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Characterization of minimally $(2,l)$-connected graphs

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ABSTRACT

For an integer $l \geqslant 2$, the l-connectivity $\kappa_l(G)$ of a graph G is defined to be the minimum number of vertices of G whose removal produces a disconnected graph with at least l components or a graph with fewer than l vertices. Let $k \geqslant 1$, a graph G is called (k,l)-connected if $\kappa_l(G) \geqslant k$. A graph G is called minimally (k,l)-connected if $\kappa_l(G) \geqslant k$ but $\forall e \in E(G)$, $\kappa_l(G-e) \leqslant k-1$. In this paper, we present a structural characterization for minimally $(2,l)$-connected graphs and classify extremal results. These extend former results by Dirac (1967) [6] and Plummer (1968) [14] on minimally $(2,2)$-connected graphs.

1. Introduction

In this paper, we consider finite graphs, and follow the notations and terms of [3], unless otherwise defined. In particular, $\omega(G)$ is the number of components of a graph G. The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal produces a disconnected graph or the trivial graph. For an integer $l \geqslant 2$, Chartrand et al. in [4] defined the *l-connectivity* $\kappa_l(G)$ of a graph G to be the minimum number of vertices of G whose removal produces a disconnected graph with at least l components or a graph with fewer than l vertices. Thus $\kappa_l(G) = 0$ if and only if $\omega(G) \geqslant l$ or $|V(G)| \leqslant l-1$. Note that $\kappa_2(G) = \kappa(G)$.

For an integer $l \geqslant 2$, l-edge-connectivity can be similarly defined. In [1], Boesch and Chen defined the *l-edge-connectivity* $\lambda_l(G)$ of a connected graph G to be the minimum number of edges whose removal leaves a graph with at least l components if $|V(G)| \geqslant l$, and $\lambda_l(G) = |E(G)|$ if $|V(G)| < l$. Note that $\lambda_2(G) = \lambda(G)$.

The generalized connectivity and edge-connectivity have been studied by many. See [1,4,7–13,15], among others. Let $k \geqslant 1$, a graph G is called *(k,l)-connected* if $\kappa_l \geqslant k$. A graph G is called *minimally (k,l)-connected* if $\kappa_l(G) \geqslant k$ but $\forall e \in E(G)$, $\kappa_l(G-e) \leqslant k-1$. Let G be a (k,l)-connected graph, and $e \in E(G)$. An edge $e \in E(G)$ is *essential* if $G-e$ is not (k,l)-connected. A graph G is called *(k,l)-edge-connected* if $\lambda_l(G) \geqslant k$. A graph G is *minimally (k,l)-edge-connected* if $\lambda_l(G) \geqslant k$ but for any edge $e \in E(G)$, $\lambda_l(G-e) \leqslant k-1$. Therefore, a $(2,2)$-connected graph is just a 2-connected graph, and a $(2,2)$-edge-connected graph is a 2-edge-connected graph.

Let $\mathscr{F}(n,k,l)$ be the set of all connected and minimally (k,l)-connected graphs with n vertices. We define $F(n,k,l) = \max\{|E(G)|\colon\ G \in \mathscr{F}(n,k,l)\}$ and $f(n,k,l) = \min\{|E(G)|\colon\ G \in \mathscr{F}(n,k,l)\}$. Let $\mathscr{S}(n,k,l) = \{i \in \mathbb{N}\colon f(n,k,l) \leqslant i \leqslant F(n,k,l)$ and $\exists G \in \mathscr{F}(n,k,l)$ such that $|E(G)| = i\}$, which is referred as the *(n,k,l)-spectrum* of $\mathscr{F}(n,k,l)$. We further define $Ex(n,k,l) = \{G\colon\ G \in \mathscr{F}(n,k,l),\ |E(G)| = F(n,k,l)\}$ and $Sat(n,k,l) = \{G\colon\ G \in \mathscr{F}(n,k,l),\ |E(G)| = f(n,k,l)\}$.

Chaty and Chein presented a structural characterization of minimally $(2,2)$-edge-connected graphs [5]. Hennayake

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et al. [9] then generalized it to minimally (k,k)-edge-connected graphs by presenting a structural characterization of all minimally (k,k)-edge-connected graphs. A structural characterization of minimally $(2,2)$-connected graphs was obtained independently by Dirac [6] and by Plummer [14]. A purpose of this paper is to give a characterization of minimally $(2,l)$-connected graphs when $l > 2$ (Theorem 3.2 and Theorem 3.5) by presenting the structures of such graphs.

The value of $F(n,2,2)$ was discovered independently by Dirac [6] and by Plummer [14] (Theorem 2.1 in this paper). Another purpose of this paper is to determine $F(n,2,l)$ and $f(n,2,l)$ when $l > 2$. The families $Ex(n,2,l)$, $Sat(n,2,l)$ and $\mathscr{I}(n,2,l)$ will also be determined in the paper. These extend former results by Dirac [6] and Plummer [14] on minimally $(2,2)$-connected graphs.

In Section 2, we will present some preliminaries as preparations for the proofs. Sections 3 and 4 are devoted to the investigations of the structural characterization of minimally $(2,l)$-connected graphs, and of $F(n,2,l)$, $f(n,2,l)$, $Ex(n,2,l)$, $Sat(n,2,l)$ and $\mathscr{I}(n,2,l)$, respectively.

2. Preliminaries

We start with a theorem by Dirac and Plummer. These results were obtained by Dirac and by Plummer independently. A *chord* of a cycle C in a graph G is an edge in $E(G) \setminus E(C)$ both of whose ends lie on C.

Theorem 2.1. *(Dirac [6] and Plummer [14], see also [2].)*

(i) *A 2-connected graph is minimally 2-connected if and only if no cycle has a chord.*
(ii) *A minimally 2-connected graph of order $n \geqslant 4$ has the size at most $2n-4$. Furthermore, $F(n,2,2) = 2n-4$ and $Ex(n,2,2) = \{K_{2,n-2}\}$ for $n \geqslant 4$.*

A *divalent path* P in a graph G is a path all of whose internal vertices have degree 2 in G. A *lane* of a graph G is a maximal divalent path in G. For convenience, a cycle is considered as a lane of itself. Let L be a lane in graph G, we define L_0 to be the set of all internal vertices of L if L is not an edge of G. If L is an edge e of G, then $L_0 = \{e\}$.

By definition, every edge of a graph G is in a divalent path of G. Hence, we have the following observation:

Observation 1. Every edge of a graph G lies in a lane in G.

A graph is *acyclic* if it does not contain a cycle. Otherwise, the graph is called *cyclic*. A *cyclic block* of a graph is a block which is not isomorphic to K_2. Let G be a connected graph with blocks $B_1, B_2, \ldots, B_s$ and cut vertices $c_1, c_2, \ldots, c_t$, where $s \geqslant 1$ and $t \geqslant 0$. The *block-cutvertex graph* of G, denoted by $bc(G)$, is the graph with vertex set $\{B_1, B_2, \ldots, B_s\} \cup \{c_1, c_2, \ldots, c_t\}$ and edge set $\{B_i c_j\colon c_j \in V(B_i)\}$ for $1 \leqslant i \leqslant s$ and $0 \leqslant j \leqslant t$. By definition, the block-cutvertex graph of graph G is a tree, and so it is also called the *block tree* of G.

The *distance* $d_G(x,y)$ of two vertices x and y in a graph G is the length of a shortest (x,y)-path in G, and if no such path exists, then the distance is set to be ∞. Let G be a graph and $U \subseteq V(G)$. The *diameter* of U in G, denoted by $diam_G(U)$, is the greatest distance $d_G(x,y)$ for $\forall x, y \in U$. If $U = V(G)$, then the diameter of G is simply denoted as $diam(G)$.

The *local connectivity* $\kappa_G(x,y)$ of two non-adjacent vertices x and y in a graph G is the minimum number of vertices separating x from y. If x and y are adjacent vertices, their local connectivity is defined as $\kappa_H(x,y)+1$, where $H = G - xy$.

3. Minimally $(2,l)$-connected graphs

In this section, we shall present a characterization of minimally $(2,l)$-connected graphs.

Lemma 3.1. *Let G be a (k,l)-connected graph. Then*

(i) $|V(G)| \geqslant k+l-1$.
(ii) *Suppose that $l' > l \geqslant 2$ and $|V(G)| \geqslant k+l'-1$. If G is (k,l)-connected, then G is (k,l')-connected, but cannot be minimally (k,l')-connected.*

Proof. (i) Suppose that $|V(G)| < k+l-1$. Let $X \subseteq V(G)$ with $|X| = k-1$. Then $|V(G-X)| < l$, and so $\kappa_l(G) \leqslant k-1$, contrary to the fact that G is (k,l)-connected.

(ii) Suppose that G is not (k,l')-connected. Then $\kappa_{l'}(G) \leqslant k-1$, and so there exists $X \subset V(G)$ with $|X| \leqslant k-1$ such that either $\omega(G-X) \geqslant l' > l$, whence $\kappa_l(G) \leqslant \kappa_{l'}(G) \leqslant k-1$, contrary to $\kappa_l(G) \geqslant k$; or $|V(G-X)| \leqslant l'-1$, whence $|V(G)| < k+l'-1$, contrary to the assumption. Hence $\kappa_{l'}(G) \geqslant k$.

To prove that G is not minimally (k,l')-connected, we argue by contradiction and assume that G is minimally (k,l')-connected. Then $\forall e \in E(G)$, $\kappa_{l'}(G-e) \leqslant k-1$. There exists an $X \subset V(G-e) = V(G)$ with $|X| \leqslant k-1$. If $\omega(G-e-X) \geqslant l'$, then $\omega(G-X) \geqslant l'-1 \geqslant l$, whence $\kappa_l(G) \leqslant k-1$, contrary to $\kappa_l(G) \geqslant k$. If $|V(G-e-X)| \leqslant l'-1$, then since $|V(G-X)| = |V(G-e-X)|$, we have $|V(G)| < k+l'-1$, contrary to $|V(G)| \geqslant k+l'-1$. Thus, G is (k,l')-connected, but not minimally (k,l')-connected. $\square$

Suppose that $l \geqslant 3$ and H is a tree such that there are at least two non-adjacent vertices $u, v \in V(H)$ satisfying $d(u) = d(v) = l-1 = \Delta(G)$. Let $\mathscr{T}(l-1)$ be the set of all such trees, and let $\mathscr{T}_n(l-1) = \{H \in \mathscr{T}(l-1)\colon |V(H)| = n\}$.

Theorem 3.2. *Let G be a tree and $l \geqslant 3$. Then G is minimally $(2,l)$-connected if and only if $G \in \mathscr{T}(l-1)$.*

Proof. First we assume that $G \in \mathscr{T}(l-1)$. Since $\Delta(G) = l-1$, $\kappa_l(G) \geqslant 2$. To prove that G is minimally $(2,l)$-connected, we need to show that $\forall e \in E(G), \kappa_l(G-e) \leqslant 1$. By assumption, G has at least one vertex v which is not incident with edge e, such that $d(v) = l-1$. Since G is a tree, both $\omega(G-v) = l-1$ and each component of $G-v$ is a tree. As e must be in a component of $G-v$, $\omega(G-e-v) = l$, whence $\kappa_l(G-e) = 1$.

We now assume that G is minimally $(2,l)$-connected to prove the necessity. Since G is a tree and $\kappa_l(G) \geqslant 2$, we have $\Delta(G) \leqslant l-1$.

Claim 1. *Let $e \in E(G)$. Then $\exists u \in V(G)$ which is not incident with e such that $d(u) = l - 1$.*

Proof of Claim 1. Since G is minimally $(2,l)$-connected, $\kappa_l(G-e) = 1$, and so $\exists u \in V(G)$ such that $\omega(G-e-u) \geqslant l$. Thus $\omega(G-u) \geqslant l-1$ and $d(u) \geqslant l-1$. Since $\Delta(G) \leqslant l-1$, $\Delta(G) = d(u) = l-1$. Note that u is not incident with e, as otherwise, $\omega(G-u) = \omega(G-e-u) \geqslant l$, contrary to the fact that G is $(2,l)$-connected. Thus Claim 1 must hold. □

By Claim 1, $\Delta(G) = l-1$ and so $\exists u \in V(G), d(u) = l-1$. Let $e' \in E(G)$ be an edge incident with u. By Claim 1, there exists a vertex $u' \in V(G)$ such that $d(u') = l-1$ and e' is not incident with u'. Thus $u' \neq u$. If u' is not adjacent to u, then the theorem holds. Hence we assume that $e'' = uu' \in E(G)$. By Claim 1, there exists a vertex $u'' \in V(G)$ such that $d(u'') = l-1$ and $u'' \notin \{u, u'\}$. Thus G has 3 vertices with degree $l-1$. Since G is a tree, at least 2 of these vertices of degree $l-1$ are non-adjacent. Hence $G \in \mathscr{T}(l-1)$. □

Corollary 3.3. *Let G be a tree. Then G is minimally $(2,3)$-connected if and only if G is a path P_n (a path with n vertices), where $n \geqslant 5$.*

Let G be a graph, and $k \geqslant 1, l \geqslant 2$ be integers. A (k,l)-*cut* of G is a set $F \subseteq V(G)$ such that $|F| = k$ and $\omega(G-F) \geqslant l$. As any $(1,l)$-cut consists of a single vertex, a $(1,l)$-cut is also called a $(1,l)$-*cut-vertex*. We shall use the notation $J^l(G)$ to denote the set of all $(1,l)$-cut-vertices of G.

Lemma 3.4. *Let $l \geqslant 3$. Suppose that G is a connected, minimally $(2,l)$-connected graph. Let B be a cyclic block of G. Then $\forall e \in E(B)$, $\exists u \in V(B)$ such that $u \in J^{l-1}(G)$ and such that u is not incident with e.*

Proof. Since G is minimally $(2,l)$-connected and $e \in E(B) \subseteq E(G)$, $\kappa_l(G-e) = 1$. Thus $\exists u \in V(G-e) = V(G)$ such that $\omega(G-e-u) \geqslant l$. Hence $\omega(G-u) \geqslant l-1$. Since G is $(2,l)$-connected, it must be the case that $\omega(G-u) = l-1$, and so u is a $(1,l-1)$-cut-vertex of graph G. We claim that $u \in V(B)$. If not, then $u \notin V(B) = V(B-e)$, and so $B-e$ is contained in a component of $G-e-u$. Hence $\omega(G-u) = \omega((G-e-u)+e) = \omega(G-e-u) \geqslant l$, contrary to the fact that G is $(2,l)$-connected. We also claim that u is not incident with edge e. If not, then $\omega(G-u) = \omega(G-e-u) \geqslant l$, contrary to the fact that G is $(2,l)$-connected. Thus the lemma must hold. □

Theorem 3.5. *Let $l \geqslant 3$. A connected graph G is minimally $(2,l)$-connected if and only if each of the following holds.*

(i) *Each cut vertex of G has degree no more than $l-1$ in the block-cutvertex graph of G.*
(ii) *If G is a tree, then $G \in \mathscr{T}(l-1)$.*
(iii) *For each cyclic block B not isomorphic to K_3 and for each lane L of B, if $J(B-L_0)$ denotes the set of all cut vertices of $B-L_0$ and $S = V(L) \cap J^{l-1}(G)$, then either $|S| \geqslant 2$ and $diam_L(S) \geqslant 2$, or $J(B-L_0) \cap J^{l-1}(G) \neq \emptyset$.*
(iv) *If a block B of G is isomorphic to K_3, then $\forall v \in V(B)$, $v \in J^{l-1}(G)$.*

Proof. Assume that G is connected and minimally $(2,l)$-connected.

(i) Since G is $(2,l)$-connected, G has no $(1,l)$-cut-vertices. Thus each cut vertex of G has degree at most $l-1$ in the block-cutvertex graph of G.

(ii) It follows from Theorem 3.2.

(iii) Since G is connected and minimally $(2,l)$-connected, $\forall e \in E(L) \subseteq E(G), \kappa_l(G-e) = 1$, and so $\exists u \in V(G-e) = V(G)$ such that $\omega(G-e-u) \geqslant l$. Thus $\omega(G-u) \geqslant l-1$ and u is a $(1,l-1)$-cut-vertex of G. Suppose first that $u \notin V(L)$. If $B-L_0$ is contained in a component of $G-u-L$, then $\omega(G-u) = \omega(G-u-L) = \omega(G-u-e) \geqslant l$, contrary to the fact that G is $(2,l)$-connected. Thus u must be a cut vertex of $B-L_0$, and so $J(B-L_0) \cap J^{l-1}(G) \neq \emptyset$, and (iii) holds.

Now assume that $u \in V(L)$. Let $e' \in E(L)$ be an edge incident with u. By Lemma 3.4, $\exists v \in V(B)$ which is not incident with e' such that $v \in J^{l-1}(G)$. Thus $v \neq u$. If $v \notin V(L)$, then $J(B-L_0) \cap J^{l-1}(G) \neq \emptyset$, and (iii) holds. Thus we may assume that $v \in V(L)$. If u and v are non-adjacent, then $|S| \geqslant 2$ and $diam_L(S) \geqslant 2$, and (iii) holds. If u and v are adjacent in L, then let $e'' = uv$. By Lemma 3.4, $\exists x \in V(B)$ such that $x \in J^{l-1}(G)$ and such that x is not incident with e''. Thus $x \notin \{u, v\}$. If $x \notin V(L)$, then $J(B-L_0) \cap J^{l-1}(G) \neq \emptyset$, and (iii) holds. Hence we assume that $x \in V(L)$. Then $u, v, x \in V(L)$. Now we claim that L is not isomorphic to K_3. Otherwise, if L is isomorphic to K_3, and by the definition of a lane, there is at most one vertex in L whose degree is greater than 2 in B. If $V(B-L) \neq \emptyset$ then $\kappa(B) = 1$, contrary to the fact that B is a cyclic block. if $V(B-L) = \emptyset$, which means L is B itself, contrary to the fact that B is not isomorphic to K_3. Hence L is not isomorphic to K_3 and so at least one of vertices u, v is non-adjacent to x. Hence (iii) must hold.

(iv) By Lemma 3.4, $\forall e \in E(K_3)$, the non-adjacent vertex is in $J^{l-1}(G)$. Thus, $\forall v \in V(K_3)$, $v \in J^{l-1}(G)$.

We now prove the sufficiency. By Theorem 3.2, we may assume that G is not a tree. By (i), G has no $(1,l)$-cut-vertices. Thus $\kappa_l(G) \geqslant 2$ and so G is $(2,l)$-connected. We need to prove

$$\forall e \in E(G), \quad \kappa_l(G-e) \leqslant 1. \tag{1}$$

Pick an edge $e \in E(G)$. There are 3 cases:

Case 1. The edge e lies in a cyclic block B which is isomorphic to K_3.

Let v be the vertex in B such that v is not incident with e. By (iv), v is a $(1,l-1)$-cut-vertex of G. Thus $\omega(G-v) \geqslant l-1$. Since B is isomorphic to K_3, e must be a cut edge of a component H of $G-v$. Hence $\omega(G-e-v) \geqslant l$, and so $\kappa_l(G-e) = 1$. Thus (1) holds.

Case 2. Edge e lies in a cyclic block B which is not isomorphic to K_3.

Let L be the lane in B such that $e \in E(L)$. Then either $J(B-L_0) \cap J^{l-1}(G) \neq \emptyset$, or $|S| \geqslant 2$ and $diam_L(S) \geqslant 2$.

Assume first that $|S| \geqslant 2$ and $diam_L(S) \geqslant 2$. Then L has at least 2 non-adjacent vertices which are $(1, l-1)$-cut-vertices of G. Hence there is a vertex $v \in V(L)$ such that $v \in J^{l-1}(G)$ and such that v is not incident with e. Thus $\omega(G-v) \geqslant l-1$. Since $e \in E(L)$ and L is a lane in B, by the definition of a lane, e must be a cut edge of a component of $G-v$. Thus $\omega(G-e-u) \geqslant l$, and so $\kappa_l(G-e) = 1$. Hence (1) holds. Therefore, by (iii), we assume that $J(B-L_0) \cap J^{l-1}(G) \neq \emptyset$. Let $v \in J(B-L_0) \cap J^{l-1}(G)$. Since $v \in J^{l-1}(G)$, $\omega(G-v) \geqslant l-1$ and e is in a component H of $G-v$. Let x and y be the end vertices of lane L. Since v is a cut vertex of $B-L_0$, $\kappa_G(x, y) = 2$, whence e is a cut edge of the component H in $G-v$. Then $\omega(G-e-v) \geqslant l$, whence $\kappa_l(G-e) = 1$, and so (1) holds.

Case 3. The edge e does not lie in any cyclic block of G.

Since G is not a tree, G must have a cyclic block B. By (iii) and (iv), whether B is isomorphic to K_3 or not, G has a $(1, l-1)$-cut-vertex v which is not incident with e. Hence $\omega(G-v) \geqslant l-1$ and e lies in a component H of $G-v$. Since e does not lie in any cyclic block of G, e must be a cut edge of H. Thus $\omega(G-e-v) \geqslant l$, whence $\kappa_l(G-e) = 1$, and so (1) holds. □

Corollary 3.6. *Let G be a connected, minimally $(2, l)$-connected graph. Then every cyclic block of G is minimally 2-connected.*

Proof. Let B be a cyclic block of G. By Theorem 2.1, to prove B is minimally 2-connected, it suffices to show that each cycle in B has no chords. Assume that there is a cycle C in B with a chord $e = xy$. By the definition of a lane, e is a lane of B. By Theorem 3.5 (iii), it must be the case that $J(B-e) \cap J^{l-1}(G) \neq \emptyset$, and let $v \in J(B-e) \cap J^{l-1}(G)$. Since B is 2-connected and v is a cut vertex of $B-e$, x and y must be in different components of $B-e-v$, whence $\kappa_{B-e}(x, y) = 1$. But since e is a chord of cycle C in B, $\kappa_{B-e}(x, y) \geqslant 2$. We get a contradiction. Hence, every cyclic block of G is minimally 2-connected. □

4. $F(n, 2, l)$, $f(n, 2, l)$, $Ex(n, 2, l)$, $Sat(n, 2, l)$ and $\mathscr{I}(n, 2, l)$

In this section, we shall determine the value of $F(n, 2, l)$ and $f(n, 2, l)$, and discover the family of $Ex(n, 2, l)$, $Sat(n, 2, l)$ and $\mathscr{I}(n, 2, l)$.

Lemma 4.1. *Let G be a connected, minimally $(2, l)$-connected graph. Let $l \geqslant 3$ and $|V(G)| = n$.*

(i) *If G is acyclic, then $2l-1 \leqslant n$;*
(ii) *If G is cyclic, then $2l \leqslant n$.*

Proof. (i) By Theorem 3.2, there are two non-adjacent vertices u and v such that $d(u) = d(v) = l-1$. Hence $2(l-1) - 1 + 2 \leqslant n$, that is $2l-1 \leqslant n$.

(ii) By Corollary 3.6, there must be a cyclic block which is minimally 2-connected. There are two cases here. If the cyclic block is a K_3, then by Theorem 3.5, all the three vertices of K_3 are $(1, l-1)$-cut-vertices, and hence there are at least $3(l-2)+3$ vertices. Thus $n \geqslant 3(l-2)+3 = 3l-3 \geqslant 2l$, since $l \geqslant 3$. If the cyclic block is not a K_3, then the block has at least 4 vertices, and by Theorem 3.5, at least 2 of them are $(1, l-1)$-cut-vertices. Hence $n \geqslant 2(l-2)+4$, that is $2l \leqslant n$. □

Lemma 4.2. *Let G be a connected, minimally $(2, l)$-connected graph with $|V(G)| = n$ and $|E(G)| = m$. Then*

(i) *$n-1 \leqslant m \leqslant 2n-2l$.*
(ii) *$m = 2n-2l$ holds if and only if one of the following holds:*
   (a) *G is a tree and $n = 2l-1$; or*
   (b) *G has only one cyclic block, the cyclic block is isomorphic to $K_{2,n-2l+2}$, and G has exactly two non-adjacent $(1, l-1)$-cut-vertices; or*
   (c) *$l = 3$, $n = 6$ and the only cyclic block of G is isomorphic to K_3.*

Proof. If G is a tree, then $m = n-1$. By Lemma 4.1, $2l-1 \leqslant n$. Hence $m = n-1 \leqslant n-1+n-(2l-1) \leqslant 2n-2l$, where equality holds if and only if $n = 2l-1$. Thus the lemma must hold.

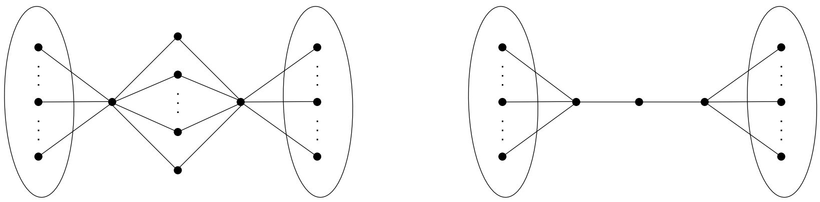
Now we assume that G is cyclic. Since G is connected, $m \geqslant n$. We still need to prove $m \leqslant 2n-2l$. Suppose that G has t cyclic blocks which are not isomorphic to K_3, denoted by $H_1, H_2, \ldots, H_t$, and s cyclic blocks which are isomorphic to K_3. Let n' be the total number of vertices of all cyclic blocks, ans so $n' = 3s + (n_1 + n_2 + \cdots + n_t)$. Each H_i has n_i vertices and m_i edges, for $i = 1, 2, \ldots, t$. By Corollary 3.6, each cyclic block is a minimally 2-connected graph. By Theorem 2.1, $m_i \leqslant 2n_i - 4$ for $i = 1, 2, \ldots, t$. Then $m = 3s + m_1 + m_2 + \cdots + m_t + (t+s-1) + n - (3s + n_1 + n_2 + \cdots + n_t) \leqslant 3s + (n_1 + n_2 + \cdots + n_t) + n - 3t - 2s - 1 = n' + n - 3t - 2s - 1$. Let $M = n' + n - 3t - 2s - 1$. We have the following claim.

Claim. *When M reaches the maximum value, there is exactly one cyclic block in the graph.*

Proof of the claim. Without loss of generality, we may assume that $n' \geqslant 4$. If the number of cyclic blocks is 1, then by Corollary 3.6 and Theorem 2.1, the maximum value of M is $2n' - 4 + (n - n') = n' + n - 4$. If the number of cyclic blocks is at least 2, then $t + s \geqslant 2$. The maximum value of M is $n' + n - 3t - 2s - 1 = n' + n - 2(t+s) - t - 1 < n' + n - 4$. This completes the proof of the claim. □

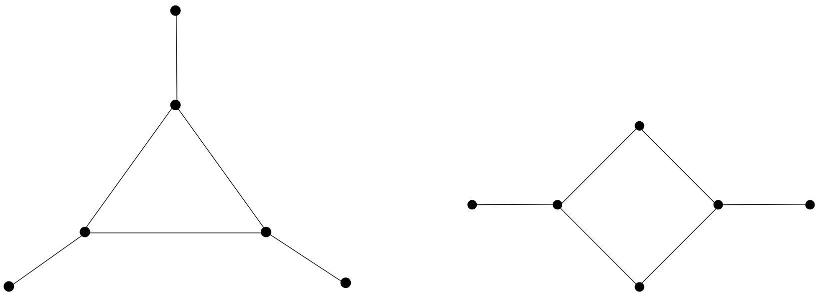
Case 1. $t \neq 0$. By the claim, when M reaches the maximum value, $t = 1$, $s = 0$ and $M = n' + n - 4 = n_1 + n - 4$. By Theorem 3.5, there are at least two $(1, l-1)$-cut-vertices in a minimally $(2, l)$-connected graph. Hence $n_1 \leqslant n - 2(l-2)$. Thus $m \leqslant 2n - 2l$, and (i) must hold. The equality holds if and only if $t = 1$, $s = 0$, $n_1 = n - 2(l-2)$ and $m_1 = 2n_1 - 4$. By Theorem 2.1, $m_1 = 2n_1 - 4$ if and only the cyclic block is isomorphic to $K_{2,n-2l+2}$. And $n_1 = n - 2(l-2)$ holds if and only if there are exactly two vertices which are not in the cyclic block, i.e., G has exactly two non-adjacent $(1, l-1)$-cut-vertices, by Theorem 3.5. Thus (ii) must hold.

Case 2. $t = 0$. By the claim, when M reaches the maximum value, $t = 0$, $s = 1$ and $M = n' + n - 3 = n$. By Lemma 4.1,



(a) $G_{2,s,t}$ (b) $T_{3,t}$

Fig. 1. Some classes of graphs.



(a) $G_{3,3}$ (b) $G_{2,2,1}$

Fig. 2. Extremal graphs for $F(6,2,3)$.

$M = n \leqslant 2n-2l$, and the equality holds if and only if $n = 2l$. Since the only cyclic block is a K_3, by Theorem 3.5, each vertex of the cyclic block is a $(1, l-1)$-cut-vertex, and thus the number of vertices in the graph is $n = 3 + 3(l-2) = 3l-3$. Hence $M = 2n-2l$ holds if and only if $n = 2l$ and $n = 3l-3$, i.e., $l = 3$ and $n = 6$. □

Let $K_{2,s}$ be a complete bipartite graph with bipartition (A, B) such that $|A| = 2$ and $|B| = s$. Let $G_{2,s,t}$ denote the graph obtained from $K_{2,s}$ by joining each vertex in set A to t new vertices, respectively, as shown in Fig. 1(a). Let u and v be two non-adjacent vertices of P_3. Let $T_{3,t}$ denote the graph obtained from P_3 by joining each of u, v to t new vertices, respectively, as shown in Fig. 1(b). Graph $G_{3,3}$ is shown in Fig. 2(a).

Theorem 4.3.

(i) $F(n,2,l) = 2n-2l$.
(ii) $Ex(5,2,3) = \{P_5\}$; $Ex(6,2,3) = \{G_{3,3}, G_{2,2,1}\}$; $Ex(n,2,3) = \{G_{2,n-4,1}\}$ *for* $n \geqslant 7$.
(iii) *When* $l \geqslant 4$ *and* $n = 2l-1$, $Ex(n,2,l) = \{T_{3,l-2}\}$.
(iv) *When* $l \geqslant 4$ *and* $n \geqslant 2l$, $Ex(n,2,l) = \{G_{2,n-2l+2,l-2}\}$.

Proof. When $l = 2$, by Theorem 2.1, $F(n,2,2) = 2n-4$ and $Ex(n,2,2) = \{K_{2,n-2}\}$. So we assume that $l \geqslant 3$. By Lemma 4.2, $F(n,2,l) \leqslant 2n-2l$. In order to prove $F(n,2,l) = 2n-2l$, it suffices to show that there exists a connected, minimally $(2,l)$-connected graph with n vertices and $2n-2l$ edges. When $l = 3$, by Lemma 4.1, $n \geqslant 5$ and G is tree if $n = 5$. By Corollary 3.3, $Ex(5,2,3) = \{P_5\}$. If $n = 6$, G is cyclic and by Lemma 4.2, $Ex(6,2,3) = \{G_{3,3}, G_{2,2,1}\}$. If $n \geqslant 7$, $\forall G \in Ex(n,2,3)$, by Lemma 4.2, the only cyclic block of G is $K_{2,n-2l+2}$, and G has exactly two non-adjacent $(1, l-1)$-cut-vertices. Hence, $Ex(n,2,3) = \{G_{2,n-4,1}\}$.

When $l \geqslant 4$, by Lemma 4.1, $n \geqslant 2l-1$. If $n = 2l-1$, then G is a tree, and by Theorem 3.2, $\forall G \in Ex(n,2,l)$, $G \in \mathscr{T}(l-1)$. Then there are two non-adjacent vertices with degree $l-1$. Since $n = 2l-1$, G must be $T_{3,l-2}$. If $n \geqslant 2l$, then by Lemma 4.2, $Ex(n,2,l) = \{G_{2,n-2l+2,l-2}\}$. Thus, the theorem holds. □

Theorem 4.4.

(i) $f(n,2,l) = n-1$.
(ii) $Sat(n,2,l) = \mathscr{T}_n(l-1)$.

Proof. By Lemma 4.2, $f(n,2,l) \geqslant n-1$. In order to prove $f(n,2,l) = n-1$, it suffices to show that there's a connected, minimally $(2,l)$-connected graph G such that $|V(G)| = n$ and $|E(G)| = n-1$. Graph g must be a tree, since $|E(G)| = |V(G)| - 1$. By Theorem 3.2, $G \in \mathscr{T}(l-1)$. Thus (i) holds. Since G has n vertices, $Sat(n,2,l) = \mathscr{T}_n(l-1)$. (ii) must hold. □

Theorem 4.5. $\mathscr{I}(n,2,l) = \{i \in \mathbb{N}\colon n-1 \leqslant i \leqslant 2n-2l\}$.

Proof. It suffices to show that for each $m \in \mathbb{N} \cap [n-1, 2n-2l]$, there is a graph $G \in \mathscr{F}(n,2,l)$ such that $|E(G)| = m$. For each m, we will construct a minimally $(2,l)$-connected graph with n vertices and m edges. When $m = n-1$, $G = P_n$. When $n \leqslant m \leqslant 2n-2l$, we construct a minimally $(2,l)$-connected graph G as follows: Let C be a cycle with $2n-m-2(l-2)$ vertices, and u_1, u_2 are two non-adjacent vertices in C. Let V_1 and V_2 be two sets of $(l-2)$ vertices, and $V_1 \cap V_2 = \emptyset$. Then G is the graph

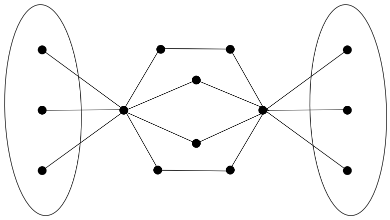


Fig. 3. An example when $l = 5$, $n = 14$ and $m = 16$ in the proof of Theorem 4.5.

obtained from C by joining u_i to each vertex in V_i respectively for $i = 1, 2$, and joining u_1 and u_2 by $m - n$ disjoint paths. These disjoint paths are $m - n$ copies of P_3. Obviously, $|E(G)| = m$ and $|V(G)| = n$. By Theorem 3.5, G is a minimally $(2, l)$-connected graph. An example is shown in Fig. 3 when $l = 5$, $n = 14$ and $m = 16$. □

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