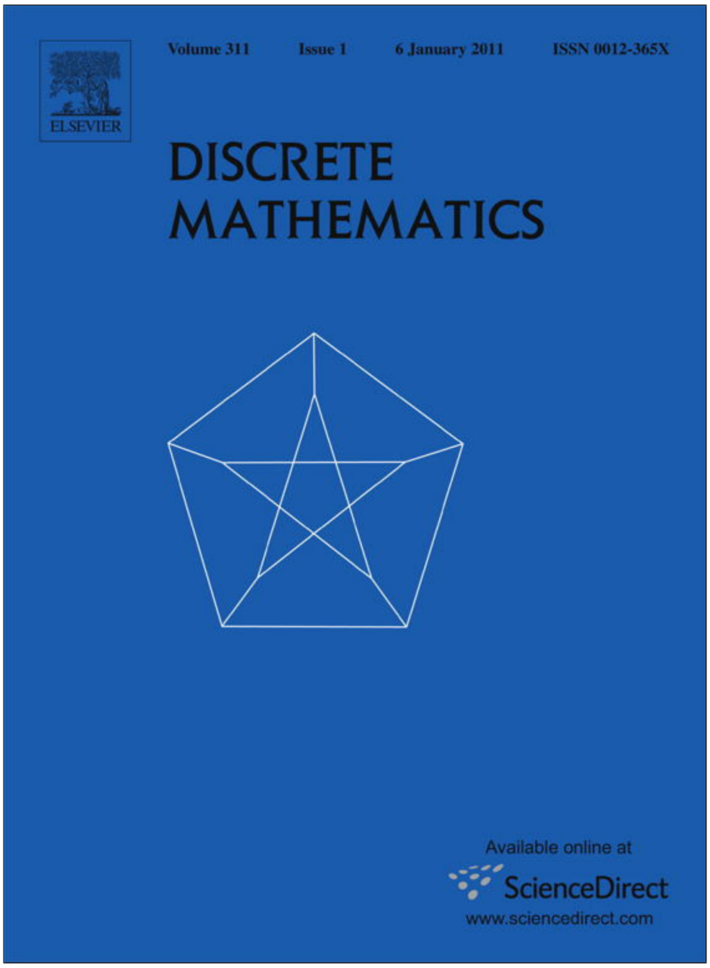
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Group connectivity in line graphs

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ABSTRACT

Tutte introduced the theory of nowhere zero flows and showed that a plane graph G has a face k-coloring if and only if G has a nowhere zero A-flow, for any Abelian group A with $|A| \geq k$. In 1992, Jaeger et al. [9] extended nowhere zero flows to group connectivity of graphs: given an orientation D of a graph G, if for any $b : V(G) \mapsto A$ with $\sum_{v\in V(G)} b(v) = 0$, there always exists a map $f : E(G) \mapsto A - \{0\}$, such that at each $v \in V(G)$,

$$\sum_{e=vw \text{ is directed from } v \text{ to } w} f(e) - \sum_{e=uv \text{ is directed from } u \text{ to } v} f(e) = b(v)$$

in A, then G is A-connected. Let Z_3 denote the cyclic group of order 3. In [9], Jaeger et al. (1992) conjectured that every 5-edge-connected graph is Z_3-connected. In this paper, we proved the following.

(i) Every 5-edge-connected graph is Z_3-connected if and only if every 5-edge-connected line graph is Z_3-connected.
(ii) Every 6-edge-connected triangular line graph is Z_3-connected.
(iii) Every 7-edge-connected triangular claw-free graph is Z_3-connected.

In particular, every 6-edge-connected triangular line graph and every 7-edge-connected triangular claw-free graph have a nowhere zero 3-flow.

1. Introduction

Graphs considered in this paper are finite and loopless. Undefined terms and notations can be found in [2]. In particular, the minimum degree, the connectivity and the edge-connectivity of a graph G are denoted by $\delta(G)$, $\kappa(G)$ and $\kappa'(G)$, respectively, and a subgraph H of G is a **clique** if H is isomorphic to a complete graph. If $X \subseteq V(G)$ (or $X \subseteq E(G)$), then $G[X]$ denotes the subgraph of G induced by X. However, a nontrivial 2-regular connected graph will be called a **circuit** instead of a cycle. A circuit of n edges is also referred as an n-**circuit**. For a vertex $v \in V(G)$, $N_G(v) = \{v' \in V(G)|vv' \in E(G)\}$ is the **neighborhood** of v in G, and $N_G[v] = N_G(v) \cup \{v\}$ is the **closed neighborhood** of v in G. Define

$$E_G(v) = \{e \in E(G)|e \text{ is incident with } v \text{ in } G\}.$$

When G is understood from the context, the subscript G in $E_G(v)$ might be omitted. For graphs G and H, by $H \subseteq G$ we mean that H is a subgraph of G.

Let G be a graph with an orientation $D = D(G)$. If an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then define **tail** $(e) = u$ and **head** $(e) = v$. For a vertex $v \in V(G)$, let

$$E_D^+(v) = \{e \in E(G) \mid v = \text{tail}(e)\}, \quad \text{and} \quad E_D^-(v) = \{e \in E(G) \mid v = \text{head}(e)\}.$$

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Throughout this paper, **Z** denotes the set of all integers, A denotes an (additive) Abelian group with identity 0, and $A^* = A - \{0\}$. For $m \in \mathbf{Z}$ with $m \geq 2$, Z_m denotes the cyclic group of order m, as well as the set of all integers modulo m. For a graph G, define $F(G, A) = \{f | f : E(G) \mapsto A\}$ and $F^*(G, A) = \{f | f : E(G) \mapsto A^*\}$. For an $f \in F(G, A)$, let $\partial f : V(G) \mapsto A$ be given by, for all $v \in V(G)$,

$$\partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e),$$

where "$\sum$" refers to the addition in A.

A map $b : V(G) \mapsto A$ is an A-**valued zero sum map** on G if $\sum_{v \in V(G)} b(v) = 0$. The set of all A-valued zero sum maps on G is denoted by $Z(G, A)$. An $f \in F(G, A)$ is an A-**flow** of G if $\partial f = 0$. An A-flow is a nowhere zero A-flow (A-**NZF** for short) if $f \in F^*(G, A)$. If f is a **Z**-NZF satisfying for all $e \in E(G)$, $|f(e)| < k$, then f is a **nowhere zero** k-**flow** (k-**NZF** for short). Tutte [20] indicated that, for a finite Abelian group A, a graph G has an A-NZF if and only if G has an $|A|$-NZF.

Given a $b \in Z(G, A)$, an $f \in F^*(G, A)$ is a **nowhere zero** (A, b)-**flow** ((A, b)-**NZF** for short) if $\partial f = b$. A graph G is A-**connected** if for all $b \in Z(G, A)$, G always has an (A, b)-NZF. Let $\langle A \rangle$ denote the family of graphs that are A-connected. The **group connectivity number** of a graph G is defined as

$$\Lambda_g(G) = \min\{k | G \in \langle A \rangle \text{ for every Abelian group } A \text{ with } |A| \geq k\}.$$

In [8,9], it is shown that whether G has an A-NZF or whether $G \in \langle A \rangle$ is independent of the choice of the orientation of G. These are undirected graph properties.

In 1950s, Tutte initiated the theory of nowhere zero flows as a mechanism to attack the then 4-color-conjecture. The following fascinating conjectures of Tutte and Jaeger on nowhere zero flows remain open as of today.

Conjecture 1.1 (*Tutte [20,21], See Also [8]*).

(i) *(Tutte) Every graph G with $\kappa'(G) \geq 2$ has a 5-NZF.*
(ii) *(Tutte) Every graph G with $\kappa'(G) \geq 2$ and without a subgraph contractible to the Petersen graph has a 4-NZF.*
(iii) *(Tutte) Every graph G with $\kappa'(G) \geq 4$ has a 3-NZF.*
(iv) *(Jaeger) There exists an integer $k \geq 4$ such that every k-edge-connected graph has 3-NZF.*

As the nowhere zero flow problem is the corresponding homogeneous case of the group connectivity problem, Jaeger et al. [9] proposed the following conjectures, which, as suggested by a result of Kochol [10], are stronger than the corresponding conjectures above.

Conjecture 1.2 (*Jaeger et al., [9]*). *Let G be a graph.*

(i) *If $\kappa'(G) \geq 3$, then $\Lambda_g(G) \leq 5$.*
(ii) *If $\kappa'(G) \geq 5$, then $\Lambda_g(G) \leq 3$.*
(iii) *There exists an integer $k \geq 5$ such that if $\kappa'(G) \geq k$, then $\Lambda_g(G) \leq 3$.*

In [22], Xu and Zhang proposed a triangulated version of the 3-flow conjecture. Let J_3 denote the family of all connected graphs such that $G \in J_3$ if and only if every edge of G lies in a K_3 of G. A graph in J_3 will also be referred as a J_3 **graph**.

Conjecture 1.3 (*Xu and Zhang, [22]*). *If $\kappa'(G) \geq 4$ and if $G \in J_3$, then G has a 3-NZF.*

Devos (Problem 1 in [15]) suggested that if $\kappa'(G) \geq 4$ and if $G \in J_3$, then $\Lambda_g(G) \leq 3$. But a counterexample to this stronger version was given in [15], where a modified version of the conjecture is proposed: If $\kappa'(G) \geq 5$ and if $G \in J_3$, then G has a 3-NZF.

There have been lots of researches conducted to attack Conjectures 1.1 and 1.2. See [8,23] for literature surveys. Jaeger [7] was the first to show that every 2-edge-connected graph has an 8-NZF, and that every 4-edge-connected graph has a 4-NZF. Later Seymour [18] proved that every 2-edge-connected graph has a 6-NZF. Jaeger et al. [9] further showed that if G is a 3-edge-connected graph, then $\Lambda_g(G) \leq 6$. More recently, Sudakov [19] showed that almost every random graph with minimum degree at least 2 has a 3-NZF. As for highly connected graphs, Lai and Zhang [16] first proved that every $4 \log_2 |V(G)|$-edge-connected graph has a 3-NZF. More recently in [14], it is proved that every $3 \log_2 |V(G)|$-edge-connected graph is Z_3-connected. In this paper, we proved the following:

Theorem 1.4. (i) *Every 5-edge-connected graph is Z_3-connected if and only if every 5-edge-connected line graph is Z_3-connected.*
(ii) *Every 6-edge-connected triangular line graph is Z_3-connected.*
(iii) *Every 7-edge-connected triangular claw-free graph is Z_3-connected.*

In particular, every 6-edge-connected triangular line graph has a nowhere zero 3-flow, and every 7-edge-connected triangular claw-free graph has a nowhere zero 3-flow.

This paper is organized as follows: In Section 2, we present some of the backgrounds and mechanisms to be used in the proofs. Theorem 1.4(i) is proved in Section 3. In order to prepare a proof for Theorem 1.4(iii), we also show that Ryjáček's line graph closure [17] can also be applied to convert the study of the group connectivity of claw-free graphs into that of line graphs. In Section 4, we shall assume the truth of a technical theorem to prove Theorem 1.4(ii) and (iii). The last section is devoted to the proof of the technical theorem.

2. Preliminaries

Let G be a graph and let $X \subseteq E(G)$ be an edge subset. The **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. For convenience, we use G/e for $G/\{e\}$ and $G/\emptyset = G$; and if H is a subgraph of G, we write G/H for $G/E(H)$.

Proposition 2.1 (*Proposition 3.2 of [11]*). *Let A be an Abelian group with $|A| \geq 3$. Then $\langle A \rangle$ satisfies each of the following:*

(C1) $K_1 \in \langle A \rangle$,
(C2) *if $G \in \langle A \rangle$ and if $e \in E(G)$, then $G/e \in \langle A \rangle$,*
(C3) *if H is a subgraph of G and if both $H \in \langle A \rangle$ and $G/H \in \langle A \rangle$, then $G \in \langle A \rangle$.*

Let H_1 and H_2 be two subgraphs of a connected graph G. We say that G is a **parallel connection** of H_1 and H_2, denoted by $H_1 \oplus_2 H_2$, if $E(H_1) \cup E(H_2) = E(G)$, $|V(H_1) \cap V(H_2)| = 2$, and $|E(H_1) \cap E(H_2)| = 1$.

For $k \in \mathbf{Z}$ with $k \geq 3$, a **wheel** W_k is the simple graph obtained from a k-circuit by adding a new vertex v, referred as **the center of the wheel**, and by joining the center to every vertex of the k-circuit. A fan $\mathbf{F}_k$ is the graph obtained from W_k by deleting an edge not incident with the center. Define F_2 to be the 3-circuit. The family $\mathcal{WF}$ can now be recursively constructed as follows:

(WF1) For all $k \geq 1$, and $n \geq 2$, $W_{2k+1}, F_n \in \mathcal{WF}$.
(WF2) If $G, H \in \mathcal{WF}$, then any parallel connection of G and H is also in $\mathcal{WF}$.

Lemma 2.2. *Let G be a graph and A be an Abelian group with $|A| \geq 3$, K_n be a complete graph of order n, and let C_n denote the circuit on n vertices (also referred as an n-circuit).*

(i) *(Lemma 2.1 of [12]) If for every edge e in a spanning tree of G, G has a subgraph $H_e \in \langle A \rangle$ with $e \in E(H_e)$, then $G \in \langle A \rangle$.*
(ii) *([9] and Lemma 3.3 of [11]) $\Lambda_g(C_n) = n + 1$.*
(iii) *(Lemma 2.8 of [3], Lemma 2.6 of [5]) For any integer $k > 1$, $\Lambda_g(W_{2k}) = 3$.*
(iv) *(Corollary 3.5 of [11]) Let $n \geq 5$ be an integer. Then $K_n \in \langle A \rangle$.*

A J_3 graph G is **triangularly connected** if for all $e, e' \in E(G)$, G has a sequence of circuits $C^1, C^2, \ldots, C^m$ in G such that each of the following holds.

(TC1) $e \in E(C^1)$ and $e' \in E(C^m)$,
(TC2) for all $1 \leq i \leq m$, $|E(C^i)| \leq 3$, and
(TC3) for all $1 \leq i \leq m - 1$, $|E(C^i) \cap E(C^{i+1})| > 0$.

The sequence $\{C^1, C^2, \ldots, C^m\}$ will be referred as an (e, e')-**triangle-path** in G. Graphs in $\mathcal{WF}$ are usually referred as *WF*-graphs. By definition, every *WF*-graph is triangularly connected.

Theorem 2.3 (*Fan et al., [5]*). *Let G be a triangularly connected graph with $|V(G)| \geq 2$. Each of the following holds.*

(i) *(Theorem 1.4 of [5]) G is Z_3-connected if and only if $G \notin \mathcal{WF}$.*
(ii) *(Lemma 2.4 of [5]) G is Z_3-connected if and only if G contains a nontrivial Z_3-connected subgraph.*

The following is an immediate corollary of Theorem 2.3 and Lemma 2.2(ii) and (iii).

Corollary 2.4. *If $G \in \mathcal{WF}$, then G does not contain any even wheel or 2-circuit.*

Given an $f \in F(G, A)$ and a subset $X \in E(G)$, $f|_X$ denotes the **restriction** of f to X. For $b \in Z(G, A)$, a graph G is (A, b)-**extensible from** v, if for all $f_1 : E(v) \mapsto A^*$ satisfying $\partial f_1(v) = b(v)$, there exists an $f \in F^*(G, A)$ with $\partial f = b$ such that $f|_{E(v)} = f_1$. If for any $b \in Z(G, A)$, G is (A, b)-extensible from v, then G is called A-**extensible from** v. By definition, if G is A-extensible from v, then $G \in \langle A \rangle$.

Lemma 2.5 (*Lemma 2.3, [13]*). *Let G be a graph and $H \cong K_4$ be a subgraph of G and $v \in V(H)$ (see Fig. 1(a) and Fig. 2(a)). If $d_G(v) = 6$ and if G has another subgraph $H' \cong K_4$ such that $V(H) \bigcap V(H') = \{v\}$, $N_H(v) = \{x_1, x_2, x_3\}$ and $N_{H'}(v) = \{y_1, y_2, y_3\}$, then let G_v be the graph obtained from G by splitting the vertex $v \in V(G)$ into v_1, v_2 (as depicted in Fig. 1(b)), and by first deleting x_3v_1, y_3v_2 and then contracting v_1x_1, v_2y_1 (depicted in Fig. 1(c)); and if $d_G(v) > 6$, then let G_v be the graph obtained from G by splitting the vertex $v \in V(G)$ into v_1, v_2, deleting the edge v_1x_3, and then contracting v_1x_1 (depicted in Fig. 2(c)).*

(i) *If $G_v \in \langle Z_3 \rangle$, then $G \in \langle Z_3 \rangle$.*
(ii) *If for some $u \in V(G) - v$, G_v is Z_3-extensible from u, then G is also Z_3-extensible from u.*

Proof. The proof for (i) is given in [13]. The proof for (ii) is similar to that for (i) and so omitted. □

Definition 2.6. Suppose that $N_G(v) = \{v_1, v_2, \ldots, v_n\}$, and let $Y = \{vv_1, vv_2\}$. As in [15], define $G_{[v,Y]}$ to be the graph obtained from $G - \{vv_1, vv_2\}$ by adding a new edge that joins v_1 and v_2.

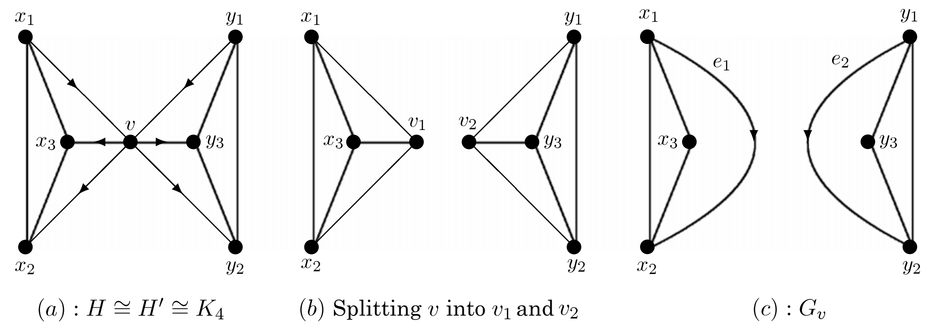

Fig. 1. Reduction in Lemma 2.5.

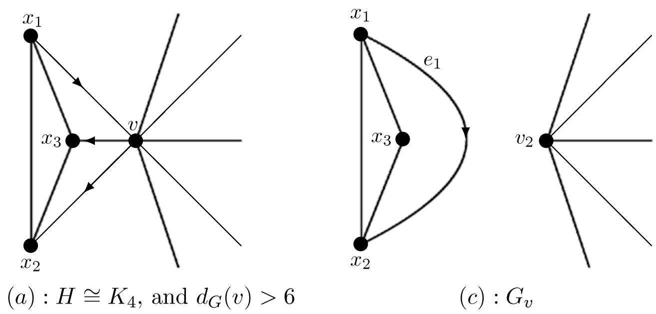

Fig. 2. Reduction in Lemma 2.5.

Lemma 2.7 (*Lemma 6, [15]*)**.** *For any Abelian group A and $b \in Z(G, A)$, if $G_{[v,Y]}$ has an (A, b)-NZF, then G has an (A, b)-NZF. Moreover, if $G_{[v,Y]}$ is A-extensible from a vertex u with $u \neq v$, then G is also A-extensible from u.*

Lemma 2.8 (*Lemma 7, [15]*)**.** *Let A be an Abelian group, G be a graph and $H \in \langle A \rangle$ be a connected subgraph of G. We define $G^* = G/H$ and denote by v_H the vertex in G^* onto which H is contracted. For any $b \in Z(G, A)$, define $b' : V(G^*) \mapsto A$ by $b'(v_H) = \sum_{u \in V(H)} b(u)$ and $b'(v) = b(v)$ for $v \neq v_H$. If G^* admits an (A, b')-NZF f^*, then f^* can be extended to an (A, b)-NZF of G.*

3. Line graphs and claw-free graphs

We shall follow [4] to define a line graph. The **line graph** of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where for an integer $k \in \{0, 1, 2\}$, two vertices in $L(G)$ are joined by k edges in $L(G)$ if and only if the corresponding edges in G are sharing k common vertices in G. In other words, if e_1 and e_2 are adjacent but not parallel in G, then e_1 and e_2 are joined by one edge in $L(G)$; if e_1 and e_2 are parallel edges in G, then e_1 and e_2 are joined by two (parallel) edges in $L(G)$. Note that our definition for line is slightly different from the one defined in [2] (called an edge graph there). But when G is a simple graph, both definitions are the same. The main reason for us to adopt this definition in [4] instead of the traditional definition of a line graph is explained in the introduction section of [13].

For an integer $i > 0$ and for a graph G, define

$$D_i(G) = \{v \in V(G) : d_G(v) = i\}.$$

A vertex $v \in V(G)$ is **locally connected** if $G[N_G(v)]$ is connected. A graph G is **claw-free** if G does not have an induced subgraph isomorphic to $K_{1,3}$. It is well known ([1,6]) that every line graph is a claw-free graph.

Following the definition given by Ryjáček ([17]), a graph H is the **closure** of a claw-free graph G, denoted by $H = cl(G)$, if

(CL1) there is a sequence of graphs $G_1, \ldots, G_t$ such that $G_1 = G$, $G_t = H$, $V(G_{i+1}) = V(G_i)$ and $E(G_{i+1}) = E(G_i) \bigcup \{uv : u, v \in N_{G_i}(x_i), uv \notin E(G_i)\}$ for some $x_i \in V(G_i)$ with connected non-complete $G_i[N_{G_i}(x_i)]$, for $i = 1, \ldots, t-1$, and
(CL2) No vertex of H has a connected non-complete neighborhood.

Lemma 3.1. *Let G be a claw-free graph.*

(i) *For any $v \in V(G)$, either $G[N_G(v)]$ is an edge disjoint union of two cliques or v is a locally connected vertex.*
(ii) *If v is a locally connected vertex of G, then $G[N_G[v]]$ is triangularly connected.*

Proof. (i) follows from the definition of claw-free graphs immediately.
(ii) Let $e = xy$, $e' = uw \in E(G[N_G[v]])$, where $y, w \in N_G(v)$ and e and e' are not contained in the same triangle. Since v is locally connected, there is a path $P = v_1v_2 \ldots v_s$ joining $y = v_1$ and $w = v_s$, where $v_i \in N_G(v)$, for $i = 2, \ldots, s-1$,

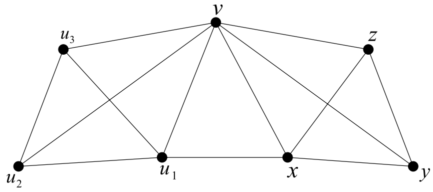

Fig. 3. The graph L_1 in Lemma 3.5.

in such a way that if $x \neq v$, then $x = v_2$, and if $u \neq v$, then $u = v_{s-1}$. Since $vv_i \in E(G)$, and since e is in the 3-circuit $G[\{v, v_1, v_2\}]$ and e' is in the 3-circuit $G[\{v, v_{s-1}, v_s\}]$, the 3-circuits $G[\{v, v_i, v_{i+1}\}]$, $1 \le i \le s-1$, is an (e, e')-triangle-path. Therefore $G[N_G(v)]$ is triangularly connected. □

Theorem 3.2. *The following are equivalent.*

(i) *Every 5-edge-connected graph is Z_3-connected.*
(ii) *Every 5-edge-connected line graph is Z_3-connected.*

Proof. As (i) trivially implies (ii), it suffices to show that (ii) implies (i). Let G be a graph with $\kappa'(G) \ge 5$ and let $S(G)$, the **subdivided graph** of G, be the graph obtained from G by replacing each edge $e = uv$ of G by a 2-path uv_ev, where v_e is a new vertex. Let e' be the edge in $L(S(G))$ that has uv_e and v_ev as its ends, and let $E' = \{e' \in E(L(S(G))) | e \in E(G)\}$. It then follows that $L(S(G))/[E(L(S(G))) - E'] = G$. (See Claims 1 and 2 within the proof of Theorem 3.4 in [4]). Moreover, If $\kappa'(G) \ge 5$, then $\kappa'(L(S(G))) \ge 5$, and so $L(S(G)) \in \langle Z_3 \rangle$ follows by (ii). As $L(S(G))/[E(L(S(G))) - E'] = G$, by Proposition 2.1(C2), $G \in \langle Z_3 \rangle$, and so (i) must hold. □

Theorem 3.3. *Let A be an Abelian group with $|A| \ge 4$ and G be a claw-free graph with $\delta(G) \ge 3$. Each of the following holds:*

(i) *Suppose that a vertex $v \in V(G)$ is locally connected, and $x, y \in N_G(v)$ are not adjacent. If $G + xy$ is A-connected, then G is A-connected.*
(ii) *If $cl(G)$ is A-connected, then G is A-connected.*

Proof. By the definition of the closure of a claw-free graph, $cl(G)$ contains G as a spanning connected subgraph. Thus Theorem 3.3(ii) follows from Theorem 3.3(i) and Lemma 2.2(i). Therefore, it suffices to prove Theorem 3.3(i).

Let G be a claw-free graph and let $v \in V(G)$ be a locally connected vertex. By Lemma 3.1(ii), every edge in the graph $G[N_G[v]]$ lies in a 3-circuit. As $|A| \ge 4$, by Lemma 2.2(ii) with $n = 3$, every edge of $G[N_G[v]]$ lies in an A-connected subgraph of $G[N_G[v]]$. It follows by Lemma 2.2(i) that $G[N_G[v]] \in \langle A \rangle$. Let $G' = G+xy$. Then $G'[N_{G'}[v]] = G[N_G[v]]+xy$. As $G[N_G[v]] \in \langle A \rangle$, it follows by Lemma 2.2(i) that $G'[N_{G'}[v]] \in \langle A \rangle$. Hence if $G' \in \langle A \rangle$, then by Proposition 2.1(C2), $G'/G'[N_{G'}[v]] \in \langle A \rangle$. As $G/G[N_G[v]] = G'/G'[N_{G'}[v]] \in \langle A \rangle$, and as $G[N_G[v]] \in \langle A \rangle$, it follows by Proposition 2.1(C3) that $G \in \langle A \rangle$. □

Lemma 3.4. *Let G be a claw-free graph with $\delta(G) \ge 3$ and $v \in V(G)$ be locally connected. Then $G[N_G(v)]$ has a Hamilton path.*

Proof. Arguing by contradiction, we assume that $G[N_G(v)]$ does not have a Hamilton path. As every connected graph on 3 vertices has a Hamilton path, we assume $d_G(v) \ge 4$.

Let $P = x_1x_2 \dots x_p$ be a longest path in $G[N_G(v)]$. As $V(P) \neq N_G(v)$, we can pick $x \in N_G(v) - V(P)$. As P is longest, $xx_1, xx_p \notin E(G)$. Since $G[\{x, x_1, x_p, v\}] \ncong K_{1,3}$, we must have $x_1x_p \in E(G)$. Since $G[N_G(v)]$ is connected, $G[N_G(v)]$ has a path P' from x to a vertex $x_{i_0} \in V(P)$, internally disjoint from $V(P)$. It follows that $xP'x_{i_0}x_{i_0+1} \dots x_px_1x_2 \dots x_{i_0-1}$ is a longer path, contrary to the assumption that P is a longest path in $G[N_G(v)]$. □

Lemma 3.5. *Let G be a claw-free graph with $\delta(G) \ge 6$ and $v \in V(G)$ be a locally connected vertex. Each of the following holds.*

(i) *If $d_G(v) \ge 6$ and if $G[N_G[v]] \in \mathcal{WF}$, then $G[N_G[v]]$ contains the graph L_1 depicted in Fig. 3 as an induced subgraph. Moreover, if $d_G(v) = 6$, then $G[N_G[v]] = L_1$.*
(ii) *If $d_G(v) \ge 7$, then $G[N_G[v]]$ is Z_3-connected.*

Proof. (i) Suppose $d_G(v) = m \ge 6$. By Lemma 3.4, $G[N_G(v)]$ has a path $P = v_1v_2 \dots v_m$, where $v_i \in N_G(v)$, $1 \le i \le m$.

We claim that $G[N_G[v]]$ has a K_4 with $v \in V(K_4)$. If not, then $L = G[\{v, v_1, v_3, v_5\}] \ncong K_4$, and so both $v_1v_3 \notin E(G)$ and $v_3v_5 \notin E(G)$. Since $G[\{v, v_1, v_3, v_5\}] \ncong K_{1,3}$, we must have $v_1v_5 \in E(G)$. Similarly, $v_2v_6 \in E(G)$ as $G[\{v, v_2, v_4, v_6\}] \ncong K_4$. It follows that $G[\{v, v_1, v_2, v_5, v_6\}]$ consists a W_4, contrary to Corollary 2.4 as $G[N_G[v]] \in \mathcal{WF}$. Thus $G[N_G[v]]$ must have a K_4.

Let $H_1 \cong K_4$ be a subgraph of $G[N_G[v]]$ with $v \in V(H_1)$. Let $W = N_G(v) - V(H_1)$. Note that for all $w \in W$, if w is adjacent to two vertices in $V(H_1) - \{v\}$, then $W_4 \subseteq G[V(H_1) \cup \{w\}]$, contrary to Corollary 2.4. Since $|W| \ge 3$, and since every $w \in W$ is adjacent to at most one vertex in $V(H_1)$, it follows from the fact that P is a Hamilton path that there must be $x, y, z \in W$ such that $xz, yz \in E(G)$. Let $V(H_1) - \{v\} = \{u_1, u_2, u_3\}$. With these notations, we further claim that $K_3 \subseteq G[W]$.

Assume that $G[W]$ contains no K_3's. Then $xy \notin E(G)$. Since for all $u_i \in V(H_1) - \{v\}$, $G[\{v, x, y, u_i\}] \ncong K_{1,3}$, u_i must be adjacent to x or y. Hence we may assume that there are two u_i's, say u_1, u_2, that are adjacent to the same vertex in $\{x, y\}$, say

x. It follows that $G[\{v, u_1, u_2, u_3, x\}]$ contains a W_4, contrary to Corollary 2.4. Thus we must have both $G[\{x, y, z\}] \cong K_3$ and $G[\{v, x, y, z\}] \cong K_4$. Let $H_2 = G[\{v, x, y, z\}]$.

Now assume that $d_G(v) = 6$, and so $N_G(v) = V(H_1) \cup W$. Since v is locally connected, $G[N_G(v)]$ has an edge e, say $e = u_1x$, joining H_1 and H_2. Let $G' = G[E(H_1) \cup E(H_2) \cup \{e\}]$. Then $G' \subseteq G[N_G[v]]$. By the definition of $\mathcal{WF}$, $G' \in \mathcal{WF}$. Let $e' \in E(G[N_G[v]]) - E(G')$. If e and e' are not adjacent, say $e' = u_2y$, then $W_4 \subseteq G[\{v, u_1, u_2, x, y\}]$; if e and e' are adjacent, say $e' = u_2x$, then $W_4 \subseteq G[\{v, u_1, u_2, u_3, x\}]$, contrary to Corollary 2.4 in either case. Thus we must have $G[N_G[v]] = G'$, as desired.

(ii) By contradiction, assume that $G[N_G[v]] \notin \langle Z_3 \rangle$. By Lemma 3.1(ii), $G[N_G[v]]$ is triangularly connected. By Theorem 2.3, $G[N_G[v]] \in \mathcal{WF}$.

By (i), $G[N_G[v]]$ contains a subgraph L_1 as depicted in Fig. 3. Define H_1 and H_2 as the two 4-cliques above in $G[N_G[v]]$ with $V(H_1) \cap V(H_2) = \{v\}$, and let $W' = N_G(v) - (V(H_1) \cup V(H_2))$. Again since $G[N_G[v]]$ contains no W_4, every vertex $w' \in W'$ is adjacent to at most one vertex in $V(H_i)$, $i \in \{1, 2\}$. It follows that $G[N_G[v]]$ contains an induced subgraph $G[\{v, w', z_1, z_2\}] \cong K_{1,3}$, for some $z_i \in V(H_i) - \{v\}$, $(1 \le i \le 2)$, contrary to the assumption that G is claw-free. Thus $G[N_G[v]]$ must be Z_3-connected if $d_G(v) \ge 7$. □

Theorem 3.6. *Let G be a claw-free graph with $\delta(G) \ge 7$. If $cl(G) \in \langle Z_3 \rangle$, then $G \in \langle Z_3 \rangle$.*

Proof. For any locally connected $v \in V(G)$ with $d_G(v) \ge 7$, by Lemma 3.5(ii), $G[N_G[v]]$ is Z_3-connected. Let $H_1, \ldots, H_m$ be all the maximal Z_3-connected subgraphs of G. Suppose $G_1 = G, G_2, \ldots, G_m, G_{m+1}$ is a sequence of graphs such that, for $i = 1, 2, 3, \ldots, m$, $G_{i+1} = G_i/H_i$. Suppose $G'_1 = cl(G), G'_2, \ldots, G'_m, G'_{m+1}$ is a sequence of graphs such that, for $i = 1, 2, 3, \ldots, m$, $G'_{i+1} = G'_i/H'_i$, where H'_i is the subgraph induced by $V(H_i)$ in $cl(G)$. Note that $H_i \subseteq H'_i$.

Now we claim that $G'_{m+1} = G_{m+1}$. By the construction of G_m and G'_m, we have $V(G'_{m+1}) = V(G_{m+1})$ and $E(G_{m+1}) \subseteq E(G'_{m+1})$. We only need to show $E(G'_{m+1}) \subseteq E(G_{m+1})$. Let $e \in E(G'_{m+1})$ and $e \notin E(G_{m+1})$. Assume $e = v_1v_2$ in $cl(G)$. By the definition of closure, there is a locally connected vertex $v \in V(G)$ such that $v_1, v_2 \in N_G(v)$ and v_1 and v_2 are not adjacent. By Lemma 3.5(ii) $G[N_G[v]]$ is Z_3-connected, then $G[N[v]]$ will be contained in some H_i, and $e \in E(H'_i)$, contrary to the fact that $e \in G'_{m+1}$.

Therefore $G_{m+1} = G'_{m+1}$. Since $cl(G) = G'_1 \in \langle Z_3 \rangle$, by Proposition 2.1(C2) $G'_2 \in \langle Z_3 \rangle$. Inductively, we conclude that $G'_i \in \langle Z_3 \rangle$, $1 \le i \le m+1$. It follows that $G_{m+1} = G'_{m+1} \in \langle Z_3 \rangle$. Since $H_m \in \langle Z_3 \rangle$, by Proposition 2.1(C3) $G_m \in \langle Z_3 \rangle$. Inductively, we conclude that $G_i \in \langle Z_3 \rangle$, $1 \le i \le m-1$. In particular, $G = G_1 \in \langle Z_3 \rangle$. □

4. Group connectivity of J_3 line graphs and J_3 claw-free graphs

The main result of this section is the following.

Theorem 4.1. *Each of the following holds.*

(i) *Every 6-edge-connected J_3 line graph is Z_3-connected.*
(ii) *Every 7-edge-connected J_3 claw-free graph is Z_3-connected.*

An edge cut X of G is **essential** if $G-X$ has at least two nontrivial components. For any integer $k > 0$, a graph is **essentially k-edge-connected** if G has no essential edge cut X with $|X| < k$. By this definition, if a graph G is k-edge-connected, then G is also essentially k-edge-connected. An edge cut X of G is a **cyclical edge cut** if neither side of $G-X$ is acyclic; G is cyclically k-edge-connected if G has no cyclical edge cut of size less than k.

By the definition of a line graph, for all $v \in V(G)$, $E(v)$ induce a complete subgraph H_v in $L(G)$. When $u, v \in V(G)$ with $u \ne v$, if G is simple, then H_v and H_u are edge disjoint complete subgraphs of $L(G)$. Such an observation motivates the following definition.

For a connected graph G, a partition $(E_1, E_2, \ldots, E_k)$ of $E(G)$ is a **clique partition** of G if $G[E_i]$ is spanned by a maximal complete subgraph of G for each $i \in \{1, 2, \ldots, k\}$. Furthermore, $(E_1, E_2, \ldots, E_k)$ is a (≥ 3)-**clique partition** of G, if for each $i \in \{1, 2, \ldots, k\}$, $G[E_i]$ is spanned by a K_{n_i} with $n_i \ge 3$; and a (K_3, K_4)-**partition** if for each $i \in \{1, 2, \ldots, k\}$, $G[E_i]$ is spanned by a maximal subgraph of G isomorphic to a K_3 or a K_4. Note that if G is simple, and if $(E_1, E_2, \ldots, E_k)$ of $E(G)$ is a clique partition of G, then $|V(G[E_i]) \cap V(G[E_j])| \le 1$ where $i \ne j$ and $i, j \in \{1, 2, \ldots, k\}$. By the definition of a line graph, every J_3 line graph must have a (≥ 3)-clique partition. By Proposition 2.1 and Lemma 2.2(iv), it suffices to study the Z_3-connectedness of graphs with a (K_3, K_4)-partition.

For an integer $m > 0$, mK_2 denotes the graph with 2 vertices and m parallel edges. Define $\mathcal{F}^0 = \{G : G$ has a (K_3, K_4)-partition$\}$, and $\mathcal{F}$ to be the family of graphs such that $G \in \mathcal{F}$ if and only if either $G \in \mathcal{F}_0$, or G is obtained from a member $G' \in \mathcal{F}_0$ by contracting some edges in $E(G')$.

Let $H_1 \cong K_4$ and H_0, H_2, H_3 be contractions of H_1, where $H_0 = 4K_2$. Let $H_4 \cong 2K_2$ be the graph obtained from K_3 by contracting an edge (see Fig. 4 for H_i, $0 \le i \le 4$). Then for every graph $G \in \mathcal{F}$, $E(G)$ is partitioned into $E_1, E_2, \ldots, E_k$, such that $G[E_j] \in \{H_0, H_1, H_2, H_3, K_3, H_4\}$, for $j = 1, 2, \ldots, k$.

We shall prove the following stronger result, which implies Theorem 4.1.

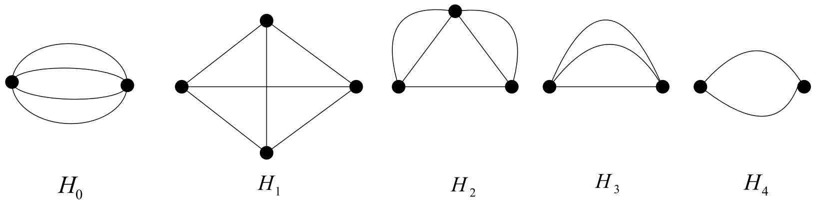

Fig. 4. H_0, H_1, H_2, H_3, H_4.

Theorem 4.2. *Let $G \in \mathcal{F}$ be an essentially 6-edge-connected graph with $|D_3(G) \cup D_4(G) \cup D_5(G)| \leq 1$. Each of the following holds.*

(i) *For any $u \in D_6(G) \cup D_7(G) \cup D_8(G)$, G is Z_3-extensible from u.*
(ii) *If $D_6(G) \cup D_7(G) \cup D_8(G) = \emptyset$, then G is Z_3-connected.*

Assuming the truth of Theorem 4.2, we can derive the following results. A graph G is Z_3-**reduced** if G does not have a nontrivial subgraph in $\langle Z_3 \rangle$.

Theorem 4.3. *Every 6-edge-connected graph with a (≥ 3)-clique partition is Z_3-connected.*

Proof. Let G be a counterexample with $|V(G)|$ minimized. As the theorem holds trivially if $|V(G)| \leq 6$, we assume that $|V(G)| \geq 7$. By the minimality of G, G is Z_3-reduced. By Lemma 2.2(iv), G must have a (K_3, K_4)-partition, and so $G \in \mathcal{F}$. Thus $G \in \langle Z_3 \rangle$ by Theorem 4.2. □

Proof of Theorem 4.1. (i) Let G be a 6-edge-connected J_3 line graph. By the definition of a line graph, and since G is a J_3 graph, G is a 6-edge-connected graph with a (≥ 3)-clique partition. It follows by Theorem 4.3 that G is Z_3-connected.
(ii) Let G be a 7-edge-connected J_3 claw-free graph, and let $cl(G)$ be its closure. Then $cl(G)$ is a 7-edge-connected J_3 line graph. By Theorem 4.1(i), $cl(G)$ is Z_3-connected. By Theorem 3.6, G is Z_3-connected. This completes the proof of Theorem 4.1. □

5. The proof of Theorem 4.2

Throughout this section, for a graph G and for $W \subseteq E(G)$, any map $g : W \mapsto Z_3$ is viewed as a map $g : E(G) \mapsto Z_3$ such that $g(e) = 0$, for all $e \in E(G) - W$.

By contradiction, assume that there exists a graph $G \in \mathcal{F}$ such that

$$G \text{ is a counterexample to Theorem 4.2 with} |V(G)| + |E(G)| \text{minimized.} \tag{1}$$

Thus either

$$D_6(G) \cup D_7(G) \cup D_8(G) = \emptyset, \quad \text{and} \quad G \notin \langle Z_3 \rangle, \tag{2}$$

or

$$\text{there exists } u \in D_6(G) \cup D_7(G) \cup D_8(G) \text{ such that } G \text{ is not } Z_3\text{-extensible from } u. \tag{3}$$

For a graph Γ, let $N(\Gamma) = |V(\Gamma)| + |E(\Gamma)|$. We have the following claims.

Claim 1. *If (2) holds, then G is Z_3-reduced; if (3) holds, then $G - u$ is Z_3-reduced.*

Assume (3) holds. Suppose $G - u$ has a nontrivial subgraph H with $H \in \langle Z_3 \rangle$. Since $G \in \mathcal{F}$, $G/H \in \mathcal{F}$. As H is nontrivial, $N(G/H) < N(G)$. Since G is essentially 6-edge-connected, G/H is also essentially 6-edge connected. By (1), G/H satisfies (i). It follows by Lemma 2.8 that G is A-extensible from u, contrary to (1). The proof for the case when (2) holds is similar. This proves Claim 1.

By Lemma 2.2(ii) and Proposition 2.1, any Z_3-reduced graph does not have H_0, H_2, H_3 and H_4 as a subgraph. Thus by Claim 1,

$$G \text{ (when (2) holds) or } G - u \text{ (when (3) holds) does not have } H_0, H_2, H_3, \text{ or } H_4 \text{ as a subgraph.} \tag{4}$$

Claim 2. *G is cyclically 9-edge-connected.*

Suppose that G has a minimal cyclical edge-cut X with $|X| < 9$. Let G_1 and G_2 be the two components of $G - X$. Since G is essentially 6-edge connected and since both G_1 and G_2 are nontrivial, we have $6 \leq |X| \leq 8$. Let v_{G_i} be the new vertex in G/G_i onto which G_i is contracted, for $i = 1, 2$. Then

$$E_{G/G_1}(v_{G_1}) = E_{G/G_2}(v_{G_2}) = X.$$

Case 1. (2) holds.

Let $b \in Z(G, Z_3)$. Define $b_2 : V(G/G_2) \mapsto Z_3$ by

$$b_2(v) = \begin{cases} \sum_{z \in V(G_2)} b(z), & \text{if } v = v_{G_2} \\ b(v), & \text{otherwise.} \end{cases}$$

Then $b_2 \in Z(G/G_2, Z_3)$ as $b \in Z(G, Z_3)$. By (1) and since $N(G/G_2) < N(G)$, G/G_2 has a (Z_3, b)-NZF f_2. Now define $b_1 : V(G/G_1) \mapsto Z_3$ by

$$b_1(v) = \begin{cases} \sum_{z \in V(G_1)} b(z), & \text{if } v = v_{G_1} \\ b(v), & \text{otherwise.} \end{cases}$$

Then $b_1 \in Z(G/G_1, Z_3)$ as $b \in Z(G, Z_3)$. Define $g = f_2|_X : X \mapsto Z_3^*$. Then

$$\partial g(v_{G_1}) = -\partial f_2(v_{G_2}) = -b_2(v_{G_2}) = -\sum_{z \in V(G_2)} b(z) = \sum_{z \in V(G_1)} b(z) = b_1(v_{G_1}).$$

Since $6 \le d_{G/G_1}(v_{G_1}) \le 8$, and by (1), G/G_1 is Z_3-extensible from v_{G_1}. Therefore there is a (Z_3, b)-NZF f_1 of G/G_1 such that $f_1|_X = g = f_2|_X$. Then $f = f_1 + f_2 - f_2|_X$ is a (Z_3, b)-NZF of G, contrary to (1).

Case 2. (3) holds.

Let $b \in Z(G, Z_3)$. Assume $u \in V(G_1)$ and $f_0 : E(u) \mapsto Z_3^*$ such that $\partial f_0(u) = b(u)$.

Define $b_2 : V(G/G_2) \mapsto Z_3$ by

$$b_2(v) = \begin{cases} \sum_{z \in V(G_2)} b(z), & \text{if } v = v_{G_2}, \\ b(v), & \text{otherwise.} \end{cases}$$

Then $b_2 \in Z(G/G_2, Z_3)$ as $b \in Z(G, Z_3)$. By (1) and since $N(G/G_2) < N(G)$, G/G_2 is Z_3-extensible from u, and so G/G_2 has a (Z_3, b)-NZF f_2 such that $f_2|_{E(u)} = f_0$.

Now define $b_1 : V(G/G_1) \mapsto Z_3$ by

$$b_1(v) = \begin{cases} \sum_{z \in V(G_1)} b(z), & \text{if } v = v_{G_1}, \\ b(v), & \text{otherwise.} \end{cases}$$

Then $b_1 \in Z(G/G_1, Z_3)$ as $b \in Z(G, Z_3)$. For v_{G_1}, define $g = f_2|_X : X \mapsto Z_3^*$. Then

$$\partial g(v_{G_1}) = -\partial f_2(v_{G_2}) = -b_2(v_{G_2}) = -\sum_{z \in V(G_2)} b(z) = \sum_{z \in V(G_1)} b(z) = b_1(v_{G_1}).$$

By (1), by $N(G/G_1) < N(G)$, and since $6 \le d_{G/G_1}(v_{G_1}) \le 8$, G/G_1 is Z_3-extensible from v_{G_1}. Therefore G/G_1 has a (Z_3, b_1)-NZF f_1 satisfying $f_1|_X = g = f_2|_X$. Thus $f = f_1 + f_2 - f_2|_X$ is a (Z_3, b)-NZF of G such that $f|_{E(u)} = f_2|_{E(u)} = f_0$, contrary to (1). This proves Claim 2.

Let $\mathcal{H} = \{H_0, H_1, H_2, H_3, K_3, H_4\}$. For a graph $G \in \mathcal{F}$, a subgraph $H \subseteq G$ is $\mathcal{H}$-**maximal** if $H \in \{H_0, H_1, H_2, H_3, K_3, H_4\}$ and H is not properly contained in another subgraph of G that is also a member in $\{H_0, H_1, H_2, H_3, K_3, H_4\}$. By the definition of $\mathcal{F}$, if $G \in \mathcal{F}$, then every edge must be in an $\mathcal{H}$-maximal subgraph of G.

Claim 3. $D_3(G) \cup D_4(G) \cup D_5(G) \neq \emptyset$.

By contradiction, assume that

$$D_3(G) \cup D_4(G) \cup D_5(G) = \emptyset. \quad (5)$$

Let $v \in V(G)$ such that if (3) holds, then choose v so that u and v are not in the same $\mathcal{H}$-maximal subgraph of G. Thus $d_G(v) \ge 6$. Since $G \in F$ and by (4), v must be in an $\mathcal{H}$-maximal subgraph H of G such that $H \in \{K_3, K_4\}$.

Case 1. Suppose $v \in V(H)$ where $H \cong K_4$ with $V(H) = \{v, x_1, x_2, x_3\}$. Let G_v be the graph as defined in Lemma 2.5, and we shall use the notations in Figs. 1 and 2.

By the definition of G_v, $N(G_v) < N(G)$ and $G_v \in \mathcal{F}$. If G_v is essentially 6-edge-connected, then by (1), G_v satisfies (i) or (ii). By Lemma 2.5, G satisfies (i) or (ii) respectively, contrary to (1).

Thus G_v has a minimal essential edge cut X with $|X| < 6$. Let G_1, G_2 be the two components of $G - X$. Since G is essentially 6-edge-connected, $\{x_1, x_2, x_3\}$ and $N_G(v) - \{x_1, x_2, x_3\}$ must be in distinct components of $G_v - X$. By the assumption that $G \in \mathcal{F}$ and by (4), neither G_1 nor G_2 is acyclic. It follows that in G, $X \cup \{vx_1, vx_2, vx_3\}$ is a cyclical edge-cut with at most 8 edges, contrary to Claim 2. This precludes Case 1 of Claim 3.

Case 2. Suppose $v \in V(H)$ where $H \cong K_3$ with $V(H) = \{v, v_1, v_2\}$. Let $Y = \{vv_1, vv_2\}$ and $G_{[v,Y]}$ be the graph defined in Definition 2.6. Then $N(G_{[v,Y]}) < N(G)$. By the choice of H, $G_{[v,Y]} \in \mathcal{F}$. If $G_{[v,Y]}$ is essentially 6-edge-connected, then by (1), $G_{[v,Y]}$ satisfies (i) or (ii). By Lemma 2.7, G satisfies (i) or (ii) respectively, contrary to (1).

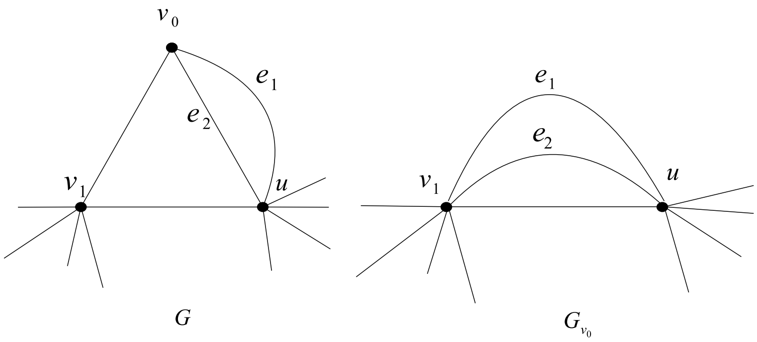

Fig. 5. Case 1a in the proof of Theorem 4.2.

Thus $G_{[v,Y]}$ must have a minimal essential edge cut X with $|X| < 6$. Let G_1, G_2 be the two components of $G_{[v,Y]} - X$. Using the notation in Definition 2.6, since G is essentially 6-edge-connected, v and $\{v_1, v_2\}$ must be separated by X in $G_{[v,Y]}$. We may assume that $\{v_1, v_2\} \subseteq V(G_1)$ and $N_G[v] - \{v_1, v_2\} \subseteq V(G_2)$. Note that $G_1[\{v_1, v_2\}]$ is a 2-circuit, and by (4) and since $d_G(v) \geq 6$, G_2 cannot be acyclic. It follows that $X \cup \{vv_1, vv_2\}$ is a cyclical 7-edge-cut of G, contrary to Claim 2. This precludes Case 2 of Claim 3, and completes the proof for Claim 3.

Claim 4. $\kappa(G) \geq 2$.

By contradiction, assume that G has two subgraphs G_1, G_2 with $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{w\}$. Without loss of generality, if (3) holds, we may further assume that $u \in V(G_1)$. By (1), $G_2 \in \langle Z_3 \rangle$, contrary to Claim 1. This proves Claim 4.

By Claim 3, we assume that

$$D_3(G) \cup D_4(G) \cup D_5(G) = \{v_0\}.$$

Let $b \in Z(G, Z_3)$ and $f_0 : E(u) \mapsto Z_3^*$ be such that $\partial f_0(u) = b(u)$. Without loss of generality, we assume that all edges in $E_G(u)$ are oriented away from u.

In the rest of the proof, we shall assume the existence of $u \in D_6(G) \cup D_7(G) \cup D_8(G)$ to prove that G is Z_3-extensible from u. We shall also show that no matter whether the degree of v_0 in G is 3, 4 or 5, a contradiction will be obtained. The proof for the case when $D_6(G) \cup D_7(G) \cup D_8(G) = \emptyset$ is similar.

By (3), in each of the cases below, we always assume that there exists a $b \in Z(G, Z_3)$ and an $f_0 : E_G(u) \mapsto Z_3^*$ with $\partial f_0(u) = b(u)$, such that Theorem 4.2(i) fails.

Case 1. $v_0 \in D_3(G)$.

Since $v_0 \in D_3(G)$, G has an $\mathcal{H}$-maximal subgraph H with $v_0 \in V(H)$. By Claim 4 and by $v_0 \in D_3(G)$, $H \in \{H_1, H_2\}$. By (4), if $H = H_2$, then u must be the degree 4 vertex in H_2.

Case 1a. $H \cong H_2$.

Denote $V(H) = \{v_0, u, v_1\}$ where $u \in D_4(H)$ and $G_{v_0} = G/\{v_0v_1\}$ (see Fig. 5). Then $N(G_{v_0}) < N(G)$. Since $G \in \mathcal{F}$ and G is essentially 6-edge-connected, $G_{v_0} \in F$ and G_{v_0} is essentially 6-edge connected. By (1), G_{v_0} satisfies (i).

Define $b' : V(G_{v_0}) \mapsto Z_3$ by

$$b'(v) = \begin{cases} b(v_0) + b(v_1), & \text{if } v = v_1 \\ b(v), & \text{otherwise.} \end{cases}$$

As $\sum_{v \in V(G_0)} b'(v) = \sum_{v \in V(G)} b(v) = 0$, $b' \in Z(G_{v_0}, Z_3)$. Since G_{v_0} is Z_3-extensible from u, there exists $g \in F^*(G_{v_0}, Z_3)$ such that $\partial g = b'$ and $g|_{E(u)} = f_0$. Assume that the edge v_0v_1 is oriented from v_0 to v_1. Define $f : E(G) \mapsto Z_3^*$ by

$$f(e) = \begin{cases} b(v_0) + g(e_1) + g(e_2), & \text{if } e = v_0v_1 \\ g(e), & \text{otherwise.} \end{cases}$$

Then for all $v \in V(G)$,

$$\partial f(v) = \begin{cases} b(v_0) + g(e_1) + g(e_2) - g(e_1) - g(e_2) = b(v_0) & \text{if } v = v_0, \\ (b'(v_1) + g(e_1) + g(e_2)) - (b(v_0) + g(e_1) + g(e_2)) = b(v_1) & \text{if } v = v_1, \\ b'(v) = b(v), & \text{otherwise.} \end{cases}$$

It follows that $\partial f = b$, and $f|_{E(u)} = g|_{E(u)} = f_0$. Therefore G is Z_3-extensible from u, contrary to (1). This completes the proof for Case 1a.

Case 1b. $H = H_1 \cong K_4$ and $u \in V(H)$.

Let $V(H) = \{v_0, u, v_2, v_3\}$. Define G_{v_0} to be the graph obtained from $G - v_0v_2$ by replacing uv_0v_3 by one edge e_0 (see Fig. 6). Then $N(G_{v_0}) < N(G)$.

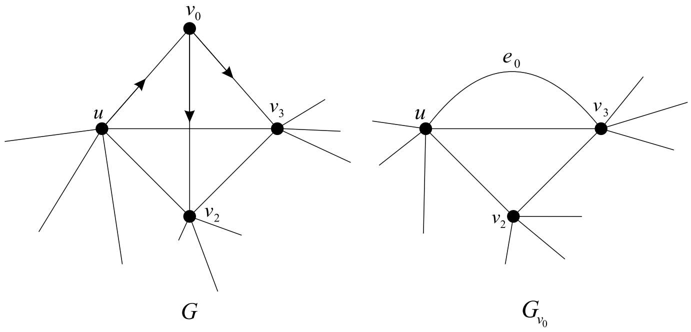

Fig. 6. Case 1b in the proof of Theorem 4.2.

Suppose that G_{v_0} has an essential edge-cut X with $|X| < 6$. Since G is essentially 6-edge-connected, X must separate v_0 and v_2. It follows by (4) that $X \cup \{v_0v_2\}$ is a cyclical edge-cut of G with $|X \cup \{v_0v_2\}| \le 6$, contrary to Claim 2. Thus G_{v_0} is essentially 6-edge-connected and so by (1),

$$G_{v_0} \text{ is } Z_3\text{-extensible from } u. \tag{6}$$

We shall show that f_0 can be extended to $f \in F^*(G, Z_3)$ to find a contradiction to (1).

Case 1b1. $b(v_0) = 0$. Define $b' : V(G_{v_0}) \mapsto Z_3$ by

$$b'(v) = \begin{cases} b(v_2) - f_0(uv_0), & \text{if } v = v_2, \\ b(v_3) + f_0(uv_0), & \text{if } v = v_3, \\ b(v), & \text{otherwise.} \end{cases}$$

Since $\sum_{v \in V(G_{v_0})} b'(v) = \sum_{v \in V(G)} b(v) = 0$, $b' \in Z(G_{v_0}, Z_3)$. By (6), there exists $g \in F^*(G_{v_0}, Z_3)$ such that $\partial g = b'$, and $g|_{E(u)} = f_0$. Assume that v_0v_2 is oriented from v_0 to v_2 and v_0v_3 is oriented from v_0 to v_3. Define $f : E(G) \mapsto Z_3$ by

$$f(e) = \begin{cases} g(uv_0), & \text{if } e = v_0u, \\ -g(uv_0), & \text{if } e = v_0v_2, \\ 2g(uv_0), & \text{if } e = v_0v_3, \\ g(e), & \text{otherwise.} \end{cases}$$

Since $g \in F^*(G_{v_0}, Z_3)$, $f \in F^*(G, Z_3)$. For each $v \in V(G)$,

$$\partial f(v) = \begin{cases} 2g(uv_0) - g(uv_0) - g(uv_0) = 0 = b(v_0), & \text{if } v = v_0, \\ \partial g(v_2) - (-g(uv_0)) = b'(v_2) + g(uv_0) = b(v_2), & \text{if } v = v_2, \\ b'(v_3) + g(uv_0) - 2g(uv_0) = b(v_3), & \text{if } v = v_3, \\ \partial g(v) = b'(v) = b(v), & \text{otherwise.} \end{cases}$$

Thus $\partial f = b$ and $f|_{E(u)} = g|_{E(u)} = f_0$. Hence G is Z_3-extensible from u, contrary to (1).

Case 1b2. $b(v_0) \ne 0$.

Define $b' : V(G_{v_0}) \mapsto Z_3$ by

$$b'(v) = \begin{cases} b(v_2) + b(v_0), & \text{if } v = v_2, \\ b(v), & \text{otherwise.} \end{cases}$$

Then $b' \in Z(G_{v_0}, Z_3)$. By (6), G_{v_0} has an $g : E(G_{v_0}) \mapsto Z_3^*$ such that $\partial g = b'$ and $g|_{E(u)} = f_0$. Assume that v_0v_2 and v_0v_3 are oriented away from v_0. Define $f : E(G) \mapsto Z_3^*$ by

$$f(e) = \begin{cases} b(v_0), & \text{if } e = v_0v_2, \\ g(v_0u), & \text{if } e = v_0u, v_0v_3, \\ g(e), & \text{otherwise.} \end{cases}$$

Since $g \in F^*(G_{v_0}, Z_3)$ and since $b(v_0) \ne 0$, $f \in F^*(G, Z_3)$. For each $v \in V(G)$,

$$\partial f(v) = \begin{cases} b(v_0) + g(v_0u) - g(v_0u) = b(v_0), & \text{if } v = v_0, \\ \partial g(v_2) - b(v_0) = b'(v_2) - b(v_0) = b(v_2), & \text{if } v = v_2, \\ \partial g(v) = b'(v) = b(v), & \text{otherwise.} \end{cases}$$

Therefore $\partial f = b$ and $f|_{E(u)} = g|_{E(u)} = f_0$. Thus G is Z_3-extensible from u, contrary to (1).

Case 1c. $H = H_1 \cong K_4$ and $u \notin V(H)$.

Let $V(H) = \{v_0, v_1, v_2, v_3\}$. Then $d_G(v_i) \ge 6$ for $i = 1, 2, 3$. Let G_{v_1} be the graph obtained from G by first splitting the vertex $v_1 \in V(G)$ into v_1, v_1' (where v_1' is adjacent to v_0, v_2, v_3), deleting the edge $v_1'v_2$, and then contracting $v_1'v_3$

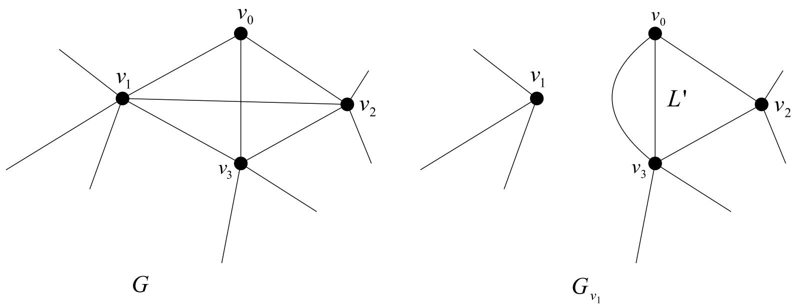

Fig. 7. Case 1c in the proof of Theorem 4.2.

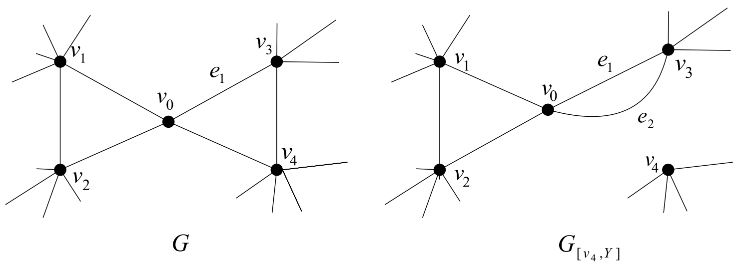

Fig. 8. Case 2a.

(see Fig. 7). As before, if G_{v_1} has an essential edge cut X with $|X| < 6$, then X must separate v_1 and $\{v_0, v_2, v_3\}$, and so $X \cup \{v_1v_0, v_1v_2, v_1v_3\}$ is a cyclical edge cut of G. It follows by Claim 2 that G_{v_1} is essentially 6-edge-connected.

Let $L' = G_{v_1}[\{v_0, v_2, v_3\}]$. As L' is a 3 vertex graph with 4 edges, $L' \in \langle Z_3 \rangle$. Let $G' = G_{v_1}/L'$ with a new vertex $v_{L'}$. Define $b_1 : V(G_{v_1}) \mapsto Z_3$ such that $b_1(v) = b(v)$, for all $v \in V(G_{v_1})$. As $b \in Z(G, Z_3)$, $b_1 \in Z(G_{v_1}, Z_3)$. Define $b' : V(G') \mapsto Z_3$ to be

$$b'(v) = \begin{cases} b_1(v_0) + b_1(v_2) + b_1(v_3), & \text{if } v = v_{L'}, \\ b_1(v), & \text{otherwise.} \end{cases}$$

Then as $b_1 \in Z(G_{v_1}, Z_3)$, $b' \in Z(G', Z_3)$.

As G_{v_1} is essentially 6-edge-connected, so is G'. By (1), G' satisfies (i). For any (Z_3, b')-NZF g of G', by Lemma 2.8, g can be extended to a (Z_3, b_1)-NZF f_1 of G_{v_1}, and by Lemma 2.5, f_1 can be extended to a (Z_3, b)-NZF f of G. Therefore G satisfies (i), a contrary to (1).

Case 2. $v_0 \in D_4(G)$.

Since $G \in \mathcal{F}$, either G has two $\mathcal{H}$-maximal subgraphs H', H'' isomorphic to K_3, with $v_0 \in V(H') \cap V(H'')$, or G has an $\mathcal{H}$-maximal subgraph $H \cong H_2$ with $v_0 \in V(H)$, as by Claim 4, $H \cong H_0$ is impossible.

Case 2a. Suppose $v_0 \in V(H') \cap V(H'')$ for two maximal subgraph $H' \cong H'' \cong K_3$ (see Fig. 8).

Let $N_G(v_0) = \{v_1, v_2, v_3, v_4\}$. Without loss of generality, we may assume that $V(H'') = \{v_0, v_3, v_4\}$ and $u \notin V(H'')$. Let $Y = \{v_4v_0, v_4v_3\}$ and define $G_{[v_4,Y]}$ as in Definition 2.6. Denote the two parallel edges joining v_0 and v_3 by e_1, e_2. Let $G_{v_4} = G_{[v_4,Y]}/\{e_1, e_2\}$. Then $N(G_{v_4}) < N(G)$. As before, if G_{v_4} has an essential edge cut X with $|X| < 6$, then X must separate v_4 and v_0 in G_{v_4}, and so $X \cup \{v_4v_0, v_4v_3\}$ is a cyclical edge cut of G. It follows by Claim 2 that G_{v_4} is essentially 6-edge-connected. By (1), G_{v_4} satisfies Theorem 4.2(i). By Lemma 2.7, G also satisfies Theorem 4.2(i), contrary to (1).

Case 2b. Suppose v_0 is contained in a subgraph $H \cong H_2$.

Since $G \in \mathcal{F}$, $d_G(v_0) = d_H(v_0) = 4$, G must have a 2-circuit which does not contain u as a vertex, contrary to (4). This precludes Case 2.

Case 3. $v_0 \in D_5(G)$.

Since $G \in \mathcal{F}$, by the definition of $\mathcal{F}$, G must have two $\mathcal{H}$-maximal subgraphs H', H'' such that $H' \in \{K_3, H_4\}$ and $H'' \in \{H_1, H_2, H_3\}$ with $v_0 \in V(H') \cap D_3(H'')$. By (4), H' and H'' cannot both have multiple edges, and so

$$(H', H'') \in \{(K_3, H_1), (H_4, H_1), (K_3, H_2), (K_3, H_3)\}. \tag{7}$$

If $(H', H'') = (K_3, H_3)$, (see Fig. 9), then let $V(K_3) = \{v_0, v_1, v_2\}$ and $V(H_3) = \{v_0, v_3\}$. By (4), $u = v_3$. Let $V_1 = \{v_0, u\}$, $V_2 = V(G) - V_1$, and W be the set of edges with one end in V_1 and the other in V_2. Since $d_G(u) \leq 8$, $|W| \leq 2 + d_G(u) - 3 < 8$, and so X is a cyclical edge cut of G with at most 7 edges, contrary to Claim 2.

Assume that $(H', H'') = (K_3, H_1)$. Let $V(K_3) = \{v_0, v_1, v_2\}$, and define $Y = \{v_0v_1, v_0v_2\}$. Define $G_{[v_0,Y]}$ as in Definition 2.6. Then $N(G_{[v_0,Y]}) < N(G)$. If $G_{[v_0,Y]}$ has an essential edge cut X with $|X| < 6$, then X must separate $V(K_3) - \{v_0\}$ and $V(H_1) - \{v_0\}$ in $G_{[v_0,Y]}$, and so $X \cup \{v_0v_1, v_0v_2\}$ is a cyclical edge cut of G. It follows by Claim 2 that $G_{[v_0,Y]}$ is essentially 6-edge-connected. By (1), $G_{[v_0,Y]}$ satisfies (i). By Lemma 2.7, G also satisfies (i) of Theorem 4.2, contrary to (1).

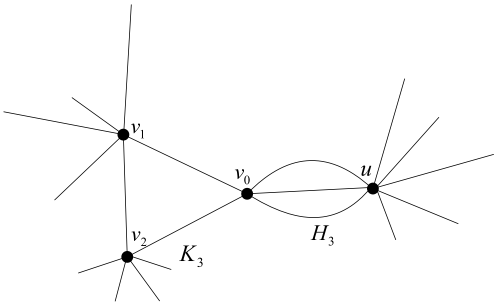

Fig. 9. $(H', H'') = (K_3, H_3)$ in Case 3.

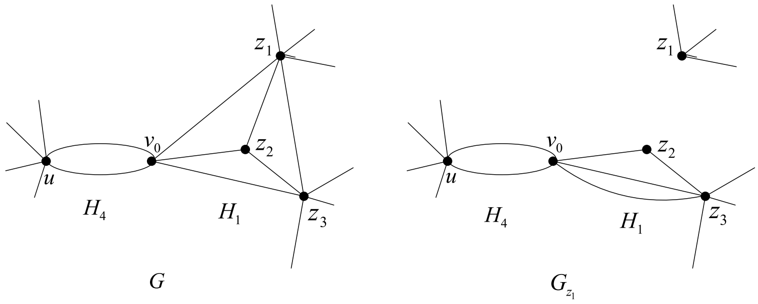

Fig. 10. $(H', H'') = (H_4, H_1)$ in Case 3.

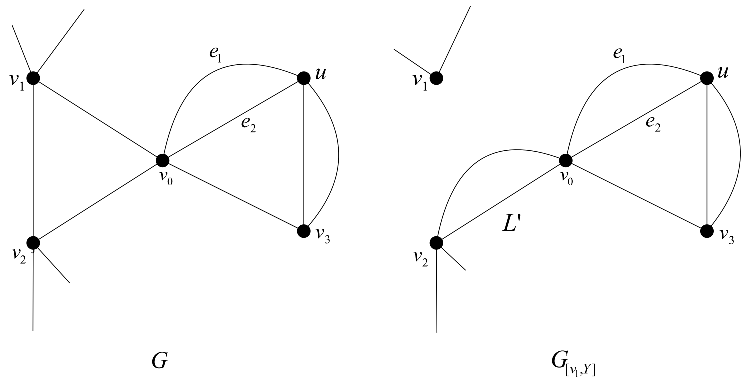

Fig. 11. $(H', H'') = (K_3, H_2)$ in Case 3.

Next, we assume that $(H', H'') = (H_4, H_1)$. Then by (4), we denote $V(H_1) = \{v_0, z_1, z_2, z_3\}$ and $V(H_4) = \{v_0, u\}$ (see Fig. 10). Let G_{z_1} be the graph obtained from G by first splitting the vertex $z_1 \in V(G)$ into z_1, z_1' (where z_1' is adjacent to v_0, z_2, z_3), deleting the edge $z_1'z_2$, and then contracting $z_1'z_3$. If G_{z_1} has an essential edge cut X with $|X| < 6$, then X must separate z_1 and v_0, z_2, z_3 in G_{z_1}, and so $X \cup \{z_1v_0, z_1z_2, z_1z_3\}$ is a cyclical edge cut of G. It follows by Claim 2 that G_{z_1} is essentially 6-edge-connected. Let $L' = G_{z_1}[\{v_0, z_2, z_3\}]$. As L' is a 3 vertex graph with 4 edges, $L' \in \langle Z_3 \rangle$. Let $G' = G_{z_1}/L'$. As G_{z_1} is essentially 6-edge-connected, so is G'. By (1), G' satisfies (i). By Lemma 2.8, G_{z_1} satisfies (i). It follows by Lemma 2.5 that G satisfies (i), a contrary to (1).

Therefore, we must have $(H', H'') = (K_3, H_2)$. Since $v_0 \in V(H') \cap V(H'')$, we may assume that $V(H') = \{v_0, v_1, v_2\}$. By (4), u must be the only vertex of degree 4 in H''. Let e_1 and e_2 denote the two parallel edges joining v_0 and u (see Fig. 11).

Note that $d_G(v_1) \geq 6$. Let $Y = \{v_1v_0, v_1v_2\}$. Define $G_{[v_1,Y]}$ as in Definition 2.6. By the definition of $\mathcal{F}$, $G_{[v_1,Y]} \in \mathcal{F}$. If $G_{[v_1,Y]}$ has an essential edge cut X with $|X| < 6$, then X must separate v_1 and v_0 (see Fig. 10) in $G_{[v_1,Y]}$, and so $X \cup \{v_1v_0, v_1v_2\}$ is a cyclical edge cut of G. It follows by Claim 2 that $G_{[v_1,Y]}$ is essentially 6-edge-connected.

Let $L' = G_{[v_1,Y]}[\{v_0, v_2\}]$, which is a 2-circuit, and so $L' \in \langle Z_3 \rangle$. Let $G' = G_{[v_1,Y]}/L'$. As $G_{[v_1,Y]}$ is essentially 6-edge-connected, so is G'. By (1), G' satisfies (i). By Lemma 2.8, $G_{[v_1,Y]}$ satisfies (i). It follows by Lemma 2.7 that G satisfies (i), contrary to (1). This completes the proof for Case 3.

As all the cases lead to contradictions, the theorem is established.

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