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EVERY N_2-LOCALLY CONNECTED CLAW-FREE GRAPH WITH MINIMUM DEGREE AT LEAST 7 IS Z_3-CONNECTED

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Dedicated to the 70th Birthday of Professor Feng Tian

Let G be a 2-edge-connected undirected graph, A be an (additive) abelian group and $A^* = A - \{0\}$. A graph G is A-connected if G has an orientation $D(G)$ such that for every function $b : V(G) \mapsto A$ satisfying $\sum_{v\in V(G)} b(v) = 0$, there is a function $f : E(G) \mapsto A^*$ such that for each vertex $v \in V(G)$, the total amount of f values on the edges directed out from v minus the total amount of f values on the edges directed into v equals $b(v)$. Let Z_3 denote the group of order 3. Jaeger *et al.* conjectured that there exists an integer k such that every k-edge-connected graph is Z_3-connected. In this paper, we prove that every N_2-locally connected claw-free graph G with minimum degree $\delta(G) \geq 7$ is Z_3-connected.

Keywords:

Mathematics Subject Classification:

1. Introduction

We consider finite graphs which permit multiple edges but no loops, and refer to [2] for undefined terminologies and notations. In particular, the minimum degree, the maximum degree of a graph G are denoted by $\delta(G)$, $\Delta(G)$, respectively. If G is a simple graph, then G^c denotes the complement of G. For a subset $X \subseteq V(G)$ or $X \subseteq E(G)$, $G[X]$ denotes the subgraph of G induced by X. Unlike in [2], a 2-regular connected nontrivial graph is called a *circuit*, and a circuit on k vertices is also referred as a *k-circuit*. Throughout this paper, A denotes an (additive) abelian group with identity 0. For an integer $m \geq 1$, Z_m denotes the set of all integers modulo m, as well as the cyclic group of order m.

Let G be a graph with an orientation $D = D(G)$. For a vertex $v \in V(G)$, we use $E^+(v)$ (or $E^-(v)$, respectively) to denote the set of edges with tails (or heads, respectively) at v. Following [8], define $F(G, A) = \{f : E(G) \mapsto A\}$ and $F^*(G, A) = \{f : E(G) \mapsto A - \{0\}\}$. Given an $f \in F(G, A)$, the *boundary* of f is a map $\partial f : V(G) \mapsto A$ defined by

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e) \quad \forall\, v \in V(G),$$

where "$\sum$" refers to the addition in A.

A map $b : V(G) \mapsto A$ is called an *A-valued zero sum map* on G if $\sum_{v \in V(G)} b(v) = 0$. The set of all A-valued zero sum maps on G is denoted by $Z(G, A)$. A graph G is *A-connected* if G has an orientation D such that for every function $b \in Z(G, A)$, there is a function $f \in F^*(G, A)$ such that $\partial f = b$. Define

$$\Lambda_g(G) = \min\{k : \text{for any abelian group } A \text{ with } |A| \geq k, G \text{ is } A\text{-connected}\}.$$

An $f \in F(G, A)$ is an *A-flow* of G if $\partial f = 0$. If an A-flow $f \in F^*(G, A)$, then f is an A-nowhere-zero flow (abbreviated as an *A-NZF*). When $A = \mathbf{Z}$ is the group of integers and f is a Z-NZF, if for $\forall\, e \in E(G)$, $|f(e)| < k$, then f is a *nowhere-zero k-flow* (abbreviated as a *k-NZF*). It is noted in [8] that for a graph G, the property of being A-connected or having an A-NZF is independent of the choice of the orientation of G. Moreover, Tutte [24] showed that, for a finite abelian group A, a graph G has an A-NZF if and only if G has an $|A|$-NZF. The following conjectures on nowhere-zero flows, were first proposed by Tutte and supplemented by Jaeger.

Conjecture 1.1 (Tutte [24, 25], see also [7]).

(i) *Every graph G with $\kappa'(G) \geq 4$ has a 3-NZF.*
(ii) *There exists an integer $k \geq 4$ such that every k-edge-connected graph has 3-NZF.*

As the nowhere-zero flow problem is the corresponding homogeneous case of the group connectivity problem, Jaeger *et al.* proposed the following conjectures, which, as suggested by a result of Kochol [9], are stronger than the corresponding conjectures above.

Conjecture 1.2 (Jaeger *et al.* [8]). *Let G be a graph.*

(i) *If $\kappa'(G) \geq 5$, then $\Lambda_g(G) \leq 3$.*
(ii) *There exists an integer $k \geq 5$ such that if $\kappa'(G) \geq k$, then $\Lambda_g(G) \leq 3$.*

Many researchers have been studying these conjectures and a number of results toward these conjectures have been obtained. Steinberg and Younger [22], and independently Thomassen [23] proved that within the family of projective planar graphs, 4-edge-connectedness is sufficient for the existence of a 3-NZF. Lai and Li [12] proved that every 5-edge-connected planar graph G satisfies $\Lambda_g(G) = 3$. Several researchers proved sufficient degree conditions for the existence of a 3-NZF or

Z_3-connectedness. See [4, 5, 19, 27], and [28], among others. In [15] (see also [18]), it is shown that when the edge connectivity of a simple graph G on n vertices is at least $3\log_2(n)$, then G is Z_3-connected. Recent studies also show that among certain triangulated graphs, high edge-connectivity will assure the existence of 3-NZF, or stronger, Z_3-connectedness. See [3, 17, 26], among others.

The *line graph* of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. For a graph G, an induced subgraph H isomorphic to $K_{1,3}$ is called a *claw* of G, and the only vertex of degree 3 of H is called the *center* of the claw. A graph G is *claw-free* if it does not have an induced subgraph isomorphic to $K_{1,3}$. Beineke ([1]) and Robertson ([20] and [6]) showed that every line graph is also a claw-free graph.

Theorem 1.3. *Let G be a graph and let $L(G)$ be the line graph of G.*

(i) *Every line graph of a 4-edge-connected graph is Z_3-connected* (*see* [14, Corollary 1.5]).
(ii) *Every 2-edge-connected, locally 3-edge-connected graph is Z_3-connected* (*see* [11, Theorem 3.1]).
(iii) *Every 5-edge-connected graph is Z_3-connected if and only if every 5-edge-connected line graph is Z_3-connected* (*see* ([13])).

These recent researches motivate the current project. We are to investigate which families of claw-free graphs with certain connectivity property would imply Z_3-connectedness.

In [21], Ryjáček introduced the N_2-locally connected graphs. Let G be a graph. Denoted by $N(v, G) = \{z \in V(G) : vz \in E(G)\}$ be the neighborhood of v in G. For notational convenience, we shall also use $N(v, G)$ to denote the subgraph of G induced by $N(v, G)$. When the context is clear, we can write $N(v)$ for abbreviation. Let $N_2(v, G) = \{u : 0 < d(u, v) \leq 2\}$ be a vertex set, where $d(u, v)$ denotes the distance between u and v. We call $N_2(v, G)$ the second type neighborhood. A vertex v is *N_2-locally connected* if its second type neighborhood $N_2(v)$ is connected; and G is called *N_2-locally connected* if every vertex of G is N_2-locally connected. It follows from the definitions that every locally connected graph is N_2-locally connected.

A result related to Hamilton connectivity of N_2-locally connected is as follows:

Theorem 1.4 ([16, Theorem 1.4]). *Every 3-connected N_2-locally connected claw-free graph is Hamiltonian.*

The condition that a graph is N_2-locally connected does not imply high edge connectivity. Consider the graph G shown in Fig. 1, where each K_n represents a complete graph on n vertices. Then G is an N_2-locally connected claw-free graph with $\kappa'(G) = 2$.

Our main result of this paper can be stated as follows.

1st Reading

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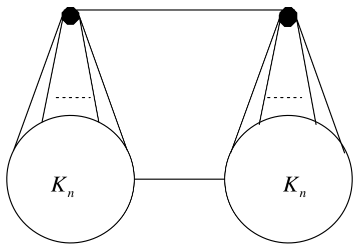

Fig. 1. An N_2-locally connected claw-free graph with $\kappa'(G) = 2$.

Theorem 1.5. *Every N_2-locally connected claw-free graph with $\delta(G) \geq 7$ is Z_3-connected.*

In Sec. 2, we present some of the preliminaries that will be needed in the proofs. The last section is devoted to the proof of the main theorem.

2. Preliminaries

Let G be a graph and $X \subseteq E(G)$. The *contraction* G/X is the graph obtained by identifying two ends of each edge in X and then deleting the resulting loops. If H is a subgraph of G, G/H is the graph $G/E(H)$.

Theorem 2.1 ([10, Proposition 3.2]). *For any Abelian group A, $\langle A \rangle$ is a family of connected graphs satisfying each of the following*:

(C1) $K_1 \in \langle A \rangle$,
(C2) *if $e \in E(G)$ and if $G \in \langle A \rangle$, then $G/e \in \langle A \rangle$, and*
(C3) *if $H \in \langle A \rangle$ and if $G/H \in \langle A \rangle$, then $G \in \langle A \rangle$.*

Let C_n denote the n-circuit, and K_n denote the complete graph on n vertices. The proof for part (iii) of the next lemma is an easy exercise using part (i) and Theorem 2.1(C3).

Lemma 2.2 ([8, 18, Proposition 3.2]). *Let G be a graph and A be an Abelian group with $|A| \geq 3$. Then $\langle A \rangle$ satisfies each of the following*:

(i) $\Lambda_g(C_n) = n + 1$ (*see* [10, Lemma 3.3]).
(ii) *Let $n \geq 5$ be an integer. Then $K_n \in \langle A \rangle$* (*see* [10, Corollary 3.5]).
(iii) *If G contains K_3 as a spanning subgraph and has at least four edges, then G is Z_3-connected. Consequently, if $n \geq 3$ and if $e \in E(K_n)$, then K_n/e is Z_3-connected.*

Next, we will give an example that shows the condition N_2-locally connected in Theorem 1.5 is necessary. We need another theorem. In [11], it is shown that for a given Abelian group A, every graph G has a unique subgraph $M_A(G)$ such

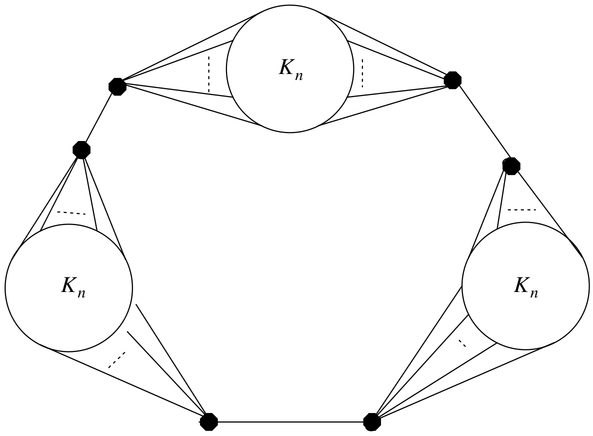

Fig. 2. Figure for Example 1.

that each component of $M_A(G)$ is a maximally A-connected subgraph of G. The contraction $G/M_A(G)$ is *the A-reduction* of G.

Theorem 2.3 ([11, Corollary 2.3]). *Let G be a graph. Then each of the following holds.*

(i) $G \in \langle A \rangle$ *if and only if* $G/M_A(G) \cong K_1$.
(ii) $G/M_A(G)$ *does not have nontrivial subgraph that is A-connected.*

Example 2.4. Let G be the graph shown in Fig. 2. Each K_n in Fig. 2 represents a complete graph with $n \geq 6$. Then G is a claw-free graph with $\delta(G) \geq 7$, and G is not N_2-locally connected. By Lemma 2.2, K_n and C_2 is Z_3-connected. After we contract K_n and C_2 successively, the resulting graph is a C_3. Since C_3 is not Z_3-connected, by Theorem 2.3(i), G is not Z_3-connected.

3. Proof of the Main Theorem

Lemma 3.1. *Let G be a nonempty claw-free graph with $\delta(G) \geq 2$, and for any $v \in V(G)$, let $H = G[N(v)]$ denote the subgraph of G induced by $N(v)$. Then $N(v)$ is a complete subgraph or can be partitioned into V_1 and V_2 such that $G[V_1]$ and $G[V_2]$ are complete subgraphs.*

Proof. Let H^c be the complement of $H = G[N(v)]$. And $|N(v)| \geq 2$ by $\delta(G) \geq 2$. If $E(H^c) = \emptyset$, then $G[N(v)]$ is a clique. Any partition (V_1, V_2) of $N(v)$ has the property that $G[V_i]$ is a complete graph, $i = 1, 2$. If $E(H^c) \neq \emptyset$, since G is a claw-free graph, every path in H^c has length at most 1. Thus H^c is the union of disjoint edges (and some isolated vertices). Let V_1 denote the vertex set that contains exactly one end of these disjoint edges, and let $V_2 = N(v) - V_1$. Then the subgraphs induced by V_1

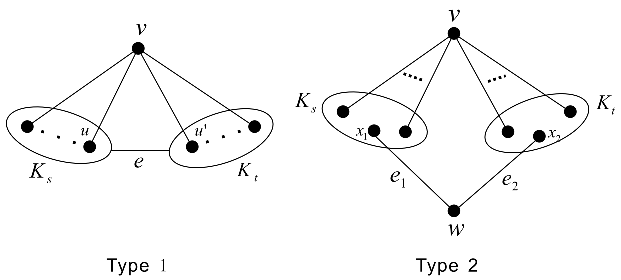

Fig. 3. Two types of vertex v.

and V_2 in H^c are independent sets in H^c, and so $G[V_1]$ and $G[V_2]$ are both complete graphs. □

Note that the partition in Lemma 3.1 may not be unique, especially when $G[N(v)]$ itself is a complete graph. Since G is a claw-free graph with $\delta(G) \geq 2$, by Lemma 3.1, for any $v \in V(G)$, the subgraph $H = G[N(v)]$ induced by $N(v)$ contains two edge-disjoint cliques as subgraphs. Since G is N_2-locally connected, we can classify v into the following two types.

Type 1: Two cliques of H are connected in the induced graph $G[N(v)]$ (see Fig. 3).

Type 2: Two cliques of H are disconnected in the induced graph $G[N(v)]$ (see Fig. 3).

Definition of $\boldsymbol{Q_v}$. For any $v \in V(G)$, if v is of Type 1, let $Q_v = G[N(v) \cup \{v\}]$ be the subgraph induced by $N(v) \cup \{v\}$ in G; if v is of Type 2, let Q_v be the subgraph induced by $N(v) \cup \{v, w\}$ in G where $w \in V(G)$ is a vertex which is adjacent to both K_s and K_t. Note that w has neighbors in each of the two different cliques of H.

Let A be an Abelian group with $|A| \geq 3$ and let G' be the A-reduction of G. By Theorem 2.3(ii), G' does not have nontrivial subgraph that is Z_3-connected. By the definition of contraction, $E(G') \subseteq E(G)$.

Lemma 3.2. *Let G be an N_2-locally connected claw-free graph with $\delta(G) \geq 7$, and let $A = Z_3$. If v is a vertex of Type 1, then $E(Q_v) \subset E(M_A(G))$, and so $E(Q_v) \cap E(G') = \emptyset$.*

Proof. Suppose that v is of Type 1. Denote the two adjacent complete graphs in $G[N(v)]$ by K_s and K_t with $s \geq t$, and let $e = uu'$ be an edge joining K_s and K_t, with $u \in V(K_s)$ and $u' \in V(K_t)$. As $\delta(G) \geq 7$, $s \geq 4$ (see Fig. 3 for an illustration). Let $H' = H[V(K_s) \cup \{u', v\}]$ and let $H_1 = H[V(K_s) \cup \{v\}]$. Since $H_1 \cong K_{s+1}$ with $s + 1 \geq 5$, it follows by Lemma 2.2(ii) that H_1 is A-connected. Since H'/H_1 is a 2-circuit, by Lemma 2.2(i) that H'/H_1 is also A-connected. Hence by Theorem 2.1(C3) that H' is A-connected. By the definition of Type 1 vertices, and since v is of Type 1, every vertex of Q_v/H' lies in a 2-circuit, and so by Lemma 2.2(i), Q_v/H' must be A-connected. Since H' is A-connected, it follows by Theorem 2.1(C3) that Q_v is A-connected. Thus $E(Q_v) \subset E(M_A(G))$, and so by the definition of G', $E(Q_v) \cap E(G') = \emptyset$. □

Proof of Theorem 1.5. Let G be an N_2-locally connected claw-free graph with $\delta(G) \geq 7$. Let $G' = G/M_{Z_3}(G)$ denote the Z_3-reduction of G. By Theorem 2.3, if we can prove $G' \cong K_1$, then we have G is Z_3-connected.

We prove by way of contradiction. Suppose that $E(G') \neq \emptyset$. Then we have the following claim.

Claim. Let $e = uv$ be any edge in $E(G')$. Then each of the following holds.

(i) The vertex v must be of Type 2 in G.
(ii) Moreover, if K_s and K_t are the two nonadjacent complete graphs with $s \geq t$ in $G[N(v)]$, then $E(G[V(K_s) \cup \{v\}]) \subseteq E(M_{Z_3}(G))$ and so $E(G[V(K_s) \cup \{v\}]) \cap E(G') = \emptyset$. (See Figure 3 for an illustration.)
(iii) $E(G[V(K_t) \cup \{v\}]) \subseteq E(G')$.

Proof of the Claim. (i) By Lemma 3.2, vertex v cannot be of Type 1 in G, as in this case $E(Q_v) \cap E(G') = \emptyset$. Hence v is of Type 2.

(ii) Since G is N_2-locally connected, there is a vertex w connecting to both K_s and K_t via two edges $e_1 = wx_1$ and $e_2 = wx_2$, where $x_1 \in V(K_s)$ and $x_2 \in V(K_t)$. As $\delta(G) \geq 7$ and $s \geq t$, $s \geq 4$. Then the subgraph $G[V(K_s) \cup \{v\}]$ is isomorphic to K_{s+1}. Since $s + 1 \geq 5$, by Lemma 2.2 (ii), $G[V(K_s) \cup \{v\}]$ is Z_3-connected, and so by the definition of $E(M_{Z_3}(G))$ and G', $E(G[V(K_s) \cup \{v\}]) \subseteq E(M_{Z_3}(G))$ and $E(G[V(K_s) \cup \{v\}]) \cap E(G') = \emptyset$.

(iii) If there exists an edge $f = xy \in G[V(K_t) \cup \{v\}]$ and $f \notin E(G')$, then $f \in M_{Z_3}(G)$. As $G[V(K_t) \cup \{v\}] \cong K_{t+1}$, by Lemma 2.2(iii), $E(G[V(K_t) \cup \{v\}]) \subseteq E(M_A(G))$, and so $e = uv \in E(M_A(G))$, contradicts to the assumption of this claim. This finishes the proof of the claim.

Since $E(G') \neq \emptyset$, let $e = uv \in E(G')$. By Claim(i), v is of Type 2 in G. Denote the two nonadjacent complete graphs in $G[N(v)]$ by K_s and K_t with $s \geq t$. Since v is of Type 2 in G, there exists a vertex $w \in V(G)$ that connecting K_s and K_t via the edges $e_1 = wx_1$ and $e_2 = wx_2$, such that $x_1 \in V(K_s)$ and $x_2 \in V(K_t)$, see Fig. 3. If one $e_i \in E(M_{Z_3}(G))$ with $i \in \{1, 2\}$, then by Claim(ii), $E(G[K_s \cup \{v\}]) \subseteq E(M_{Z_3}(G))$. It follows that the two edges vx_2 and e_{3-i} will be in a 2-cycle in G', contrary to Claim(iii).

Hence both e_1, e_2 are in $E(G')$. By Claim(i), w must be of a Type 2 vertex of G. By Claim(ii), one of e_1 or e_2 must be in $E(M_{Z_3}(G))$, contrary to the fact that both e_1, e_2 are in $E(G')$. This contradiction established the theorem. □

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