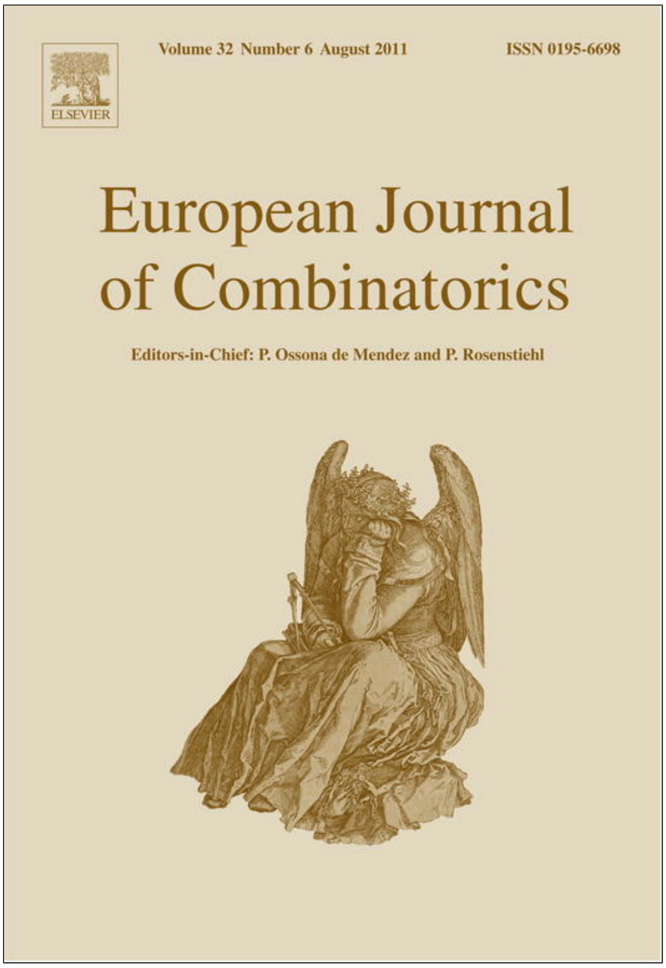
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Obstructions to a binary matroid being graphic

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A R T I C L E I N F O

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A B S T R A C T

Bixby and Cunningham showed that a 3-connected binary matroid M is graphic if and only if every element belongs to at most two non-separating cocircuits. Likewise, Lemos showed that such a matroid M is graphic if and only if it has exactly $r(M) + 1$ non-separating cocircuits. Hence the presence in M of either an element in at least three non-separating cocircuits, or of at least $r(M) + 2$ non-separating cocircuits, implies that M is non-graphic. We provide lower bounds on the size of the set of such elements, and on the number of non-separating cocircuits, in such non-graphic binary matroids. A computationally efficient method for finding such lower bounds for specific minor-closed classes of matroids is given. Applications of this method and other results on sets of obstructions to a binary matroid being graphic are given.

1. Introduction

There has been much interest in studying non-separating cocircuits in binary matroids as these cocircuits model vertices in graphs (see, for example, [1–4,6,5,7,8,11,13]). These are cocircuits D in a connected matroid M such that $M \setminus D$ is connected. A circuit C in a connected matroid M is *non-separating* or *contractible* if and only if the contraction M/C is connected. A circuit in a graph is called *induced* if it does not have any chords in the graph. It is straightforward to check that a circuit C in a 2-connected graph G is induced and non-separating if and only if C is a non-separating cocircuit of

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$M^*(G)$ unless $M(G)/C$ is a loop (i.e., G is a cycle plus a single chord). We mostly follow Oxley [9] for terminology. We use $\mathcal{R}^*(M)$ to denote the set of non-separating cocircuits of M. Two sets are said to *meet* when their intersection is non-empty. The following result of Tutte [11] is a fundamental characterization of 3-connected planar graphs.

Theorem 1.1. *Let G be a 3-connected graph. Then G is planar if and only if every edge meets at most (in fact, exactly) two non-separating induced circuits.*

If $M = M(G)$ is a 3-connected graphic matroid, then every edge of G belongs to exactly two non-separating cocircuits. The next result of Bixby and Cunningham [1] generalized Tutte's theorem.

Theorem 1.2. *Let M be a 3-connected binary matroid with $r(M) \geq 3$.*

(i) *Each element of M belongs to at least two non-separating cocircuits.*
(ii) *The family of non-separating cocircuits of M spans the cocycle space of M.*
(iii) *M is graphic if and only if each element of M belongs to exactly two non-separating cocircuits.*

By (iii) above, an element of a 3-connected binary matroid M that belongs to at least three non-separating cocircuits is an obstruction to the matroid being graphic. Denote the set of such elements by $X(M) = \{e \in E(M) : |\{C^* \in \mathcal{R}^*(M) : e \in C^*\}| \geq 3\}$. Then the last part of the theorem can be restated as *a 3-connected binary matroid M is graphic if and only if $X(M) = \emptyset$.*

The next theorem is due to Lemos [4,5]. Part (ii) of the theorem was conjectured by Wu.

Theorem 1.3. *Let M be a 3-connected binary matroid with $r(M) \geq 1$.*

(i) *The family of non-separating cocircuits of M avoiding any element spans a subspace of the cocycle space of M with dimension $r(M) - 1$.*
(ii) *M is graphic if and only if each element of M avoids exactly $r(M) - 1$ non-separating cocircuits.*

In particular, a non-graphic 3-connected matroid M has an element e avoiding at least $r(M)$ non-separating cocircuits. As this element belongs to at least two non-separating cocircuits, it follows that M has at least $r(M) + 2$ non-separating cocircuits. Therefore, the following result is an immediate consequence of the last theorem.

Corollary 1.4. *Let M be a 3-connected binary matroid. Then M is graphic if and only if it contains exactly $r(M) + 1$ non-separating cocircuits.*

An element of a 3-connected binary matroid M that avoids at least $r(M)$ non-separating cocircuits is an obstruction to M being graphic according to Theorem 1.3. Denote the set of such elements by $Y(M) = \{e \in E(M) : |\{C^* \in \mathcal{R}^*(M) : e \notin C^*\}| \geq r(M)\}$. Thus the last part of Theorem 1.3 may be restated as *a 3-connected binary matroid M is graphic if and only if $Y(M) = \emptyset$.*

In this paper, we consider the following two problems for a non-graphic 3-connected binary matroid M.

(i) Give a lower bound on the number of non-separating cocircuits for M.
(ii) Study the sets of obstructions to M being graphic.

Next we state the main results of the paper.

Theorem 1.5. *If M is a 3-connected non-regular binary matroid, then $|X(M) \cap Y(M)| \geq 7$ and $|\mathcal{R}^*(M)| \geq r(M) + 3$.*

For each integer $r \geq 4$, we will construct a 3-connected non-regular binary matroid M having rank r such that $|X(M)| = 7$. Moreover, all the matroids in this infinite family have exactly $r(M) + 3$ non-separating cocircuits. For regular non-graphic matroids, it is possible that $X(M) \cap Y(M) = \emptyset$, and M may have exactly $r(M) + 2$ non-separating cocircuits. We will construct an infinite family of regular matroids M with exactly $r(M) + 2$ non-separating cocircuits.

Theorem 1.6. *If M is a 3-connected non-graphic regular matroid, then*

$$|X(M)| \geq 3.$$

We show that the bound given in this theorem is sharp in the last section of the paper. If M is a matroid and $N = M/e$ for some element e of M, then M is called a *lift* of N. Let M be a 3-connected binary matroid. We define $\delta(M) = |\mathcal{R}^*(M)| - r(M)$. Let $\mathcal{F}$ be a minor-closed class of binary matroids. For $N \in \mathcal{F}$, we define $\mathcal{F}_N$ to be the class of 3-connected corank-preserving lifts of N belonging to $\mathcal{F}$ plus the matroid N. We define $\Delta_{\mathcal{F}}(N) = \min\{\delta(M) : M \in \mathcal{F}_N\}$. The next theorem is a general result that allows one to compute lower bounds on the number of non-separating cocircuits in many interesting classes of matroids.

Theorem 1.7. *Let M and N be 3-connected matroids belonging to a minor-closed class of binary matroids $\mathcal{F}$ such that $r(N) \geq 3$. If N is a minor of M, then*

$$\delta(M) \geq \Delta_{\mathcal{F}}(N).$$

In particular, $|\mathcal{R}^*(M)| \geq r(M) + \Delta_{\mathcal{F}}(N)$.

For some subclasses of non-regular matroids, the next two results show that the bounds in Theorem 1.5 can be improved.

Theorem 1.8. *If M is a 3-connected binary matroid having a minor isomorphic to $PG(r, 2)^*$ for a fixed integer r exceeding one, then*

$$|\mathcal{R}^*(M)| \geq r(M) + \frac{4}{3}(2^{r-2} - 1)(2^{r+1} - 1) + r + 1.$$

Moreover, $|X(M) \cap Y(M)| \geq 2^{r+1} - 1$.

This result is also sharp. We will construct an infinite family of matroids that attain both bounds (with $X(M) \subseteq Y(M)$ for each matroid in this family). Moreover, for each integer s such that $s \geq r(PG(r, 2)^*)$, there is a matroid in this family with rank equal to s. In particular, the lower bound for $|\mathcal{R}^*(M)|$ cannot be improved by any other function involving only $r(M)$ and r.

Theorem 1.9. *If M is a 3-connected cographic matroid with a minor isomorphic to $M^*(K_n)$ for some $n \geq 5$, then*

$$|\mathcal{R}^*(M)| \geq r(M) + \binom{n-1}{3}.$$

Moreover, $|X(M) \cap Y(M)| \geq \binom{n}{2}$.

This result is also sharp. We will construct an infinite family of matroids that attains both bounds (with $X(M) \subseteq Y(M)$ for each matroid in this family).

The rest of the paper is arranged as follows. In Section 2, we give the proofs of our main results. In Section 3, we construct several infinite families of matroids to show that the bounds in our main results are best possible.

2. Proofs of the main results

In this section, we present the proofs of our main results. We will first give several lemmas needed to prove our main results. We begin with some notation. For an element e of a 3-connected binary matroid M, we define

$$\gamma_M(e) = |\{C^* \in \mathcal{R}^*(M) : e \in C^*\}| - 2.$$

By Theorem 1.2, $\gamma_M(e) \geq 0$. We also set

$$\Gamma(M) = \sum_{e\in E(M)} \gamma_M(e) = \left(\sum_{C\in\mathcal{R}^*(M)} |C|\right) - 2|E(M)|.$$

By Theorem 1.2, M is graphic if and only if $\Gamma(M) = 0$. Observe that

$$\Gamma(M) = \sum_{e\in X(M)} \gamma_M(e),$$

and hence $\Gamma(M)$ is an upper bound for $|X(M)|$.

Whittle [12] gave the following consequence of Seymour's splitter theorem [10]. The cosimplification of a matroid M is denoted by $\mathrm{co}(M)$.

Lemma 2.1. *Suppose that N is a 3-connected minor of a 3-connected matroid M. If $r^*(M) > r^*(N)$, then there is an element e of $E(M)$ such that $\mathrm{co}(M \setminus e)$ is 3-connected with an N-minor.*

The next lemma is due to Lemos [4].

Lemma 2.2. *Suppose that e is an element of a 3-connected binary matroid M such that $\mathrm{co}(M\backslash e)$ is 3-connected. Then it is possible to choose the ground set of $\mathrm{co}(M \setminus e)$ such that, for each $C^* \in \mathcal{R}^*(\mathrm{co}(M\backslash e))$, $C^*\Delta X \in \mathcal{R}^*(M)$, where $X = \emptyset, X = \{e\}$, or $X = T^* - e$, for some triad T^* meeting both e and C^*.*

The next lemma is a special case of Lemos [5, Lemma 3.1].

Lemma 2.3. *Suppose that e is an element of a 3-connected binary matroid M such that $\mathrm{co}(M\backslash e)$ is 3-connected. Then $\delta(M) \geq \delta(\mathrm{co}(M\backslash e))$.*

Lemma 2.4. *Let M and N be 3-connected matroids belonging to a minor-closed class of binary matroids $\mathcal{F}$ such that $r(N) \geq 3$. If N is a minor of M, then*

$$|X(M)| \geq \min\{|X(H)| : H \in \mathcal{F}_N\}$$
$$|X(M) \cap Y(M)| \geq \min\{|X(H) \cap Y(H)| : H \in \mathcal{F}_N\}$$
$$\Gamma(M) \geq \min\{\Gamma(H) : H \in \mathcal{F}_N\}.$$

Proof. We prove the result by induction on $r^*(M)$. If $r^*(M) = r^*(N)$, then M belongs to $\mathcal{F}_N$ (see the notation after Theorem 1.6 for the definition of $\mathcal{F}_N$) and the result follows by definition. Assume that $r^*(M) > r^*(N)$. By Lemma 2.1, there is an element e of M such that $\mathrm{co}(M \setminus e)$ is a 3-connected matroid having N as a minor.

By Lemma 2.2, it is possible to choose the ground set of $\mathrm{co}(M\backslash e)$ such that, for each $C^* \in \mathcal{R}^*(\mathrm{co}(M\backslash e))$, $C^*\Delta X \in \mathcal{R}^*(M)$, where $X = \emptyset$, $X = \{e\}$, or $X = T^* - e$, for some triad T^* meeting both e and C^*.

For $f \in X(\mathrm{co}(M \setminus e))$, let $C_1^*, C_2^*, \ldots, C_k^*$ be the non-separating cocircuits of $\mathrm{co}(M\backslash e)$ that contain f. For $i \in \{1, 2, \ldots, k\}$, choose X_i as described in the previous paragraph such that $C_i^*\Delta X_i$ is a non-separating cocircuit of M. Note that $f \in C_i^*\Delta X_i$ unless $f \in X_i$. In this case, there is a triad T^* of M such that $\{e, f\} \subseteq T^*$, say $T^* = \{e, f, f'\}$. Now there are two cases to consider.

Case 1. T^* does not exist.

Hence $f \in X(M)$ and

$$\gamma_M(f) \geq \gamma_{\mathrm{co}(M\backslash e)}(f). \tag{1}$$

Moreover, by Lemos [5, Lemma 3.1], when $f \in Y(\mathrm{co}(M\backslash e))$, we have also that $f \in Y(M)$.

Case 2. T^* exists.

In particular, T^* is unique, as is f'. Thus $C_i^*\Delta X_i$ is a non-separating cocircuit of M that contains f or f'. As T^* is a non-separating cocircuit of M that contains both f and f', it follows that

$$|\{C^* \in \mathcal{R}^*(M) : f \in C^*\}| + |\{C^* \in \mathcal{R}^*(M) : f' \in C^*\}| \geq k + 2$$
$$= |\{C^* \in \mathcal{R}_f^*(\mathrm{co}(M \setminus e)) : f \in C^*\}| + 2$$

and so

$$\gamma_M(f) + \gamma_M(f') \geq \gamma_{\mathrm{co}(M\backslash e)}(f). \tag{2}$$

As $\gamma_{\mathrm{co}(M\backslash e)}(f) \geq 1$, it follows that $\gamma_M(f) \geq 1$ or $\gamma_M(f') \geq 1$. Thus f or f' belongs to $X(M)$. Moreover, by Lemos [5, Lemma 3.1], when $f \in Y(\mathrm{co}(M \setminus e))$, we have also that $\{f', f\} \subseteq Y(M)$.

For both of the cases, we conclude that, for each element f of $X(\mathrm{co}(M\backslash e))$ or $X(\mathrm{co}(M\backslash e)) \cap Y(\mathrm{co}(M\backslash e))$, there is an element f'' belonging to the same series class as f in $M\backslash e$ such that f'' belongs to respectively $X(M)$ or $X(M) \cap Y(M)$. Thus $|X(M)| \geq |X(\mathrm{co}(M\backslash e))|$ and $|X(M) \cap Y(M)| \geq |X(\mathrm{co}(M\backslash e)) \cap Y(\mathrm{co}(M\backslash e))|$. The first two inequalities follow by induction. By (1) and (2), we conclude that, for each series class S of $M \setminus e$,

$$\sum_{g\in S} \gamma_M(g) \geq \gamma_{\mathrm{co}(M\backslash e)}(f),$$

where $f \in S \cap E(\mathrm{co}(M \setminus e))$. Hence $\Gamma(M) \geq \Gamma(\mathrm{co}(M \setminus e))$. The third inequality also follows by induction. □

Let $\mathcal{B}$ and $\mathcal{R}$ denote, respectively, the class of binary matroids and the class of regular matroids.

Lemma 2.5. *If M is a 3-connected binary matroid having a minor isomorphic to $P(r, 2)^*$ for a fixed r exceeding one, then $|X(M) \cap Y(M)| \geq 2^{r+1} - 1$. Moreover,*

$$\Gamma(M) \geq (2^{r+1} - 1)(2^r - 3).$$

Proof. As $PG(r, 2)^*$ has no 3-connected corank-preserving binary lift, it follows that $\mathcal{B}_{PG(r,2)^*} = \{PG(r, 2)^*\}$. By Lemma 2.4,

$$|X(M) \cap Y(M)| \geq |X(PG(r, 2)^*) \cap Y(PG(r, 2)^*)| = |E(PG(r, 2)^*)| = 2^{r+1} - 1,$$
$$\text{and} \quad \Gamma(M) \geq \Gamma(PG(r, 2)^*) = (2^{r+1} - 1)(2^r - 3).$$

This completes the proof of the lemma. □

Proof of Theorem 1.5. The result holds for F_7. Assume that M is not isomorphic to F_7. By [10, (7.6)], F_7 is a splitter of the class of 3-connected binary matroids without an F_7^*-minor. We deduce that M has an F_7^*-minor. As $F_7 = PG(2, 2)$, the result follows from Lemma 2.5. Moreover, as $X(M) \cap Y(M) \neq \emptyset$, we conclude that M has at least $r(M) + 3$ non-separating cocircuits because there is an element e that is contained in at least three non-separating cocircuits and that avoids at least $r(M)$ such cocircuits. □

Let A and B be the partite sets of $K_{3,3}$. We use $K'_{3,3}, K''_{3,3}, K'''_{3,3}$ to denote, respectively, the simple graph obtained by adding one, two, or three edges connecting vertices in A. We use $K'_{3,3+}, K''_{3,3+}, K'''_{3,3+}, K''_{3,3++}$ to denote, respectively, the simple graphs obtained from $K'_{3,3}, K''_{3,3}, K'''_{3,3}, K''_{3,3}$ by adding one, one, one, and two edges connecting vertices in the set B.

Proof of Theorem 1.6. If M is isomorphic to $M^*(K_5)$, then the result follows. Assume that M is not isomorphic to $M^*(K_5)$. By [10, (7.5)], $M^*(K_5)$ is a splitter for the class of regular matroids without a minor isomorphic to $M^*(K_{3,3})$. Therefore, M has a minor isomorphic to $M^*(K_{3,3})$. It is easily verified that the unique 3-connected regular corank-preserving lifts of $M^*(K_{3,3})$ are $M^*(K'_{3,3})$, $M^*(K''_{3,3})$, $M^*(K'''_{3,3})$, $M^*(K'_{3,3+})$, $M^*(K''_{3,3+})$, $M^*(K'''_{3,3+})$, $M^*(K''_{3,3++})$, R_{10}, $M(K_6 \setminus e)$, and $M(K_6)$. Thus we have that $\min\{|X(H)| : H \in \mathcal{R}_{M^*(K_{3,3})}\} = |X(M^*(K'''_{3,3}))| = 3$. The result follows from Lemma 2.4. □

Proposition 2.6. *If M is a 3-connected cographic matroid with a minor isomorphic to $M^*(K_n)$, for some $n \geq 5$, then*

$$|X(M) \cap Y(M)| \geq \binom{n}{2}.$$

Moreover, $\Gamma(M) \geq \frac{n(n-1)(n-4)}{2}$.

Proof. Note that C^* is a non-separating cocircuit of $M^*(K_n)$ if and only if C^* is a contractible circuit of $M(K_n)$, and this is true if and only if C^* is a triangle of K_n. As $M^*(K_n)$ has no 3-connected rank-preserving lift in the class of cographic matroids $\mathcal{F}$, it follows that $\mathcal{F}_{M^*(K_n)} = \{M^*(K_n)\}$. By Lemma 2.4

$$|X(M) \cap Y(M)| \geq |X(M^*(K_n)) \cap Y(M^*(K_n))| = |E(M^*(K_n))| = \binom{n}{2},$$

$$\text{and} \quad \Gamma(M) \geq \Gamma(M^*(K_n)) = \frac{n(n-1)(n-4)}{2}.$$

This completes the proof of the proposition. □

Proof of Theorem 1.7. If $r^*(M) = r^*(N)$, then $M = N$, or M is a corank-preserving lift of N and the result follows by definition. Assume that $r^*(M) > r^*(N)$. By Lemma 2.1, M has an element e such that $\mathrm{co}(M \setminus e)$ is a 3-connected matroid having a minor isomorphic to N. By induction, $\delta(\mathrm{co}(M \setminus e)) \geq \Delta_{\mathcal{F}}(N)$. The result follows by Lemma 2.3. □

Note that Theorem 1.7 and Lemma 2.4 reduces the problem of finding lower bounds for $|X(M)|$, $|X(M) \cap Y(M)|$, $\Gamma(M)$, and $\delta(M)$ for matroids M in $\mathcal{F}$ having a fixed minor N in the class to a computation of such numbers for the 3-connected corank-preserving lifts on N that are in the class. These numbers can be computed by a computer, for example.

Proof of Theorem 1.8. By Theorem 1.7,

$$|\mathcal{R}^*(M)| \geq r(M) + \Delta_{\mathcal{F}}(PG(r, 2)^*).$$

As $PG(r, 2)^*$ has no 3-connected lift in the class of binary matroids, it follows that $\Delta_{\mathcal{F}}(PG(r, 2)^*) = \delta(PG(r, 2)^*) = |\mathcal{R}^*(PG(r, 2)^*)| - r(PG(r, 2)^*)$. But C^* is a non-separating cocircuit of $PG(r, 2)^*$ if and only if C^* is a contractible circuit of $PG(r, 2)$ if and only if C^* is a triangle of $PG(r, 2)$. Therefore,

$$|\mathcal{R}^*(PG(r, 2)^*)| = \frac{(2^r - 1)(2^{r+1} - 1)}{3}.$$

The first part of the result follows because $r(PG(r, 2)^*) = 2^{r+1} - r - 2$. Now the theorem follows by Lemma 2.5. □

Proof of Theorem 1.9. By Theorem 1.7,

$$|\mathcal{R}^*(M)| \geq r(M) + \Delta_{\mathcal{F}}(M^*(K_n)).$$

As $M^*(K_n)$ has no 3-connected lift in the class of cographic matroids, it follows that $\Delta_{\mathcal{F}}(M^*(K_n)) = \delta(M^*(K_n)) = |\mathcal{R}^*(M^*(K_n))| - r(M^*(K_n))$. But C^* is a non-separating cocircuit of $M^*(K_n)$ if and only if C^* is a contractible circuit of $M(K_n)$ if and only if C^* is a triangle of K_n. Therefore,

$$|\mathcal{R}^*(M^*(K_n))| = \binom{n}{3}.$$

The first part of the result follows because $r(M^*(K_n)) = |E(K_n)| - [|V(K_n)| - 1]$. Now the theorem follows by Proposition 2.6. □

3. Extremal examples

In this section, we describe the construction of several families of extremal examples. The construction is the same; the only difference is as regards the matroid with which we start. If the starting matroid has rank r, then the family contains a matroid with rank s, for each $s \geq r$. The bounds of the form $r(M) + k$ cannot be replaced by any other function involving only $r(M)$.

In the following example instead of using the dual of the operation of generalized parallel connection, we consider contractible circuits. Note that a set is a non-separating cocircuit of M^* if and only if it is a contractible circuit of M.

Lemma 3.1. *Let N be a 3-connected binary matroid having a contractible triangle T. Suppose that T is also a triangle of a rank-k wheel W different from the rim, for $k \geq 3$, satisfying $E(N) \cap E(W) = T$. If $M = P_T(N, W)$ is the generalized parallel connection of N and W along T, then*

$$\mathcal{R}^*(M^*) = [\mathcal{R}^*(N^*) - \{T\}] \cup \{R, T_1, T_2, \dots, T_{k-1}\}, \tag{3}$$

where R is the rim of W and $T_1, T_2, \dots, T_{k-1}$ are the triangles of W different from T and R. Moreover, $|\mathcal{R}^*(M)| - r(M) = |\mathcal{R}^*(N)| - r(N)$ and

$$\gamma_M(e) = \begin{cases} \gamma_N(e) & \text{when } e \in E(N) \\ 0 & \text{when } e \in E(W) - T. \end{cases}$$

Therefore, $X(M) = X(N)$ and $\Gamma(M) = \Gamma(N)$.

Proof. Assume that $T = \{a, b, c\}$, where a and b are spokes of W and c belongs to the rim of W. First, we consider the contractible circuits C of M that contain an element e of $E(W) - T$. We establish that C can only be one of the circuits $R, T_1, T_2, \dots, T_{k-1}$. In particular, $\gamma_M(e) = 0$, for $e \in E(W) - T$.

First suppose that e is a spoke of W other than a and b. We show next that e belongs to exactly two contractible circuits of M, namely, the triangles of W that contain e. Let T^* be the triad of W and so of M that contains e, say $T^* = \{e, f_1, f_2\}$. For $i \in \{1, 2\}$, $T_i' = \{e, f_i, e_i\}$ is a triangle of W and so of M, for some spoke e_i of W. If C is a contractible circuit of M that contains e, then, by orthogonality, C contains f_1 or f_2, say f_1. As e_1 is a loop of $M/\{e, f_1\}$, it follows that $e_1 \in C$ and so $C = T_1'$.

Now we assume that e belongs to the rim of W and $e \neq c$. Then e belongs to exactly two contractible circuits of M, namely, R and the triangle of W different from R that contains e. Indeed, let C be a contractible circuit of M that contains e. If C contains a spoke of W different from a and b, then, by the previous paragraph, C is a triangle of W and so the unique triangle of W that contains e and is different from R. Assume that C does not contain a spoke of W different from a and b. By orthogonality, C contains all of the rim R of W with the possible exception of the element c. The set of spokes S of W is a parallel class and c is a loop of $N/(R - c)$. Therefore, $C = R$.

Now, we need to find all contractible circuits C of M such that $C \cap [E(W) - T] = \emptyset$ and so $C \subseteq E(N)$. In particular, C is a circuit of N. Note that $C \neq T$. Otherwise,

$$\begin{aligned} r_{M/T}(E(N) - T) + r_{M/T}(E(W) - T) &= [r_M(E(N)) - 2] + [r_M(E(W)) - 2] \\ &= [r(N) + r(W)] - 4 \\ &= [r(M) + 2] - 4 \\ &= r(M) - 2 \\ &= r(M/T). \end{aligned}$$

Therefore, $\{E(N) - T, E(W) - T\}$ is a 1-separation of M/T; a contradiction.

Case 1. $C \cap T = \emptyset$.

First, we prove that $r_{N/C}(T) = r_{M/C}(T) = 2$. If $r_{M/C}(T) < 2$, then $r_{M/C}(T) = 1$ because no element of M/C is a loop of M/C. Moreover, T is contained in a parallel class of M/C. If X is a 2-element subset of T, then $X \cup Y$ is a circuit of M, for some $Y \subseteq C$. Hence $(X \cup Y)\Delta T = (T - X) \cup Y$ contains a circuit D such that $T - X \subseteq D$ and $D - (T - X) \subseteq Y \subseteq C$. Thus $T - X$ contains a circuit of M/C. That is, the unique element of $T - X$ is a loop of M/C; a contradiction. Thus $r_{N/C}(T) = r_{M/C}(T) = 2$. So M/C is the generalized parallel connection of N/C with W along T. Now it is straightforward to check that N/C is connected if and only if M/C is connected.

Case 2. $C \cap T \neq \emptyset$, say $C \cap F = \{e\}$.

In this case, M/C is the generalized parallel connection of N/C and W/e along the 2-element circuit $T - e$. So again, M/C is connected if and only if N/C is connected.

From Cases 1 and 2, we conclude that for $C \subseteq E(N)$, $C \neq T$, C is a non-separating cocircuit of M^* if and only if C is a non-separating cocircuit of N^*. Thus (3) follows. This also implies that $\gamma_M(e) = \gamma_N(e)$, when $e \in E(N)$. Indeed, if $e \in T$, then just one of the cocircuits $R, T_1, T_2, \dots, T_{k-1}$ contains e; this compensates for the loss of T when one goes from the computation of $\gamma_N(e)$ to the computation of $\gamma_M(e)$. $\square$

Now, we present some consequences of this lemma. We construct a family of extremal examples having infinitely many matroids for each result listed in the introduction. We start with a small extremal example (which is easily checked), and for each r bigger that the rank of this extremal example, we construct another one consecutively.

If $N = F_7$, then the dual of M is an extremal example for Theorem 1.5 because F_7^* is an extremal example.

If $N = M(K'''_{3,3})$, then M is a 3-connected graphic matroid (use for T a triangle containing a vertex of degree 3 of $K'''_{3,3}$). The dual of M is an extremal example for Theorem 1.6.

If $N = PG(r, 2)$, for some $r \geq 2$, then M is a 3-connected non-regular binary matroid. Moreover, the dual of M is an extremal example for Theorem 1.8 and Lemma 2.5 because $PG(r, 2)^*$ is an extremal example.

If $N = M(K_n)$, for some $n \geq 4$, then M is a 3-connected graphic matroid. Moreover, the dual of M is an extremal example for Theorem 1.9 and Proposition 2.6 because $M^*(K_n)$ is an extremal example.

Thus each one of the bounds given in this paper is attained by an infinite family of non-isomorphic matroids.

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