Information Processing Letters 111 (2011) 1085–1088

Contents lists available at SciVerse ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Mod $(2p+1)$-orientations in line graphs

Hong-Jian Lai [a,c], Hao Li [b], Ping Li [c], Yanting Liang [c,d,*], Senmei Yao [c]

[a] College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, China
[b] Department of Mathematics, Renmin University of China, Beijing 100872, China
[c] Department of Mathematics, West Virginia University, Morgantown, WV 26506, United States
[d] Mathematics and Computer Science Department, St. Mary's College of Maryland, St. Mary's City, MD 20686, United States

ARTICLE INFO

Article history:
Received 17 October 2009
Received in revised form 2 September 2011
Accepted 5 September 2011
Available online 16 September 2011
Communicated by B. Doerr

Keywords:
Combinatorial problems
Nowhere zero flows
Mod $(2p+1)$-orientation
Mod $(2p+1)$-contractible
Line graph

ABSTRACT

Jaeger in 1984 conjectured that every $(4p)$-edge-connected graph has a mod $(2p+1)$-orientation. It has also been conjectured that every $(4p+1)$-edge-connected graph is mod $(2p+1)$-contractible. In [Z.-H. Chen, H.-J. Lai, H. Lai, Nowhere zero flows in line graphs, Discrete Math. 230 (2001) 133–141], it has been proved that if G has a nowhere-zero 3-flow and the minimum degree of G is at least 4, then $L(G)$ also has a nowhere-zero 3-flow. In this paper, we prove that the above conjectures on line graphs would imply the truth of the conjectures in general, and we also prove that if G has a mod $(2p+1)$-orientation and $\delta(G) \geqslant 4p$, then $L(G)$ also has a mod $(2p+1)$-orientation, which extends a result in Chen et al. (2001) [2].

1. Introduction

Graphs in this note are finite and loopless. We follow Bondy and Murty [1] for undefined notations and terminology.

Given a graph G with $E(G) \neq \emptyset$, the line graph of G, denoted by $L(G)$, has $E(G)$ as the vertex set, where two vertices e_1, e_2 are adjacent in $L(G)$ if and only if the corresponding edges e_1, e_2 are adjacent in G. Denote $\kappa'(G)$ to be the edge connectivity of G.

Let $D = D(G)$ be an orientation of an undirected graph G. If an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then define $tail(e) = u$ and $head(e) = v$. For vertex sets $U, V \subset V(G)$ with $U \cap V = \emptyset$, denote

$$E_G(U,V) = \{uv \in E(G): u \in U,\ v \in V\},$$

$$E_D^-(U,V) = \{e = uv \in E(D): head(e) = u \in U,\ tail(e) = v \in V\},$$

$$E_D^+(U,V) = \{e = uv \in E(D): tail(e) = u \in U,\ head(e) = v \in V\}.$$

Let $d_D^-(U,V) = |E_D^-(U,V)|$ and $d_D^+(U,V) = |E_D^+(U,V)|$. If $U = \{v\}$ and $V = V(G) - \{v\}$, then we use $E_G(v)$, $E_D^-(v)$ and $E_D^+(v)$ to denote the subsets of edges incident with v in G, directed into v and directed from v under orientation D, respectively, and let $d_D^-(v) = |E_D^-(v)|$ and $d_D^+(v) = |E_D^+(v)|$. The subscript D may be omitted when $D(G)$ is understood from the context. Let $D_i(G) = \{v \in V(G): d_G(v) = i\}$, for any integer $i \geqslant 0$.

For some positive integer m, denote Z_m to be the set of integers modulo m. For a graph G with an orientation D, if $f: E \mapsto \mathbf{Z}_m$, then $\forall v \in V(G)$, define

$$\partial f(v) \equiv \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e) \pmod m.$$

Thus $\partial f: V(G) \mapsto \mathbf{Z}_m$ is a map, called the **boundary of** f. A nowhere-zero k-flow (abbreviated as a k-NZF) of G is an

* Corresponding author at: Department of Mathematics, West Virginia University, Morgantown, WV 26506, United States. Also a visiting instructor at Mathematics and Computer Science Department, St. Mary's College of Maryland, St. Mary's City, MD 20686, United States.
E-mail address: lyt814@math.wvu.edu (Y. Liang).

0020-0190/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.ipl.2011.09.005

orientation D of G together with a map $f : E \mapsto \mathbf{Z}_k$ such that $\partial f(v) = 0$ at every vertex v of G and $f(e) \neq 0$ for every edge e of G.

The well-known Tutte's 3-NZF conjecture is still open, to the best of our knowledge.

Conjecture 1.1. *(See Tutte [8].) Every* 4*-edge-connected graph has a* 3*-NZF.*

In [2], the following theorem is proved.

Theorem 1.2. *(See Chen et al., Theorem 1.4 in [2].) If G has a* 3*-NZF and the minimum degree of G is at least* 4*, then $L(G)$ also has a* 3*-NZF.*

For a graph G, if G has an orientation D such that at every vertex $v \in V(G)$, $d_D^+(v) - d_D^-(v) \equiv 0 \pmod{2p+1}$, then we say that G admits a **mod** $(2p+1)$**-orientation**. The set of all graphs which have mod $(2p+1)$-orientations is denoted by M_{2p+1}. It can be proved that G has a 3-NZF if and only if $G \in M_3$ by letting $f : E \mapsto \{1, -1\}$ with $\partial f = 0$ and then reversing the orientation of e for every $e \in E(G)$ with $f(e) = -1$ to obtain a mod 3-orientation.

Jaeger extended Tutte's 3-NZF conjecture to M_{2p+1}.

Conjecture 1.3. *(See Jaeger [3] and [4].) Every $(4p)$-edge-connected graph is in M_{2p+1}.*

A function $b : V(G) \mapsto \mathbf{Z}_m$ is a **zero sum function** in $\mathbf{Z}_m$ if $\sum_{v \in V(G)} b(v) \equiv 0 \pmod m$. The set of all zero sum functions in $\mathbf{Z}_m$ of G is denoted by $Z(G, \mathbf{Z}_m)$. Define M_{2p+1}^o to be the collection of graphs such that $G \in M_{2p+1}^o$ if and only if $\forall b \in Z(G, \mathbf{Z}_{2p+1})$, there exists an orientation D of G such that $d_D^+(v) - d_D^-(v) \equiv b(v) \pmod{2p+1}$.

Several characterizations of graphs in M_{2p+1}^o have been obtained in [6]. Some investigations on M_{2p+1} and M_{2p+1}^o are performed in [5], [7] and [6]. In particular, a similar conjecture is also proposed for M_{2p+1}^o.

Conjecture 1.4. *(See [5] and [7].) Every $(4p+1)$-edge-connected graph is in M_{2p+1}^o.*

In this paper, we prove that the Conjectures 1.3 and 1.4 on line graphs would imply the truth of the conjectures in general.

Theorem 1.5. *Let k be an integer. Then the following statements are equivalent:*

(i) *For every graph G, if $\kappa'(G) \geqslant k$, then $G \in M_{2p+1}$.*
(ii) *For every graph G, if $\kappa'(L(G)) \geqslant k$, then $L(G) \in M_{2p+1}$.*

Theorem 1.6. *Let k be an integer. Then the following statements are equivalent:*

(i) *For every graph G, if $\kappa'(G) \geqslant k$, then $G \in M_{2p+1}^o$.*
(ii) *For every graph G, if $\kappa'(L(G)) \geqslant k$, then $L(G) \in M_{2p+1}^o$.*

We also extend Theorem 1.2 to the following.

Theorem 1.7. *Let G be a graph. If $G \in M_{2p+1}$ and $\delta(G) \geqslant 4p$, then $L(G) \in M_{2p+1}$.*

2. Preliminaries

In this section, we review some useful results needed in the arguments.

Proposition 2.1. *(See Proposition 2.2 in [5].) For any integer $p \geqslant 1$, M_{2p+1}^o is a family of connected graphs such that each of the following holds:*

(C1) *$K_1 \in M_{2p+1}^o$.*
(C2) *If $e \in E(G)$ and if $G \in M_{2p+1}^o$, then $G/e \in M_{2p+1}^o$.*
(C3) *If H is a subgraph of G, and if $H, G/H \in M_{2p+1}^o$, then $G \in M_{2p+1}^o$.*

Proposition 2.2. *For any integer $p \geqslant 1$, M_{2p+1} is a family of connected graphs such that each of the following holds:*

(C1) *$K_1 \in M_{2p+1}$.*
(C2) *If $e \in E(G)$ and if $G \in M_{2p+1}$, then $G/e \in M_{2p+1}$.*
(C3) *If H is a subgraph of G with $H \in M_{2p+1}^o$, then $G/H \in M_{2p+1}$ if and only if $G \in M_{2p+1}$.*

The proofs of Proposition 2.2 (C2) and (C3) are similar to those for Proposition 2.1 (C2) and (C3), and so they are omitted.

Lemma 2.3. *(See Lai et al., Proposition 2.3(v) and Example 2.5 in [6].) A complete graph $K_m \in M_{2p+1}^o$ if and only if $m = 1$ or $m \geqslant 4p+1$.*

The following lemma follows from the definition of line graph.

Lemma 2.4. *Let G be a graph with $E(G) \neq \emptyset$ and let $e \in E(G)$ such that the two ends of e are u and v. Let $G(e)$ be the graph obtained from G by replacing e by a (u, v)-path uv_ev of length* 2*. Let e' denote the edge in $L(G(e))$ that has uv_e and v_ev as its ends. Then*

$$L\big(G(e)\big)/\big\{e'\big\} = L(G).$$

Let G be a graph and let $S(G)$, the subdivided graph of G, be the graph obtained from G by replacing each edge e of G by a path of length 2 with a newly added internal vertex v_e. Note that the correspondence $e \leftrightarrow e'$ defined in Lemma 2.4 is a bijection between $E(G)$ and $\{e' \mid e \in E(G)\} \subset E(L(S(G)))$. Define

$$E'(G) = \big\{e' \in E\big(L\big(S(G)\big)\big) \bigm| e \in E(G)\big\}.$$

Then clearly,

$$L(G) = L\big(S(G)\big)/E'(G) \tag{1}$$

and

$$E\big(L\big(S(G)\big)\big) - E'(G) = \bigcup_{v \in V(G)} E\big(L\big(E_{S(G)}(v)\big)\big), \tag{2}$$

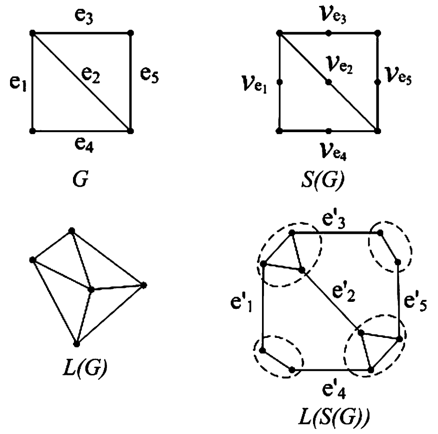

Fig. 1. Graphs $L(G)$, $S(G)$ and $L(S(G))$ for a given G.

so we have

$$L\big(S(G)\big)/\big[E\big(L\big(S(G)\big)\big)-E'(G)\big]=G. \tag{3}$$

Example 1. Let G be the graph shown in Fig. 1. And $L(G)$, $S(G)$ and $L(S(G))$ are also shown here. Note that $E'(G)=\{e_i'\colon\ 1\leqslant i\leqslant 5\}$. It's easy to check that Eqs. (1), (2) and (3) hold from the graphs.

3. Main results

Lemma 3.1. *Let T be a connected spanning subgraph of G. If for each edge $e\in E(T)$, G has a subgraph $H_e\in M^o_{2p+1}$ with $e\in E(H_e)$, then $G\in M^o_{2p+1}$.*

Proof. We argue by induction on $|V(G)|$. The lemma holds trivially if $|V(G)|=1$. Assume that $|V(G)|>1$ and pick an edge $e'\in E(T)$. Then G has a subgraph $H'\in M^o_{2p+1}$ such that $e'\in E(H')$. Let $G'=G/H'$ and let $T'=T/(E(H')\cap E(T))$. Since T is a connected spanning subgraph of G, T' is a connected spanning subgraph of G'. For each $e\in E(T')$, $e\in E(T)$, and so by assumption, G has a subgraph $H_e\in M^o_{2p+1}$ with $e\in E(H_e)$. By Proposition 2.1(C2), $H'_e=H_e/(E(H_e)\cap E(H'))\in M^o_{2p+1}$ and $e\in H'_e$. Therefore by induction $G'\in M^o_{2p+1}$. Then by Proposition 2.1(C3), and by the assumption that $H'\in M^o_{2p+1}$, $G\in M^o_{2p+1}$. $\square$

Lemma 3.2. *Let G be a graph. If $\delta(G)\geqslant 4p+1$, then $L(G)\in M^o_{2p+1}$.*

Proof. Since $\delta(G)\geqslant 4p+1$, for any $e\in L(G)$, $e\in K_m$ with $m\geqslant 4p+1$. By Lemma 2.3, $K_m\in M^o_{2p+1}$. Therefore, $L(G)\in M^o_{2p+1}$ by Lemma 3.1. $\square$

Lemma 3.3. *Let G be a graph and k be an integer. If $\kappa'(G)\geqslant k$, then*

$$\kappa'\big(L\big(S(G)\big)\big)\geqslant k.$$

Proof. By contradiction, suppose X is an edge cut of $L(S(G))$ satisfies: (1) $|X|<k$ and $|X|$ is minimized; and (2) $|X\cap E'(G)|$ is maximized subject to (1). Since $\kappa'(G)\geqslant k$, $\delta(G)\geqslant k$. Note that for any $x\in V(L(S(G)))$, $x\in K_m$ with $m\geqslant k$ and $|E_x|=1$ where $E_x=\{e=xy\in L(S(G))\mid y\notin V(K_m)\}$. Therefore, $\delta(L(S(G)))\geqslant k$.

If $X\subseteq E'(G)$, by Eq. (3), X is also an edge cut of G. Therefore, $|X|\geqslant k$, contrary to $|X|<k$.

Suppose there exists $e=uv\in X-E'(G)$, then e is in some K_m and is adjacent to some $e'=uv'$ with $v'\notin V(K_m)$. Let H be one of the components of $L(S(G))-X$. If H contains only one vertex, then $|X|\geqslant\delta(L(S(G)))\geqslant k$. If H contains at least 2 vertices, let

$$X'=\big(X-E(K_m)\big)\cup\big\{u_iv_i\in E'(G)\colon\ u_i\in K_m\cap H,\ v_i\notin K_m\big\}$$

then $|X'|\leqslant|X|$ and $|X'\cap E'(G)|>|X\cap E'(G)|$ and X' is also an edge cut of $L(S(G))$, contrary to that $|X|$ is minimized and $|X\cap E'(G)|$ is maximized.

Hence, $\kappa'(L(S(G)))\geqslant k$. $\square$

Proof of Theorem 1.5. (i) $\Rightarrow$ (ii) It is trivial.

(ii) $\Rightarrow$ (i) Since $\kappa'(G)\geqslant k$, $\kappa'(L(S(G)))\geqslant k$ by Lemma 3.3. Then by the assumption of part (ii), $L(S(G))\in M_{2p+1}$. Note that G is a contraction of $L(S(G))$ by Eq. (3). Thus $G\in M_{2p+1}$ by Proposition 2.2(C2). $\square$

Corollary 3.4. *To prove Conjecture* 1.3*, it suffices to prove that if $\kappa'(L(G))\geqslant 4p$, then $L(G)\in M_{2p+1}$, for any graph G.*

The proof of Theorem 1.6 is similar to that of Theorem 1.5.

Proof of Theorem 1.6. (i) $\Rightarrow$ (ii) It is trivial.

(ii) $\Rightarrow$ (i) Since $\kappa'(G)\geqslant k$, $\kappa'(L(S(G)))\geqslant k$ by Lemma 3.3. Then by the assumption of part (ii), $L(S(G))\in M^0_{2p+1}$. Note that G is a contraction of $L(S(G))$ by Eq. (3). Thus $G\in M^0_{2p+1}$ by Proposition 2.1(C2). $\square$

Corollary 3.5. *To prove Conjecture* 1.4*, it suffices to prove that if $\kappa'(L(G))\geqslant 4p+1$, then $L(G)\in M^0_{2p+1}$, for any graph G.*

Lemma 3.6. *If $G\in M_{2p+1}$ and $\delta(G)=4p$, then $L(G)\in M_{2p+1}$.*

Proof. By Proposition 2.2(C2) and Eq. (1), it suffices to prove that $L(S(G))\in M_{2p+1}$.

Since $G\in M_{2p+1}$, G has an orientation D such that at every vertex $v\in V(G)$,

$$d^+_D(v)-d^-_D(v)\equiv 0 \pmod{2p+1}. \tag{4}$$

Note that by Eq. (3), D is an orientation of a subgraph of $L(S(G))$. By Eq. (2), $E(L(S(G)))-E'(G)$ is a disjoint union of $K_{d(v)}$ with $v\in V(G)$. By Eqs. (2) and (4), under the orientation D,

$$d^+_D\big(K_{d(v)},L\big(S(G)\big)-K_{d(v)}\big)-d^-_D\big(K_{d(v)},L\big(S(G)\big)-K_{d(v)}\big)\equiv 0 \pmod{2p+1}, \tag{5}$$

and for any vertex $u\in K_{d(v)}$

$$\big|E_G\big(u,L\big(S(G)\big)-K_{d(v)}\big)\big|=1. \tag{6}$$

If $d(v) \geqslant 4p+1$, then by Lemma 2.3, $K_{d(v)} \in M^o_{2p+1}$, and so there exists an orientation D_v of $K_{d(v)}$ such that $d^+_{D_v}(u) - d^-_{D_v}(u) \equiv 0 \pmod{2p+1}$ in $L(S(G))$ at every vertex $u \in K_{d(v)}$.

Now suppose $d(v) = 4p$ and let $H = K_{d(v)} = K_{4p}$. By Eqs. (5) and (6), there exists partition (U, V) of H where $U = \{u_1, u_2, \ldots, u_{2p}\}$ and $V = \{v_1, v_2, \ldots, v_{2p}\}$, such that under the orientation D,

$$d^+_D\big(u_i, L\big(S(G)\big) - H\big) - d^-_D\big(u_i, L\big(S(G)\big) - H\big) = 1$$

and

$$d^+_D\big(v_i, L\big(S(G)\big) - H\big) - d^-_D\big(v_i, L\big(S(G)\big) - H\big) = -1.$$

Let $M(v) = \{u_i v_i \mid u_i \in U,\ v_i \in V\}$ be a perfect matching of $H = K_{d(v)}$. Then $H - M(v)$ is a $(4p-2)$-regular graph, and so $H - M(v)$ is Eulerian. Therefore, $H - M(v)$ has an orientation $D_{M(v)}$ such that for any $x \in V(H)$, $d^+_{D_{M(v)}}(x) - d^-_{D_{M(v)}}(x) = 0 \pmod{2p+1}$ in $H - M(v)$. Then we define an orientation $D'_{M(v)}$ for $M(v)$ as $head(u_i v_i) = u_i$ and $tail(u_i v_i) = v_i$. Let D_v be the disjoint union of $D_{M(v)}$ and $D'_{M(v)}$.

Thus the disjoint union of D and all D_v with $v \in V(G)$ gives an orientation D' of $L(S(G))$. It is routine to verify that $d^+_{D'}(x) - d^-_{D'}(x) \equiv 0 \pmod{2p+1}$ at every vertex $x \in V(L(S(G)))$.

Hence, $L(S(G)) \in M_{2p+1}$. $\square$

Theorem 1.7 now follows from Lemmas 3.6 and 3.2. When $p = 1$, we obtain Theorem 1.2, restated as the following corollary.

Corollary 3.7. *If $G \in M_3$ and $\delta(G) \geqslant 4$, then $L(G) \in M_3$.*

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan/Elsevier, London/New York, 1976.
[2] Z.-H. Chen, H.-J. Lai, H. Lai, Nowhere zero flows in line graphs, Discrete Math. 230 (2001) 133–141.
[3] F. Jaeger, On circular flows in graphs, in: Finite and Infinite Sets, Eger, 1981, in: Colloq. Math. Societatis Janos Bolyai, vol. 37, North-Holland, Amsterdam, 1984, pp. 391–402.
[4] F. Jaeger, Nowhere-zero flow problems, in: L. Beineke, R. Wilson (Eds.), Selected Topics in Graph Theory, vol. 3, Academic Press, London/New York, 1988, pp. 91–95.
[5] H.-J. Lai, Mod $(2p+1)$-orientations and $K_{1,2p+1}$-decompositions, SIAM J. Discrete Math. 21 (2007) 844–850.
[6] H.-J. Lai, Y. Liang, J. Liu, J. Meng, Y. Shao, Z. Zhang, Contractible graphs with respect to mod $(2p+1)$-orientations, submitted for publication.
[7] H.-J. Lai, Y.H. Shao, H. Wu, J. Zhou, On mod $(2p+1)$-orientations of graphs, J. Combin. Theory Ser. B 99 (2009) 399–406.
[8] W.T. Tutte, A contribution to the theory of chromatical polynomials, Canad. J. Math. 6 (1954) 80–91.