(v_2) \setminus c(\{N_2 \cup \{v_1, v_1'\}\})$. As c is an $(L, 2)$-coloring of G' and by the choices of $c(v_1)$ and $c(v_2)$, c is an $(L, 2)$-coloring of G, contrary to (10). This proves Claim 3.

Claim 4. *Let $e = v_1v_2$ be an edge in G. If $3 \in \{d_1^*, d_2^*\}$, then $d_i \geq h(G)$, $i = 1, 2$.*

If not, we assume that $d_1 \leq h(G) - 1$. Let $G' = G - v_1$. Then $g(G') \leq g(G)$, and so by (10), G' has an $(L, 2)$-coloring c. Extending c by coloring v_1 with $c(v_1) \in L(v_1) \setminus c(N(v_1))$. As c is an $(L, 2)$-coloring of G' and by the choices of $c(v_1)$, c is an $(L, 2)$-coloring of G, contrary to (10). This proves Claim 4.

Claim 5. *Let $e = v_1v_2$ be an edge in G. If $4 \in \{d_1^*, d_2^*\}$, then $d_i \geq h(G) - 1$, $i = 1, 2$.*

If otherwise, we may assume that $d_1^* = 4$ and $d_1 \leq h(G) - 2$. Denote $F = v_1v_2uwv_1$ as the 4-face. Let $G' = G - v_1 + wv_2$. Then by our assumption, G' has an $(L, 2)$-coloring c, and so $c(w) \neq c(v_2)$. Extending c by letting $c(v_1) \in L(v_1) \setminus c(N(v_1) \cup \{u\})$, contrary to the choice of G. This proves Claim 5.

For notational convenience, we shall denote $h(G)$ and $g(G)$ by h and g respectively throughout the rest of the proof.

Claim 6. *Let $e = v_1v_2$ be an edge in G. Each of the following holds:*

(i) *If $3 \in \{d_1^*, d_2^*\}$, then*

$$\Phi'(e) \geq \frac{h-6}{3h}.$$

(ii) *If $3 \notin \{d_1^*, d_2^*\}$, $4 \in \{d_1^*, d_2^*\}$, then*

$$\Phi'(e) \geq \frac{h^2 - 5h + 2}{2h(h-1)}.$$

(iii) *If $d_1^*, d_2^* \geq 5$, then*

$$\Phi'(e) \geq \frac{3h^2 - 16h + 10}{5h(h-2)}.$$

By Claim 2, $\delta(G) \geq 3$. Thus $d_i \geq 3$, $d_i^* \geq 3$, $i = 1, 2$. If $3 \in \{d_1^*, d_2^*\}$, then by Claim 4, $d_i \geq h$, $i = 1, 2$. Thus $\Phi'(e) = 1 - \frac{1}{d_1} - \frac{1}{d_2} - \frac{1}{d_1^*} - \frac{1}{d_2^*} \geq 1 - \frac{1}{h} - \frac{1}{h} - \frac{1}{3} - \frac{1}{3} = \frac{h-6}{3h}$.

If $3 \notin \{d_1^*, d_2^*\}$ and $4 \in \{d_1^*, d_2^*\}$, then by Claim 5, $d_i \geq h - 1$, $i = 1, 2$. By Claim 3, at least one of the d_i's must be at least h, and so $\Phi'(e) = 1 - \frac{1}{d_1} - \frac{1}{d_2} - \frac{1}{d_1^*} - \frac{1}{d_2^*} \geq 1 - \frac{1}{h-1} - \frac{1}{h} - \frac{1}{4} - \frac{1}{4} = \frac{h^2-5h+2}{2h(h-1)}$.

If $d_1^*, d_2^* \geq 5$, then by Claim 2, $\delta(G) \geq h - 2$. By Claim 3, at least one of the d_i's must be at least $h(G)$, and so $\Phi'(e) = 1 - \frac{1}{d_1} - \frac{1}{d_2} - \frac{1}{d_1^*} - \frac{1}{d_2^*} \geq 1 - \frac{1}{h-2} - \frac{1}{h} - \frac{1}{5} - \frac{1}{5} = \frac{3h^2-16h+10}{5h(h-2)}$. This proves Claim 6.

Since

$$\frac{h-6}{3h} < \frac{h^2 - 5h + 2}{2h(h-1)} < \frac{3h^2 - 16h + 10}{5h(h-2)}. \tag{11}$$

The following claim follows from Claim 6 and (11).

Claim 7. *For each $e \in E(G)$,*

$$\Phi'(e) \geq \frac{h-6}{3h}.$$

Claim 8. $|E(G)| \geq \frac{1}{2}(h + 3)(h - 2)$.

If $\delta(G) \geq h$, by Claim 1, we have $|V(G)| \geq h+1$, so $|E(G)| \geq \frac{1}{2}(h+1)h > \frac{1}{2}(h+3)(h-2)$. If $\delta(G) < h$, let v be a vertex of G such that $d(v) = \delta(G)$. Let u be any neighbor of v, by Claim 3, $d(u) \geq h$. Thus there exist at least $\delta(G)$ vertices of degree at least h, and so $|E(G)| \geq \frac{1}{2}((h+1)\delta(G) + \delta(G)(h - \delta(G)))$. By Claim 2, $\delta \geq h-2$. When $\delta(G) = h-1$, we have that $|E(G)| \geq \frac{1}{2}(h+2)(h-1) > \frac{1}{2}(h+3)(h-2)$. When $\delta(G) = h-2$, we have that $|E(G)| \geq \frac{1}{2}(h+3)(h-2)$. This proves Claim 8.

By Claim 2, $\delta(G) \geq h-2 \geq 5$. So G is not a tree. By Theorems 4.1 and 4.2, G has a 2-cell embedding. By Lemma 4.3, $\Phi(G) = \sum_{e\in E(G)} \Phi(e)$. Since we let $\Phi(e) = -\Phi'(e)$, we have $-\Phi(G) = \sum_{e\in E(G)} \Phi'(e)$. Now the rest of the proof is divided into 3 cases.

Case 1. $\delta(G) \geq h$.

By Claim 1, we have $|V(G)| \geq h+1$, so $|E(G)| \geq \frac{1}{2}(h+1)h$.

$$\begin{aligned}
-\Phi(G) &= \sum_{e\in E(G)} \Phi'(e) \geq \frac{1}{2}h(h+1)\cdot\frac{h-6}{3h} = \frac{1}{24}(2h)(2h-10) - 1\\
&= \frac{1}{24}\left(7+\sqrt{1+48g}\right)\left(\sqrt{1+48g}-3\right) - 1 = \frac{1}{24}\left(48g - 20 + 4\sqrt{1+48g}\right) - 1\\
&= 2g - 2 + \frac{1}{6}\sqrt{1+48g} + \frac{1}{6} > 2g-2.
\end{aligned}$$

Case 2. $\delta(G) = h-1$.

Let v be the vertex with $d(v) = h-1$. By Claim 4, every edge e incident to v can not lie in a 3-face, otherwise we can deduce that $d(v) \geq h$. By Claim 6 and (11), $\Phi'(e) \geq \frac{h^2-5h+2}{2h(h-1)}$ holds for every edge e incident to v.

$$\begin{aligned}
-\Phi(G) &= \sum_{e\in E(G)} \Phi'(e) \geq |E(G)|\cdot\frac{h-6}{3h} + (h-1)\left(\frac{h^2-5h+2}{2h(h-1)} - \frac{h-6}{3h}\right)\\
&\geq \frac{1}{2}(h+3)(h-2)\cdot\frac{h-6}{3h} + (h-1)\left(\frac{h^2-5h+2}{2h(h-1)} - \frac{h-6}{3h}\right) = \frac{1}{6}(h^2-4h-13) + \frac{5}{h}\\
&= \frac{1}{24}(2h)(2h-8) - \frac{13}{6} + \frac{5}{h} = \frac{1}{24}\left(7+\sqrt{1+48g}\right)\left(\sqrt{1+48g}-1\right) - \frac{13}{6} + \frac{5}{h}\\
&= \frac{1}{24}\left(48g - 6 + 6\sqrt{1+48g}\right) - \frac{13}{6} + \frac{5}{h} = 2g - 2 + \frac{1}{12}\left(3\sqrt{1+48g} - 5\right) + \frac{5}{h} > 2g-2.
\end{aligned}$$

Case 3. $\delta(G) = h-2$.

Let v be the vertex with $d(v) = h-2$. By Claims 4 and 5, every edge e incident to v can lie in neither a 3-face nor a 4-face. By Claim 6(iii), $\Phi'(e) \geq \frac{3h^2-16h+10}{5h(h-2)}$ holds for every edge e incident to v.

$$\begin{aligned}
-\Phi(G) &= \sum_{e\in E(G)} \Phi'(e) \geq |E(G)|\cdot\frac{h-6}{3h} + (h-2)\left(\frac{3h^2-16h+10}{5h(h-2)} - \frac{h-6}{3h}\right)\\
&\geq \frac{1}{2}(h+3)(h-2)\cdot\frac{h-6}{3h} + (h-2)\left(\frac{3h^2-16h+10}{5h(h-2)} - \frac{h-6}{3h}\right) = \frac{1}{30}(5h^2-17h-76) + \frac{4}{h}\\
&= \frac{1}{120}(2h)(10h-34) - \frac{76}{30} + \frac{4}{h} = \frac{1}{120}\left(7+\sqrt{1+48g}\right)\left(5\sqrt{1+48g}+1\right) - \frac{76}{30} + \frac{4}{h}\\
&= \frac{1}{120}\left(240g + 12 + 36\sqrt{1+48g}\right) - \frac{76}{30} + \frac{4}{h} = 2g - 2 + \frac{1}{30}\left(9\sqrt{1+48g} - 13\right) + \frac{4}{h} > 2g-2.
\end{aligned}$$

Thus in each case we have $-\Phi(G) > 2g-2$, contrary to (9). This completes the proof of Theorem 1.2. □

The corollary below follows immediately from Theorem 1.2 and (3).

Corollary 4.4. *If G is a graph with genus $g(G) \geq 1$, then $\chi_2(G) \leq \frac{1}{2}\left(7+\sqrt{1+48g(G)}\right)$.*

Note that a well-known result by Franklin [8], Ringel [16] and Youngs [19] (see also Theorem 8-8 [18]) states that, for $g(G) \geq 1$, $\chi(G) \leq \frac{1}{2}\left(7+\sqrt{1+48g(G)}\right)$ is indeed best possible, except for Klein bottle. By formula (1) and (3), $\chi(G) \leq \chi_2(G) \leq ch_2(G)$. So Theorem 1.2 and Corollary 4.4 is also best possible.

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References

[1] S. Akbari, M. Ghanbari, S. Jahanbekam, On the list dynamic coloring of graphs, Discrete Appl. Math. 157 (2009) 3005–3007.
[2] M. Alishahi, On the dynamic coloring of graphs, Discrete Appl. Math. 159 (2011) 152–156.
[3] K. Appel, W. Haken, Every plane map is four colorable, part 1, discharging, Illinois J. Math. 21 (1977) 429–490.
[4] K. Appel, W. Haken, J. Kock, Every plane map is four colorable, part 2, reducibility, Illinois J. Math. 21 (1977) 491–567.
[5] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, 2008.
[6] C. Ding, S. Fan, H.-J. Lai, Upper bound on conditional chromatic number of graphs, J. Jinan University 29 (2008) 7–14.
[7] L. Esperet, Dynamic list coloring of bipartite graphs, Discrete Appl. Math. 158 (2010) 1963–1965.
[8] P. Franklin, A six colour problem, J. Math. Phys. Massachusetts Inst. Tech. 13 (1934) 363–369.
[9] P.J. Heawood, Map colour theorem, Quart. J. Pure Math. 24 (1890) 332–338.
[10] H.-J. Lai, J. Lin, B. Montgomery, Zhisui Tao, S. Fan, Conditional colorings of graphs, Discrete Math. 306 (2006) 1997–2004.
[11] H.-J. Lai, B. Montgomery, H. Poon, Upper bounds of dynamic chromatic number, Ars Combin. 68 (2003) 193–201.
[12] Y. Lin, Upper bounds of conditional chromatics number, Master Thesis, Jinan University, 2008.
[13] B. Montgomery, Ph.D. Dissertation, West Virginia University, 2001.
[14] O. Ore, The Four Color Problem, Academic Press, New York, 1967.
[15] T. Parsons, G. Pica, T. Pisanski, A. Ventre, Orientably simple graphs, Math. Slovaca 37 (1987) 391–394.
[16] G. Ringel, Fărbungsprobleme auf Frachen und Graphen, Veb Deutscher Verlag Der Wissenschaften, Berlin, 1959.
[17] N. Robertson, D.P. Sanders, P. Seymour, R. Thomas, The four-colour theorem, J. Combin. Theory Ser. B 70 (1) (1997) 2–44.
[18] A.T. White, Graphs, Groups and Surfaces, American Elsevier, New York, 1973.
[19] J.W.T. Youngs, Minimal imbedding and the genus of a graph, J. Math. Mech. 12 (1963) 303–315.