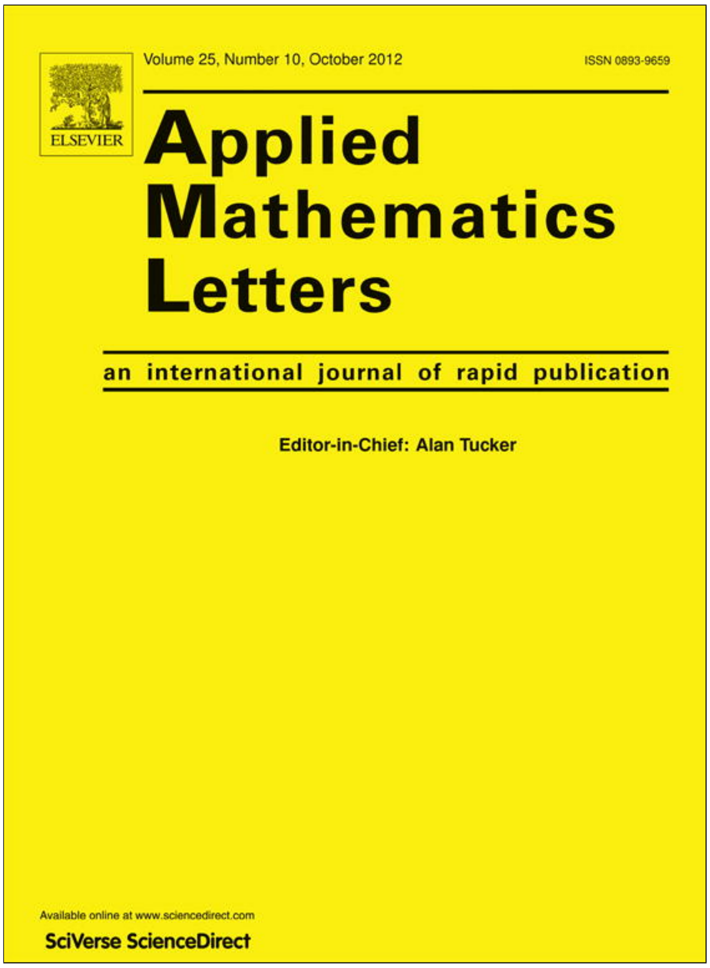
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Multigraphic degree sequences and supereulerian graphs, disjoint spanning trees

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ABSTRACT

A sequence $d = (d_1, d_2, \ldots, d_n)$ is multigraphic if there is a multigraph G with degree sequence d, and such a graph G is called a realization of d. In this paper, we prove that a nonincreasing multigraphic sequence $d = (d_1, d_2, \ldots, d_n)$ has a realization with a spanning eulerian subgraph if and only if either $n = 1$ and $d_1 = 0$, or $n \geq 2$ and $d_n \geq 2$, and that d has a realization G such that $L(G)$ is hamiltonian if and only if either $d_1 \geq n-1$, or $\sum_{d_i=1} d_i \leq \sum_{d_j\geq 2}(d_j - 2)$. Also, we prove that, for a positive integer k, d has a realization with k edge-disjoint spanning trees if and only if either both $n = 1$ and $d_1 = 0$, or $n \geq 2$ and both $d_n \geq k$ and $\sum_{i=1}^{n} d_i \geq 2k(n-1)$.

1. Introduction

This paper studies finite and undirected graphs without loops, but multiple edges are allowed. When we say "graph" in this paper, it always means "multigraph", unless otherwise stated. Undefined terms can be found in [1]. In particular, for a graph G, $L(G)$ denotes its **line graph**. Let X be a set of vertices, $G - X$ denotes the graph obtained from G by deleting X, and if $X = \{v\}$, we often use $G - v$ for $G - \{v\}$. Let S be a set of edges, $G - S$ and $G + S$ denote the graphs obtain from G by deleting S and adding S, respectively. Particularly if $S = \{e\}$, we often use $G - e$ for $G - \{e\}$ and $G + e$ for $G + \{e\}$. A vertex $v \in V(G)$ is called a **pendent vertex** if $d(v) = 1$. Let $D_1(G)$ denote the set of all pendent vertices of G. An edge $e \in E(G)$ is called a **pendent edge** if one of its ends is a pendent vertex. A path in a graph G is called a **pendent path** if one end is a pendent vertex, all internal vertices have degree 2 and the other end has degree more than 2. If $v \in V(G)$, then $N_G(v) = \{u \in V(G) : uv \in E(G)\}$; and if $T \subseteq V(G)$, then $N_G(T) = \{u \in V(G) \setminus T : uv \in E(G) \text{ and } v \in T\}$. When the graph G is understood in the context, we may drop the subscript G.

A **circuit** is a connected 2-regular graph. The notation tK_2 is defined to be the graph with 2 vertices and t multiple edges. In this paper, $2K_2$ is considered as a circuit, which is also denoted as C_2. An even subgraph of G is a **spanning eulerian subgraph** of G if it is connected and spanning. A graph G is **supereulerian** if G contains a spanning eulerian subgraph.

If a graph G has vertices $v_1, v_2, \ldots, v_n$, the sequence $(d(v_1), d(v_2), \ldots, d(v_n))$ is called a **degree sequence** of G. A sequence $d = (d_1, d_2, \ldots, d_n)$ is **graphic** if there is a simple graph G with degree sequence d, and it is **multigraphic** if there is a multigraph G with degree sequence d. In either case, such a graph G is called a **realization** of d, or a d-**realization**. A multigraphic degree sequence d is **line-hamiltonian** if d has a realization G such that $L(G)$ is hamiltonian, and d is **supereulerian** if it has a realization with a spanning eulerian subgraph.

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Hakimi [2] gave a characterization for multigraphic degree sequences as follows.

Theorem 1.1 (*Hakimi, Theorem 1 in [2]*). *If $d = (d_1, d_2, \ldots, d_n)$ is a nonincreasing sequence with $n \geq 2$ and d_i nonnegative integers for $1 \leq i \leq n$, then it is a multigraphic sequence if and only if $\sum_{i=1}^{n} d_i$ is even and $d_1 \leq d_2 + \cdots + d_n$.*

Boesch and Harary presented in [3] the following theorem which is due to Butler.

Theorem 1.2 (*Butler (Boesch and Harary, Theorem 5 in [3])*). *Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence with $n \geq 2$ and d_i nonnegative integers for $1 \leq i \leq n$. Let j be an index with $2 \leq j \leq n$. Then the sequence $\{d_1, d_2, \ldots, d_n\}$ is multigraphic if and only if the sequence $\{d_1 - 1, d_2, \ldots, d_{j-1}, d_j - 1, d_{j+1}, \ldots, d_n\}$ is multigraphic.*

The following characterizations of supereulerian degree sequences, line-hamiltonian degree sequences, and the degree sequences with realization having k edge-disjoint spanning trees have been obtained for simple graphs.

Theorem 1.3 (*Fan et al. [4]*). *Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing graphic sequence. Then d has a supereulerian realization if and only if either $n = 1$ and $d_1 = 0$, or $n \geq 3$ and $d_n \geq 2$.*

Theorem 1.4 (*Fan et al. [4]*). *Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing graphic sequence with $n \geq 3$. The following are equivalent.*

(i) *d is line-hamiltonian.*
(ii) *either $d_1 = n - 1$, or $\sum_{d_i=1} d_i \leq \sum_{d_j \geq 2}(d_j - 2)$.*
(iii) *d has a realization G such that $G - D_1(G)$ is supereulerian.*

Theorem 1.5 (*Lai et al. [5]*). *A nonincreasing graphic sequence $d = (d_1, d_2, \ldots, d_n)$ has a realization with k edge-disjoint spanning trees if and only if either $n = 1$ and $d_1 = 0$, or $n \geq 2$ and both of the following hold:*

(i) *$d_n \geq k$.*
(ii) *$\sum_{i=1}^{n} d_i \geq 2k(n - 1)$.*

In this paper, we investigate multigraphic sequences and prove the multigraphic versions for Theorems 1.3–1.5, as follows.

Theorem 1.6. *Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing multigraphic sequence. Then d has a supereulerian realization if and only if either $n = 1$ and $d_1 = 0$, or $n \geq 2$ and $d_n \geq 2$.*

Theorem 1.7. *Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing multigraphic sequence with $n \geq 3$. Then the following are equivalent.*

(i) *d is line-hamiltonian.*
(ii) *either $d_1 \geq n - 1$, or $\sum_{d_i=1} d_i \leq \sum_{d_j \geq 2}(d_j - 2)$.*
(iii) *d has a realization G such that $G - D_1(G)$ is supereulerian.*

Theorem 1.8. *Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing multigraphic sequence. Then d has a realization G with k edge-disjoint spanning trees if and only if either $n = 1$ and $d_1 = 0$, or $n \geq 2$ and both of the following hold:*

(i) *$d_n \geq k$.*
(ii) *$\sum_{i=1}^{n} d_i \geq 2k(n - 1)$.*

In Sections 2–4, we present proofs for Theorems 1.6–1.8, respectively.

2. The Proof of Theorem 1.6

Proof of Theorem 1.6. If a nonincreasing multigraphic sequence $d = (d_1, d_2, \ldots, d_n)$ has a supereulerian realization, then we must have $d_n \geq 2$ as every supereulerian graph is 2-edge-connected for $n \geq 2$.

We prove the sufficiency by induction on $m = \sum_{i=1}^{n} d_i$. Without loss of generality, we may assume that $n \geq 2$. If $n = 2$ and $d = (2, 2)$, then $m = 4$ and $2K_2$ is a supereulerian realization of d.

Suppose that the theorem holds for all such multigraphic sequences with smaller value of m. We have the following cases.

Case 1: $d_1 = d_2 = 2$. Then $d = (2, \ldots, 2)$. Therefore, C_n is a supereulerian realization of d (when $n = 2$, C_n is defined to be $2K_2$).

Case 2: $d_1 > 2$ and $d_2 = 2$. Then $d = (d_1, 2, \ldots, 2)$. By Theorem 1.1, d_1 must be even and so $d_1 \geq 4$. Since $d_1 \leq d_2 + \cdots + d_n$, we have $n \geq 3$. Suppose $d_1 = 2k$ with $k \geq 2$. Let $V = \{v_1, v_2, \ldots, v_n\}$ and circuit $C = v_1 v_{k+1} v_{k+2} \cdots v_n v_1$. And let $E = \bigcup_{i=2}^{k} \{v_1 v_i, v_i v_1\} \bigcup E(C)$. Then $G = (V, E)$ is a supereulerian realization of d.

Case 3: $d_1 \geq d_2 \geq 3$. By Theorem 1.2, $(d_1 - 1, d_2 - 1, \ldots, d_n)$ is multigraphic. Since $d_1 - 1 \geq d_2 - 1 \geq 2$, by induction, there is a supereulerian realization, say G', of $(d_1 - 1, d_2 - 1, \ldots, d_n)$. By adding an edge $v_1 v_2$ in G', we obtain a supereulerian realization of d. □

3. The Proof of Theorem 1.7

We need a theorem, which is due to Harary and Nash-Williams. The theorem shows the relationship between hamiltonian circuits in the line graph $L(G)$ and eulerian subgraph in G, and it is also true for multigraphs. A subgraph H of G is dominating if $E(G - V(H)) = \emptyset$.

Theorem 3.1 (*Harary and Nash-Williams, [6]*). *Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

Proof of Theorem 1.7. (i) $\Rightarrow$ (ii) Let G be a realization of d such that $L(G)$ is hamiltonian. By Theorem 3.1, G has a dominating eulerian subgraph H. If $d_1 \geq n-1$, then we are done. Suppose that $d_1 \leq n-2$. Then $|V(H)| \geq 2$. For any v_i with $d(v_i) = 1$, v_i must be adjacent to a vertex v_j in H and so $d_{G-E(H)}(v_j)$ is no less than the number of degree 1 vertices adjacent to v_j. Furthermore, since H is eulerian and nontrivial, $d_H(v_j) \geq 2$ and so $\sum_{d_i=1} d_i \leq \sum_{d_j \geq 2}(d_j - 2)$ holds.

(ii) $\Rightarrow$ (iii) Suppose that d is a nonincreasing multigraphic sequence satisfying (ii). If there exists a d-realization G that is a simple graph (in this case, d_1 cannot be greater than $n-1$), then d is also a nonincreasing graphic sequence. By Theorem 1.4, (iii) must hold. Hence, we may assume that every d-realization has multiple edges. If $d_n \geq 2$, then by Theorem 1.6, d has a supereulerian realization. So we also assume that $d_n = 1$. We will show that there is a d-realization G such that $\delta(G - D_1(G)) \geq 2$.

Suppose, to the contrary, that for each d-realization G, $\delta(G - D_1(G)) < 2$. As G contains multiple edges, $E(G - D_1(G))$ is not empty. Let $S = N(D_1(G))$. Then there exists $s \in S$, $|N_{G-D_1(G)}(s)| = 1$. Let $P(G) = \{s \in S : |N_{G-D_1(G)}(s)| = 1\}$ and choose G to be a graph such that $|P(G)|$ is minimized. Let $x \in P(G)$ and $d_G(x) = d_t$. Then x is not incident with multiple edges.

Since $d_G(x) = d_t$ and $|N_{G-D_1(G)}(x)| = 1$, there must be $d_t - 1$ pendent edges incident with vertex x in G. We delete these $d_t - 1$ pendent edges of x, and denote the resulting graph by G'. Then there is a pendent path P_x of x in G', and let l be the length and v_x be the other end vertex. Let G'' be the graph obtained from G' by deleting x and the internal vertices of P_x. Choose a multiple edge $e \in E(G'')$, replace e with a path of length $l+1$ and let v_e be an internal vertex. Then add $d_t - 2$ pendent edges to v_e, add one pendent edge to v_x and denote the resulting graph G_x. Then $d_{G_x}(v_e) = 2 + d_t - 2 = d_t$, and G_x is also a d-realization. Let $N_1(x)$ be the set of pendent vertices adjacent to x in G. Then $|D_1(G_x)| = |(D_1(G) - N_1(x)) \cup \{x\}| + d_t - 2 = |D_1(G)| - (d_t - 1) + 1 + d_t - 2 = |D_1(G)|$ but $|P(G_x)| < |P(G)|$, contradicting the choice of G (Note here, if G_x does not have multiple edges, then it is contrary to the assumption that every d-realization has multiple edges). By Theorem 1.6, there is a d-realization G such that $G - D_1(G)$ is supereulerian.

(iii) $\Rightarrow$ (i) If G is a realization of d such that $G - D_1(G)$ is supereulerian, then by Theorem 3.1, $L(G)$ is hamiltonian. Thus (i) holds. □

4. The Proof of Theorem 1.8

Let $\tau(G)$ be the maximum number of edge-disjoint spanning trees in a connected graph G.

Lemma 4.1. *Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing multigraphic sequence. If d has a realization G with $\tau(G) \geq k$, then either $n = 1$ and $d_1 = 0$, or $n \geq 2$ and both $d_n \geq k$ and $\sum_{i=1}^{n} d_i \geq 2k(n-1)$ hold.*

Proof. The case when $n = 1$ is trivial and so we shall assume that $n > 1$. Since G has k edge-disjoint spanning trees, $2k(|V(G)| - 1) \leq 2|E(G)| = \sum_{i=1}^{n} d_i$ and each vertex has degree at least k. □

Corollary 4.2. *Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing multigraphic sequence with $n > 2$. If d has a realization G with $\tau(G) \geq k$, then $d_1 > k$.*

Proof. Suppose not, by Lemma 4.1, $d_i = k$ for each i, $1 \leq i \leq n$. Hence $2k(n-1) \leq \sum_{i=1}^{n} d_i = kn$, whence $n \leq 2$, contrary to $n > 2$. Thus $d_1 > k$. □

Lemma 4.3. *Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing multigraphic sequence with $n > 2$. If d has a realization G with $\tau(G) \geq k$, then $d' = (d_1 - 1, d_2, \ldots, d_{j-1}, d_j + 1, d_{j+1}, \ldots, d_n)$ has a realization G' with $\tau(G') \geq k$ for any j with $2 \leq j \leq n$.*

Proof. Let v_i be the vertex with degree d_i in G, for $1 \leq i \leq n$. Then there must be a vertex v_s adjacent to v_1 where $s \neq j$. If not, then all edges incident with v_1 are between v_1 and v_j, and since G is connected, $d_j > d_1$, contrary to $d_1 \geq d_j$. Thus there is an edge e between v_1 and v_s. Let $T_1, T_2, \ldots, T_k$ be edge-disjoint spanning trees of G.

Case 1: v_1 is a leaf in T_i for each i, $1 \leq i \leq k$. Let e' be a new edge between v_s and v_j, and $G' = G - e + e'$. Then G' is a realization of d'. If $e \notin \cup_{i=1}^{k} E(T_i)$, then $T_1, T_2, \ldots, T_k$ are edge-disjoint spanning trees of G'. If $e \in E(T_l)$ where $1 \leq l \leq k$, by Corollary 4.2, $d_1 > k$, and there must be an edge e'' incident with v_1 such that $e'' \notin \cup_{i=1}^{k} E(T_i)$, then $T_1, T_2, \ldots, T_{l-1}, T_l - e + e'', T_{l+1}, \ldots, T_k$ are edge-disjoint spanning trees of G'.

Case 2: v_1 is not a leaf in T_l for some l, $1 \leq l \leq k$. Then there exists $v_t \in V(G)$ and there exists $e_t = v_1 v_t \in E(T_l)$ such that v_1 and v_j are in one component of $T_l - e_t$ while v_t is in the other component. Let e'_t be a new edge between v_j and v_t, and $G_t = G - e_t + e'_t$. Then G_t is a d'-realization, and $T_1, T_2, \ldots, T_{l-1}, T_l - e_t + e'_t, T_{l+1}, \ldots, T_k$ are edge-disjoint spanning trees of G_t. □

Lemma 4.4. *Let $d = (d_1, d_2, \dots, d_n)$ be a nonincreasing multigraphic sequence. If d has a realization G with $\tau(G) \geq k$, then $d' = (d_1, \dots, d_{i-1}, d_i + 1, d_{i+1}, \dots, d_{j-1}, d_j + 1, d_{j+1}, \dots, d_n)$ has a realization G' with $\tau(G') \geq k$, $\forall i, j$ with $1 \leq i < j \leq n$.*

Proof. Let v_i, v_j be the vertices with degree d_i and d_j in G, respectively, and e be a new edge between v_i and v_j. Let $G' = G + e$, then G' is a d'-realization with $\tau(G') \geq k$. □

Proof of Theorem 1.8. Lemma 4.1 proves the necessity. To prove the sufficiency, we prove a claim first.

Claim. *Let $d = (d_1, d_2, \dots, d_n)$ be a nonincreasing multigraphic sequence with $d_n \geq k$ and $\sum_{i=1}^n d_i \geq 2k(n-1)$. If any nonincreasing multigraphic sequence $d' = (d'_1, d'_2, \dots, d'_n)$ with $d'_n \geq k$ and $\sum_{i=1}^n d'_i = 2k(n-1)$ has a realization with k edge-disjoint spanning trees, then d has a realization with k edge-disjoint spanning trees.*

Proof of the claim: Without loss of generality, we may assume that $\sum_{i=1}^n d_i > 2k(n-1)$. Noticing that $\sum_{i=1}^n d_i$ is always even, we define an operation $(*)$ for d as follows: $(*)$: If $\sum_{i=1}^n d_i > 2k(n-1)$ and $\exists i \geq 2$ such that $d_i > k$, then let $d^{(*)} = (d_1 - 1, d_2, \dots, d_{i-1}, d_i - 1, d_{i+1}, \dots, d_n)$, and reorder $d^{(*)}$ to be a nonincreasing sequence $(d_1^{(*)}, d_2^{(*)}, \dots, d_n^{(*)})$.

By Theorem 1.2, $d^{(*)}$ is still a multigraphic sequence. We keep on doing operation $(*)$ for $d^{(*)}$ until $\sum_{i=1}^n d_i^{(*)} = 2k(n-1)$ or $d_i^{(*)} = k$ for each $i = 2, 3, \dots, n$. For the latter case, $d^{(*)} = (d_1^{(*)}, k, k, \dots, k)$ and $d_1^{(*)} + k(n-1) \geq 2k(n-1)$, i.e., $d_1^{(*)} \geq k(n-1)$. Since $d^{(*)} = (d_1^{(*)}, k, k, \dots, k)$ is still a multigraphic sequence, by Theorem 1.1, $d_1^{(*)} \leq k(n-1)$. Thus $d_1^{(*)} = k(n-1)$. Hence, in both cases, $\sum_{i=1}^n d_i^{(*)} = 2k(n-1)$, and by the assumption, $d^{(*)}$ has a realization with k edge-disjoint spanning trees. By Lemma 4.4, d has a realization with k edge-disjoint spanning trees, which completes the proof of the claim.

By the claim, it suffices to show that any multigraphic sequence $d = (d_1, d_2, \dots, d_n)$ with $d_n \geq k$ and $\sum_{i=1}^n d_i = 2k(n-1)$ has a realization G with $\tau(G) \geq k$. If $n = 2$, then tK_2 is such a d-realization where $t = k$. If $n > 2$, then by Lemma 4.3, it suffices to show that $d^0 = (k(n-1), k, k, \dots, k)$ has such a realization. Let $kK_{1,n-1}$ be the graph with vertex set $\{v_1, v_2, \dots, v_n\}$ such that for each i, $2 \leq i \leq n$, there are k multiple edges between v_1 and v_i, but there are no edges between v_i and v_j for $2 \leq i < j \leq n$. Then $kK_{1,n-1}$ is a d^0-realization with $\tau(kK_{1,n-1}) = k$. This completes the proof of the theorem. □

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