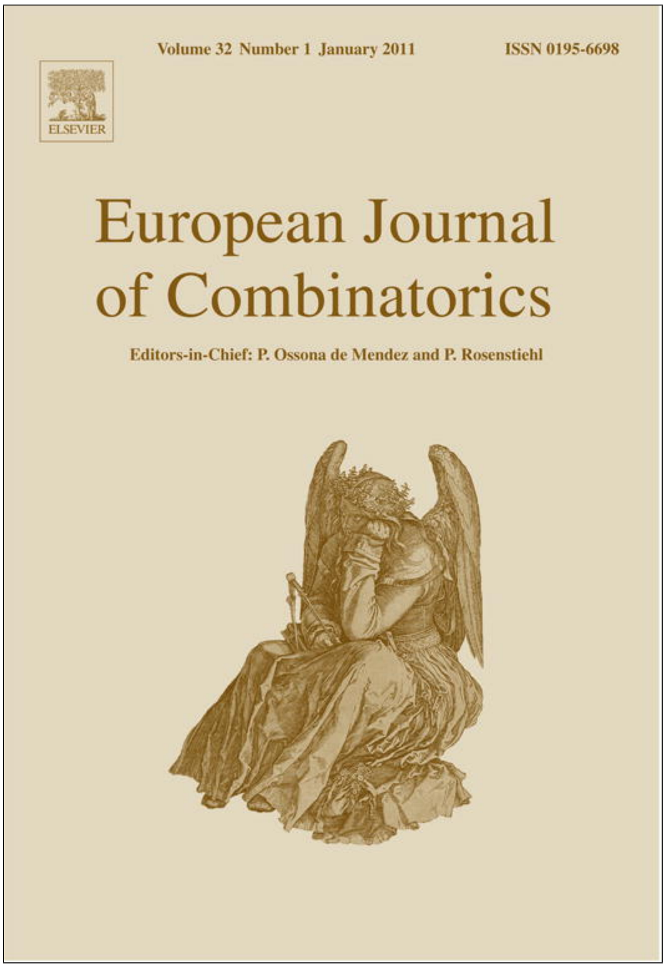
Volume 32 Number 1 January 2011
ISSN 0195-6698
ELSEVIER
European Journal of Combinatorics
Editors-in-Chief: P. Ossona de Mendez and P. Rosenstiehl

European Journal of Combinatorics 33 (2012) 182–188

Contents lists available at SciVerse ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Z_3-connectivity of 4-edge-connected 2-triangular graphs

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ARTICLE INFO

Article history:
Received 12 September 2010
Received in revised form
19 January 2011
Accepted 8 September 2011

ABSTRACT

A graph G is k-triangular if each edge of G is in at least k triangles. It is conjectured that every 4-edge-connected 1-triangular graph admits a nowhere-zero Z_3-flow. However, it has been proved that not all such graphs are Z_3-connected. In this paper, we show that every 4-edge-connected 2-triangular graph is Z_3-connected. The result is best possible. This result provides evidence to support the Z_3-connectivity conjecture by Jaeger et al that every 5-edge-connected graph is Z_3-connected.

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1. Introduction

We follow the notations and terminology of [1] except otherwise stated. For an integer $k > 0$, Z_k denotes the set of all integers modulo k, as well as the cyclic group of order k. Let G be a graph, $l > 0$ be an integer, $x \in V(G)$ and $X \subseteq V(G)$. Define $D_l(G) = \{v \in V(G) \mid d_G(v) = l\}$, $N_G(x) = \{v \in V(G) | vx \in E(G)\}$ and $G[X]$ the graph induced by X.

Broersma and Veldman introduced the concept of k-triangular graphs in [2]. A graph G is *k-triangular* if each edge of G is in at least k triangles. A 1-triangular graph is also referred to as a *triangular graph*.

Let G be a digraph, A be a nontrivial additive Abelian group and A^* be the set of nonzero elements in A. For any $v \in V(G)$, we denote the set of all edges with tails at v by $E^+(v)$ and heads at v by $E^-(v)$. Let $E(v) = E^+(v) \cup E^-(v)$.

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0195-6698/$ – see front matter. Published by Elsevier Ltd
doi:10.1016/j.ejc.2011.09.041

Following the notations in [5], we define

$$F(G, A) = \{f \mid f : E(G) \mapsto A\} \quad \text{and} \quad F^*(G, A) = \{f \mid f : E(G) \mapsto A^*\}.$$

For each $f \in F(G, A)$, the *boundary* of f is a function $\partial f : V(G) \mapsto A$ defined by

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where "$\sum$" refers to the addition in A. We define

$$Z(G, A) = \left\{ b \mid b : V(G) \mapsto A \text{ with } \sum_{v \in V(G)} b(v) = 0 \right\}.$$

An *A-nowhere-zero-flow* (abbreviated as A-NZF) in G is an $f \in F^*(G, A)$ such that $\partial f = 0$. For any given $b \in Z(G, A)$, a function $f \in F^*(G, A)$ with $\partial f = b$ is called an (A, b)-NZF.

An undirected graph G is *A-connected*, if G has an orientation G' such that for every function $b \in Z(G', A)$, there exists an (A, b)-NZF. It has been observed in [5] that whether G is A-connected is independent of the orientation of G. For an Abelian group A, let $\langle A \rangle$ denote the family of graphs that are A-connected.

The nowhere-zero flow problems were introduced by Tutte [8] and surveyed by Jaeger in [4] and by Zhang in [11]. The concept of A-connectivity was introduced by Jaeger et al. in [5], where A-NZF's were successfully generalized to A-connectivity.

The *group connectivity number* of a 2-edge-connected graph G is defined as

$$\Lambda_g(G) = \min\{k : G \text{ is } A\text{-connected for every Abelian group } A \text{ with } |A| \geq k\}.$$

In [10], it is shown that if $c(G)$ denote the circumference of G (length of a longest circuit), then $\Lambda_g(G) \leq c(G) + 1$. Thus for any 2-edge-connected graph G, $\Lambda_g(G)$ exists as a finite number.

This paper is motivated by the following conjectures.

Conjecture 1.1 (*Tutte, Unsolved Problem 48 in [1]*)**.** *Every 4-edge-connected graph admits a Z_3-NZF.*

Conjecture 1.2 (*Jaeger et al. [5]*)**.** *If G is 5-edge-connected, then $\Lambda_g(G) \leq 3$.*

A weaker version of Conjecture 1.1 is also posed by Xu and Zhang in [9].

Conjecture 1.3 (*Xu and Zhang [9]*)**.** *Every 4-edge-connected triangular graph has a Z_3-NZF.*

It was further asked (Problem 1 in [7]) whether every 4-edge-connected triangular graph is Z_3-connected. This was shown in the negative in [7]. Moreover, a recent result in [3] by Fan et al. indicated that there exist infinitely many 3-edge-connected 2-triangular graphs that are not Z_3-connected. These motivate the authors to consider the Z_3-connectivity of 4-edge-connected 2-triangular graphs. The main results of this paper are the following.

Theorem 1.4. *Every 4-edge-connected 2-triangular graph is Z_3-connected.*

Corollary 1.5. *If G is a 4-edge-connected 2-triangular graph, then $\Lambda_g(G) \leq 3$.*

Corollary 1.6. *If G is a connected 3-triangular graph, then $\Lambda_g(G) \leq 3$. In particular, every connected 3-triangular graph is Z_3-connected.*

In Section 2, we summarize some of the useful tools in the proof. In Section 3, we assume the validity of Theorem 1.4 to prove Corollaries 1.5 and 1.6, and present examples to show the sharpness of our main results. Section 4 will be devoted to the proof of Theorem 1.4.

2. Useful lemmas

Let G be a graph and $X \subseteq E(G)$ be an edge subset. The *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. For convenience, we use G/e for $G/\{e\}$ and $G/\emptyset = G$; and if H is a subgraph of G, we write G/H for $G/E(H)$.

Lemma 2.1 (*Proposition 3.2 of [6]*). *Let A be an Abelian group, G be a graph and H be a subgraph of G. If $H \in \langle Z_3 \rangle$, then $G/H \in \langle Z_3 \rangle$ if and only if $G \in \langle Z_3 \rangle$.*

It has been observed in [5] that a cycle C is A-connected if and only if $|E(C)| < |A|$. Therefore, for a connected graph G, if every edge of G lies in a cycle of length at most k, then $\Lambda_g(G) \leq k + 1$. The case in which $k = 3$ is needed in the proof.

Lemma 2.2. *If G is connected and triangular, then $\Lambda_g(G) \leq 4$.*

Let G be a graph. A *triangle-path* in G is a sequence of distinct cycles $T_1T_2 \cdots T_m$ in G, each having length at most 3, such that for $1 \leq i \leq m - 1$,

$$|E(T_i) \cap E(T_{i+1})| = 1 \quad \text{and} \quad E(T_i) \cap E(T_j) = \emptyset \quad \text{for } |i - j| > 1. \tag{1}$$

Two edges $e, e' \in E(G)$ are *triangularly connected* if G has a triangle-path $T_1T_2 \cdots T_m$ such that $e \in E(T_1)$ and $e' \in E(T_m)$. Such a triangle-path is also referred as an (e, e')-triangle-path.

Two edges $e, e' \in E(G)$ are equivalent if they are the same, parallel or triangularly connected. One can easily verify that this is an equivalence relation. Each equivalence class is called a *triangularly connected component*. A graph G is *triangularly connected* if it has only one triangularly connected component.

A *wheel* W_n is the graph obtained from C_n by adding one vertex and joining it to each vertex of C_n. A *fan* F_n is the graph obtained from P_n by adding one vertex and joining it to each vertex of P_n. Clearly, $K_4 \cong W_3$ and $K_3 \cong F_2$.

Let G_1, G_2 be two disjoint graphs. As in [3], $G_1 \oplus_2 G_2$, called the *parallel connection* of G_1 and G_2, is defined to be the graph obtained from $G_1 \cup G_2$ by identifying exactly one edge.

Let $\mathcal{WF}$ be the family of graphs that satisfy the following conditions:

(i) $K_3, W_{2n+1} \in \mathcal{WF}$;
(ii) If $G_1, G_2 \in \mathcal{WF}$, then $G_1 \oplus_2 G_2 \in \mathcal{WF}$.

Define $\mathcal{WF}_2$ to be the family of graphs such that a graph $G \in \mathcal{WF}_2$ if and only if $G \in \mathcal{WF}$ and G is 2-triangular.

Lemma 2.3 (*[3]*). *Let G be a triangularly connected graph. Then $G \notin \langle Z_3 \rangle$ if and only if $G \in \mathcal{WF}$.*

Lemma 2.4. *Let G be a connected graph. If for every edge e_0 of G, there is a minimal edge cut X_0 of G containing e_0 with size 2, then $\delta(G) = 2$.*

Proof. By the assumptions, it is obvious that G is 2-edge-connected. Choose $X = \{e_1, e_2\}$ to be a minimal edge-cut of G such that one component of $G \setminus X$, say G_1, has the fewest vertices, that is, $|V(G_1)|$ is minimized. Denote the other component of $G \setminus X$ by G_2. If e_1, e_2 are parallel edges, by the choice of e_1, e_2, G_1 is edgeless and so G_1 has a vertex of degree 2 of G, that is, $\delta(G) = 2$. Now suppose that e_1, e_2 are not parallel edges. If G_1 has an edge e_3, then it is contained in a minimal edge cut $Y = \{e_3, e_4\}$ of G. Denote the two components of $G \setminus Y$ by G_3, G_4. By the choice of X, $Y \cap E(G_1) = \{e_3\}$, $Y \cap E(G_2) = \{e_4\}$. Since Y is a minimal edge-cut of G, $X \cap E(G_3) \neq \emptyset$ and $X \cap E(G_4) \neq \emptyset$. Therefore $Z = \{e_1, e_3\}$ (so is $\{e_2, e_3\}$) is a minimal 2-edge-cut of G such that $G \setminus Z$ has a component with fewer vertices than G_1, contradicting the choice of X. Therefore, G_1 is edgeless and so G_1 has a vertex of degree 2 of G, that is, $\delta(G) = 2$. $\square$

3. Main theorems

Proposition 3.1. *Let H be a triangularly connected 2-triangular graph such that H is not Z_3-connected. Then each of the following holds.*

(i) $H \in \mathcal{WF}$ *and furthermore* $H \in \mathcal{WF}_2$;
(ii) *For any* $v \in V(H)$, $d_H(v) = 3$ *or* $d_H(v) \geq 5$;
(iii) $|D_3(H)| \geq 4$;
(iv) H *is 3-edge-connected, essentially 4-edge connected.*

Proof. (i) Since H is triangularly connected and H is not Z_3-connected, by Lemma 2.3, $H \in \mathcal{WF}$. Furthermore, since H is 2-triangular, $H \in \mathcal{WF}_2$.

(ii) By the definition of $\mathcal{WF}_2$ and the fact that $H \in \mathcal{WF}_2$, for any $w \in V(H)$, $d_H(w) \geq 3$. Suppose, to the contrary, that there is $v \in V(H)$ such that $d_H(v) = 4$. Now consider the induced graph $H[N_H(v)]$. For any vertex $x \in N_H(v)$, it must have degree at least 2 in $H[N_H(v)]$. Otherwise, vx is contained in at most one triangle in H, a contradiction to the fact that $H \in \mathcal{WF}_2$. Since $H[N_H(v)]$ has exactly 4 vertices and each vertex has degree at least 2, $H[N_H(v)]$ contains a 4-cycle as a spanning subgraph and then the graph induced by v and its neighbors $H[\{v\} \cup N_H(v)]$ contains a W_4, contradicting the fact that $H \in \mathcal{WF}$. Therefore, for any $v \in V(H)$, $d_H(v) = 3$ or $d_H(v) \geq 5$.

(iii) Define $T(H)$ to be the graph such that the vertices of $T(H)$ are the maximal odd wheels and the maximal fans of H, and two vertices of $T(H)$ are adjacent if their corresponding graphs in G share one edge. By the definition of $\mathcal{WF}$, $T(H)$ is a tree. Furthermore, by the fact that $H \in \mathcal{WF}_2$, each pendent vertex of $T(H)$ corresponds to a K_4 of H, which has at least two vertices of degree 3 in it. Otherwise, there is at least one edge which is contained in only one triangle. Since $T(H)$ has at least two pendent vertices, there are at least 4 distinct vertices in H with degree 3.

(iv) Suppose that X is an essential edge cut of H. Since H is 2-triangularly connected, $|X| \geq 3$. Suppose that $|X| = 3$. By the fact that H is 2-triangularly connected again, all the three edges in X must be adjacent to one common vertex and therefore X is a trivial edge cut, contradicting the fact that X is an essential edge cut. So $|X| \geq 4$ and H is essentially 4-edge connected. □

Assuming the truth of Theorem 1.4, we can present the proof of Corollary 1.5, as follows:

Proof of Corollary 1.5. By Lemma 2.2, $\Lambda_g(G) \leq 4$. By Theorem 1.4, $G \in \langle Z_3 \rangle$ and so $\Lambda_g(G) \leq 3$. □

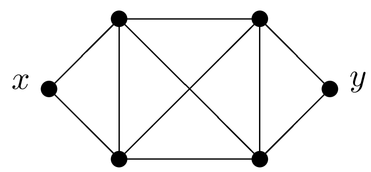
Proof of Corollary 1.6. It suffices to show that every connected 3-triangular graph is Z_3-connected. Suppose, to the contrary, that G is a minimal counterexample with $n(G) = |V(G)| + |E(G)|$ minimized.

Let $L_1, L_2, \ldots, L_s$ be the triangularly connected components of G. Then for each i, $L_i \notin \langle Z_3 \rangle$. Otherwise, assume $L_j \in \langle Z_3 \rangle$ and so G/L_j is 3-triangular. By the minimality of G, $G/L_j \in \langle Z_3 \rangle$. By Lemma 2.1, $G \in \langle Z_3 \rangle$, contradicting the choice of G.

Since for each i, L_i is not Z_3-connected, by Lemma 1.3, $L_i \in \mathcal{WF}$ and so G is a simple graph. Let X be an edge-cut of G and $e \in X$. Since G is 3-triangular, there are three distinct cycles C_1, C_2, C_3 of length 3 containing e. By the fact that $|E(C_i) \cap X| = 2$ for $i = 1, 2, 3$, we can assume that $E(C_i) \cap X = \{e, e_i\}$. Then $\{e, e_1, e_2, e_3\} \subseteq X$. Since G is simple, $|X| \geq |\{e, e_1, e_2, e_3\}| = 4$. Therefore, G is 4-edge-connected. By Theorem 1.4, $G \in \langle Z_3 \rangle$, contradicting the choice of G again. □

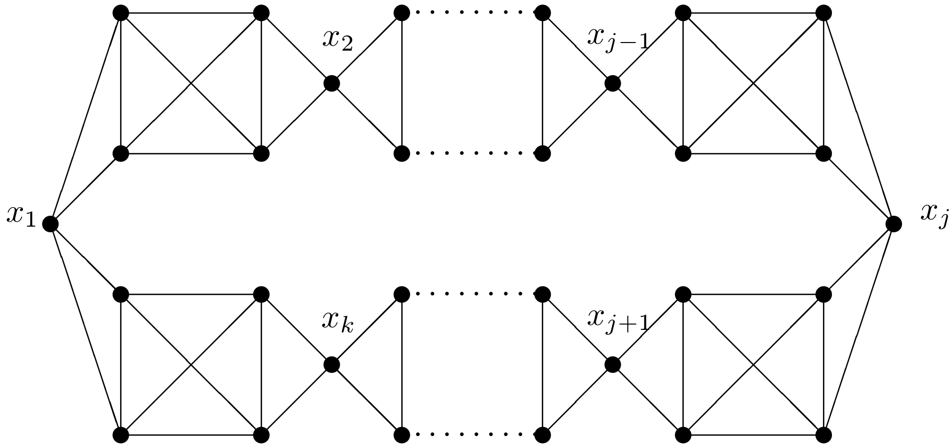
Example 3.2. Theorem 1.4 is best possible in the sense that being 2-triangular cannot be relaxed to being 1-triangular.

Let $L(x, y)$ be a graph as follows:



$L(x, y)$

For $k \geq 3$, let $L_1, L_2, \ldots, L_k$ be graphs such that for each i, $L_i(x_i, y_i) \cong L(x, y)$. Let $G(k)$ be a graph obtained from $L_1, L_2, \ldots, L_k$ by identifying y_i and x_{i+1}, where $x_{k+1} = x_1$ and $i = 1, 2, \ldots, k$.



$G(k)$

It was proved in [7] that $G(k)$ is not Z_3-connected for $k \geq 3$. Clearly, $G(k)$ is a 4-edge-connected 1-triangular graph but $G(k)$ is not Z_3-connected.

Example 3.3. Theorem 1.4 is best possible in the sense that being 4-edge-connected cannot be relaxed to being 3-edge-connected.

Let $H(k)$ be the graph obtained from k copies of K_4 by picking one edge from each copy and identifying them. It is known (Example 4.3 and Lemma 4.6 in [6]) that $H(k)$ is 3-edge-connected, 2-triangular, but $H(k)$ is not Z_3-connected. All the $H(k)$'s are members in $\mathcal{WF}$, and so Lemma 2.3 presents an alternative proof that each $H(k)$ is not Z_3-connected.

Example 3.4. Corollary 1.6 is best possible in the sense that being 3-triangular cannot be relaxed to being 2-triangular.

The graph $H(k)$ defined above is a connected 2-triangular graph, but $H(k)$ is not Z_3-connected.

4. Proof of Theorem 1.4

Let $\mathcal{F}$ be the family of 4-edge-connected 2-triangular graphs. Let $G \in \mathcal{F}$ and $H_1, H_2, \ldots, H_m$ be the triangularly connected components of G. Then for all i, j with $i \neq j$, $E(H_i) \cap E(H_j) = \emptyset$.

By way of contradiction, assume that

$$G \text{ is a counterexample with } n(G) = |V(G)| + |E(G)| \text{ minimized.} \tag{2}$$

Recall that $H_1, H_2, \ldots, H_m$ are the triangularly-connected components of G.

Claim 1. *For $1 \leq i \leq m$, $H_i \notin \langle Z_3 \rangle$.*

Proof. Assume that $H_i \in \langle Z_3 \rangle$. Let $G' = G/H_i$. By the structure of G, G' is 4-edge-connected and 2-triangular and so $G' \in \mathcal{F}$. Since $n(G') < n(G)$, by the minimality of G, $G' \in \langle Z_3 \rangle$. By Lemma 2.1, $G \in \langle Z_3 \rangle$, contrary to (2). □

Claim 2. *G is 2-connected.*

Proof. Suppose, to the contrary, that v is a vertex cut of G such that G_1 and G_2 are the two subgraphs of G such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$. By the structure of G, G_1 and G_2 are both 4-edge-connected and 2-triangular. Therefore, $G_1, G_2 \in \mathcal{F}$. Since $n(G_1) < n(G)$, $n(G_2) < n(G)$, by the minimality of G, G_1, G_2 are both Z_3-connected and therefore, G is Z_3-connected, contrary to (2). □

Define a bipartite graph $B(G) = (V_1, V_2)$ as follows: $V_1 = \{H_1, H_2, \ldots, H_m\}$, $V_2 = \{v \mid v \in \bigcup_{i=1}^{m} D_3(H_i)\}$, and $E(B(G)) = \{(H_i, v) \mid v \in V(H_i) \cap V(H_j) \cap (D_3(H_i) \cup D_3(H_j))$ for some $j\}$. By Claim 1 and Proposition 3.1(i), $H_i \in \mathcal{WF}_2$, $1 \leq i \leq m$. By Proposition 3.1 (iii), for $1 \leq i \leq m$, $|D_3(H_i)| \geq 4$.

Therefore, for each $H_i \in V_1$, $d_{B(G)}(H_i) \geq 4$. For each $v \in V_2$, by the definition of v in V_2, $d_{B(G)}(v) \geq 2$. Define $B'(G)$ to be the graph obtained from $B(G)$ as follows: for each $v \in V_2$ with $d_{B(G)}(v) = 2$, contract one edge adjacent to v. Since suppressing 2-vertices in V_2 will not result in a new degree 2 vertex in V_1 and $B(G)$ is connected, by the process of getting $B'(G)$ from $B(G)$, both $B'(G)$ is connected and $\delta(B'(G)) \geq 3$.

Let X be an essential edge cut of G such that G_1, G_2 are the two nontrivial components of $G - X$. If either $H_i \subseteq G[E(G_1) \cup X]$ or $H_i \subseteq G[E(G_2) \cup X]$ for each triangularly connected component H_i of G, then we call X a proper essential edge cut.

Claim 3. *G has a vertex w such that for some $k \neq l$, $w \in V(H_k) \cap V(H_l) \cap (D_3(H_k) \cup D_3(H_l))$, and such that every proper essential edge cut of G containing $E_{H_k}(w)$ is of size at least 9.*

Proof. Since $B'(G)$ is a connected graph with $\delta(B'(G)) \geq 3$, by Lemma 2.4, $B'(G)$ has an edge e such that for any minimal edge cut X of $B'(G)$ containing e, $|X| \neq 2$.

If $e = (H_k, H_l)$, let $w \in V(H_k) \cap V(H_l) \cap (D_3(H_k) \cup D_3(H_l))$ and, without loss of generality, assume that $w \in D_3(H_k)$. By the definition of $B(G)$ and $B'(G)$, $d_{B(G)}(w) = 2$. Suppose that $|X| = 1$, that is, $X = \{e\}$. Then $E_{H_k}(w)$ is an edge cut of G with size 3, contradicting the fact that G is 4-edge-connected. Therefore, $|X| \geq 3$. By the definition of $B'(G)$ and the fact that every minimal edge cut of $B'(G)$ containing e is of size at least 3, every minimal edge cut of $B(G)$ containing $e = (H_k, w)$ is either of size at least 3 or a trivial edge cut $\{(H_k, w), (H_l, w)\}$.

If $e = (H_k, w)$, let $w \in V(H_k) \cap V(H_l) \cap (D_3(H_k) \cup D_3(H_l))$ and without loss of generality, assume that $w \in D_3(H_k)$. Suppose that $|X| = 1$, that is, $X = \{e\}$. Then $E_{H_k}(w)$ is an edge cut of G with size 3, contradicting the fact that G is 4-edge-connected. Therefore, $|X| \geq 3$. Since every minimal edge cut of $B'(G)$ containing e is of size at least 3, every minimal edge cut of $B(G)$ containing e is of size at least 3.

Since each edge (H_i, v) in $B(G)$ corresponds to $E_{H_i}(v)$ in G with $|E_{H_i}(v)| \geq 3$ and every proper essential edge cut of G corresponds to an edge cut of $B(G)$, every proper essential edge cut of G containing $E_{H_k}(w)$ is of size at least 9. □

Let G' be the graph obtained from G by splitting w into w' and w'' with $N(w') = V(H_k) \cap N(w)$ and $N(w'') = V(H_l) \cap N(w)$. By the definition of G', G' is 2-triangular, $\delta(G') = 3$ and $D_3(G') \subseteq \{w', w''\}$. By Claim 2, G' is connected. By Proposition 3.1(iv) and the definition of G', G' is 3-edge-connected, essentially 4-edge-connected. Otherwise, G' has an essential edge cut X with $|X| = 3$. By Proposition 3.1(iv), X must be an essential edge cut of G' which is a proper essential edge cut, then $E_{H_k} \cup X$ is a proper essential edge cut of G with $|E_{H_k} \cup X| = 6$, contradicting Claim 3.

By the choice of w, the definition of G' and Proposition 3.1(ii), $d_{G'}(w'') = 3$ or $d_{G'}(w'') \geq 5$. In the following, we distinguish two cases considering $d_{G'}(w'') \geq 5$ and $d_{G'}(w'') = 3$.

Case 1: $d_{G'}(w'') \geq 5$.

Denote the vertices adjacent to w' by x_1, x_2 and x_3. Assume the three edges incident with w' all have tails at w'. Denote the edge $w'x_3$ by e'. Let H'_k be the graph obtained from H_k by deleting $w'x_1$ and then contracting $w'x_2$ and Let G'' be the graph obtained from G' by deleting $w'x_1$ and then contracting $w'x_2$. Define $G''' = G''/H'_k$. Noticing that $G''' = G''/H'_k = G'/H_k$ and that G' is 2-triangular, 3-edge-connected, essentially 4-edge-connected with $D_3(G') = \{w'\}$, G''' is 2-triangular and 4-edge-connected. Therefore, $G''' \in \mathcal{F}$. Since $n(G''') < n(G)$, by the minimality of G, $G''' \in \langle Z_3 \rangle$. Since H'_k is triangularly connected and $H'_k \notin \langle WF \rangle$, by Lemma 2.3, $H'_k \in \langle Z_3 \rangle$. By Lemma 2.1, $G'' \in \langle Z_3 \rangle$.

For any $b \in Z(G, Z_3)$, define $b'' \in Z(G'', Z_3)$ by

$$b''(z) = \begin{cases} b(z) & \text{if } z \neq w'', x_1; \\ b(w) - 1 & \text{if } z = w''; \\ b(x_1) + 1 & \text{if } z = x_1. \end{cases}$$

Since $G'' \in \langle Z_3 \rangle$, there is $f \in F^*(G'', Z_3)$ such that $\partial f = b''$.

Let $f_1 \in F^*(G, Z_3)$ be such that

$$f_1(e) = \begin{cases} 1 & \text{if } e = w'x_1; \\ 3 - f(e') & \text{if } e = w'x_2; \\ f(e) & \text{otherwise.} \end{cases}$$

It is easy to check that $\partial f_1 = b$, that is, G admits a (Z_3, b)-NZF. Since $b \in Z(G, Z_3)$ is arbitrary, $G \in \langle Z_3\rangle$, contrary to (2).

Case 2: $d_{G'}(w'') = 3$.

Denote the vertices adjacent to w' by x_1', x_2' and x_3', and the vertices adjacent to w'' by x_1'', x_2'' and x_3''. Assume the three edges incident with w' all have tails at w' and the three edges incident with w'' all have tails at w''. Denote the edge $w'x_3'$ by e' and the edge $w''x_3''$ by e''. Let H_k' be the graph obtained from H_k by deleting $w'x_1'$ and then contracting $w'x_2'$. Let H_l' be the graph obtained from H_l by deleting $w''x_1''$ and then contracting $w''x_2''$. Let G'' be the graph obtained from G' by deleting $w'x_1'$, $w''x_1''$ and then contracting $w'x_2'$ and $w''x_2''$. Define $G''' = G''/(H_k' \cup H_l')$. Noticing that $G''' = G''/(H_k' \cup H_l') = G'/(H_k \cup H_l)$ and that G' is 2-triangular, 3-edge-connected, essentially 4-edge-connected with $D_3(G') = \{w', w''\}$, it follows that G''' is 2-triangular, 4-edge-connected. Therefore, $G''' \in \mathcal{F}$. Since $n(G''') < n(G)$, by the minimality of G, $G''' \in \langle Z_3\rangle$. Since H_k' is triangularly connected and $H_k' \notin \langle WF\rangle$, by Lemma 2.3, $H_k' \in \langle Z_3\rangle$. Similarly, we can prove that $H_l' \in \langle Z_3\rangle$. By Lemma 2.1, $G'' \in \langle Z_3\rangle$.

Let $b \in Z(G, Z_3)$ and let $\alpha, \beta \in Z_3^*$ be such that $\alpha + \beta = b(w)$. This is possible since $1 + 2 \equiv 0, 2 + 2 \equiv 1, 1 + 1 \equiv 2$.

Define $b'' \in Z(G'', Z_3)$ by

$$b''(z) = \begin{cases} b(z) & \text{if } z \neq x_1', x_1''; \\ b(x_1') + \alpha & \text{if } z = x_1'; \\ b(x_1'') + \beta & \text{if } z = x_1''. \end{cases}$$

Since $G'' \in \langle Z_3\rangle$, there is $f \in F^*(G'', Z_3)$ such that $\partial f = b''$.

Let $f_1 \in F^*(G, Z_3)$ be such that

$$f_1(e) = \begin{cases} \alpha & \text{if } e = w'x_1'; \\ \beta & \text{if } e = w''x_1''; \\ 3 - f(e') & \text{if } e = w'x_2'; \\ 3 - f(e'') & \text{if } e = w''x_2''; \\ f(e) & \text{otherwise.} \end{cases}$$

It is easy to check that $\partial f_1 = b$, that is, G admits a (Z_3, b)-NZF. Since $b \in Z(G, Z_3)$ is arbitrary, $G \in \langle Z_3\rangle$, contrary to (2). □

Acknowledgments

The authors would like to thank the referees for many useful suggestions and comments which have considerably improved the presentation of this paper.

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