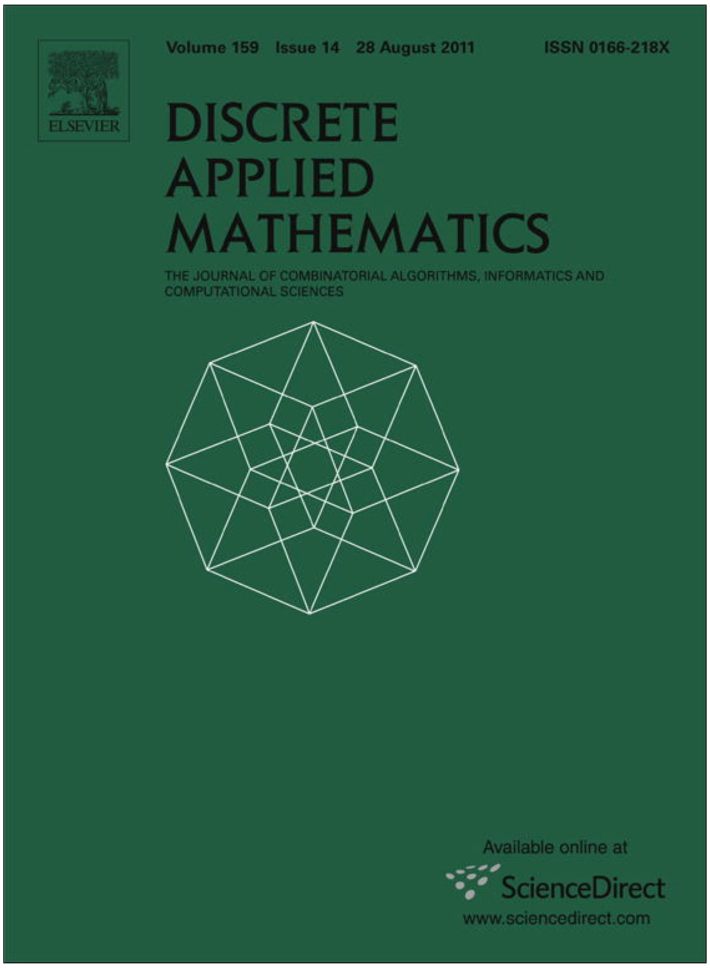
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Note

Degree sequences and graphs with disjoint spanning trees

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ABSTRACT

The design of an n processor network with a given number of connections from each processor and with a desirable strength of the network can be modeled as a degree sequence realization problem with certain desirable graphical properties. A nonincreasing sequence $d = (d_1, d_2, \ldots, d_n)$ is graphic if there is a simple graph G with degree sequence d. In this paper, it is proved that for a positive integer k, a graphic sequence d has a simple realization G which has k edge-disjoint spanning trees if and only if either both $n = 1$ and $d_1 = 0$, or $n \geq 2$ and both $d_n \geq k$ and $\sum_{i=1}^{n} d_i \geq 2k(n-1)$.

1. Introduction

We consider the problem of designing networks with n processors $v_1, v_2, \ldots, v_n$ such that, for a given sequence of positive integers $d_1, d_2, \ldots, d_n$, it is expected that each processor v_i will be connected to other processors by d_i connections. It is further expected that such networks will have certain levels of strengths. This problem can be modeled as the problem of determining whether a (graphical) degree sequence has realizations with certain graphical properties. Motivated by the research in [4], we shall consider the strength of the graph as the property of having k edge-spanning trees.

This paper studies finite and undirected graphs without loops. Undefined terms can be found in [2]. In particular, $\omega(G)$ denotes the number of components of a graph G. For a vertex $v \in V(G)$ and a subgraph K of G, $d_K(v)$ is the number of vertices in K that are adjacent to v in G. If $X \subseteq E(G)$, then $G[X]$ is the subgraph of G induced by the edge subset X, and $G(X)$ is the spanning subgraph of G with edge set X. A graph G is *nontrivial* if $E(G) \neq \emptyset$. A sequence $d = (d_1, d_2, \ldots, d_n)$ is *nonincreasing* if $d_1 \geq d_2 \geq \cdots \geq d_n$. A sequence $d = (d_1, d_2, \ldots, d_n)$ is *graphic* if there is a simple graph G with degree sequence d. In this case, this graph G is a *realization* of d. We will also call G a *d-realization*.

Many researchers have been investigating graphic degree sequences that have a realization with certain graphical properties. See [1,5–7,12–14], among others. An excellent and resourceful survey by Li can be found in [10].

In this paper, we focus on the investigation of graphic sequences that have realizations with many edge-disjoint spanning trees.

In Section 2, we develop some useful properties related to graphs with at least k edge-disjoint spanning trees. In Section 3, we present a proof for the following characterization of graphic sequences with realizations having k edge-disjoint spanning trees.

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Theorem 1.1. *A nonincreasing graphic sequence $d = (d_1, d_2, \ldots, d_n)$ has a realization G with k edge-disjoint spanning trees if and only if either $n = 1$ and $d_1 = 0$, or $n \ge 2$ and both of the following hold:*

(i) $d_n \ge k$.
(ii) $\sum_{i=1}^{n} d_i \ge 2k(n-1)$.

2. Properties of graphs with *k* edge-disjoint spanning trees

Let G be a graph, and $k \ge 2$ be an integer. Let $\tau(G)$ denote the number of edge-disjoint spanning trees of G, and $\mathcal{T}_k$ the set of all graphs with $\tau(G) \ge k$. By definition, $K_1 \in \mathcal{T}_k$, for any integer $k > 0$. In this section, we summarize and develop some useful properties on $\mathcal{T}_k$, some of which were first introduced in [11], and are later extended to matroids in [8,9].

For an edge subset $X \subset E(G)$, a *contraction* of G, denoted by G/X, is the graph obtained first from G by identifying the two ends of each edge in X, and then deleting all the resulting loops. When $X = \{e\}$, we use G/e for $G/\{e\}$. Moreover, we define $G/\emptyset = G$.

Proposition 2.1 (*Liu et al., Lemma 2.1 in [11]*)**.** *For any integer k, $\mathcal{T}_k$ is a family of connected graphs such that each of the following holds.*

(C1) $K_1 \in \mathcal{T}_k$.
(C2) *If $e \in E(G)$ and if $G \in \mathcal{T}_k$, then $G/e \in \mathcal{T}_k$.*
(C3) *If H is a subgraph of G, and if $H, G/H \in \mathcal{T}_k$, then $G \in \mathcal{T}_k$.*
(C4) *If H_1 and H_2 are two subgraphs of G such that $H_1, H_2 \in \mathcal{T}_k$ and $V(H_1) \cap V(H_2) \ne \emptyset$, then $H_1 \cup H_2 \in \mathcal{T}_k$.*

Define the density of a subgraph H of G with $|V(H)| > 1$ as follows:

$$d(H) = \frac{|E(H)|}{|V(H)| - 1}, \quad \text{if } |V(H)| > 1.$$

Theorem 2.2 (*Yao et al., Theorem 2.4 in [15]*)**.** *Let G be a multigraph. If $d(G) \ge k$, then G has a nontrivial subgraph H such that $H \in \mathcal{T}_k$.*

Let G be a nontrivial connected graph. For any positive integer r, a nontrivial subgraph H of G is $\mathcal{T}_r$*-maximal* if both $H \in \mathcal{T}_r$ and H has no proper subgraph K of G, such that $K \in \mathcal{T}_r$. A $\mathcal{T}_r$-maximal subgraph H of G is called an r*-region* if $r = \tau(H)$. Define $\overline{\tau}(G) = \max\{r : G \text{ has a subgraph as an } r\text{-region}\}$.

Lemma 2.3 (*Liu et al., Lemma 2.3 in [11]*)**.** *Let $r, r' > 0$ be integers, H, H' be an r-region and an r'-region of G, respectively. Then exactly one of the following must hold:*

(i) $V(H) \cap V(H') = \emptyset$,
(ii) *$r' = r$ and $H = H'$,*
(iii) *$r' > r$ and H is a nonspanning subgraph of H',*
(iv) *$r' < r$ and H contains H' as a non-spanning subgraph.*

Theorem 2.4 (*Theorem 2.4 in [11]*)**.** *Let G be a nontrivial connected graph. Then*

(a) *there exists a positive integer m, and an m-tuple $(i_1, i_2, \ldots, i_m)$ of positive integers with*

$$\tau(G) = i_1 < i_2 < \cdots < i_m = \overline{\tau}(G),$$

and a sequence of edge subsets

$$E_m \subset \cdots \subset E_2 \subset E_1 = E(G),$$

such that each component of the induced subgraphs $G[E_j]$ is an r-region of G for some r with $r \ge i_j$, $(1 \le j \le m)$, and such that at least one component H in $G[E_j]$ is an i_j-region of G;
(b) *if H is a subgraph of G with $\tau(H) \ge i_j$, then $E(H) \subseteq E_j$;*
(c) *the integer m and the sequence of edge subsets are uniquely determined by G.*

Lemma 2.5. *Let $k \ge 1$ be an integer, G be a graph with $\overline{\tau}(G) \ge k$. Then each of the following statements holds.*

(i) *The graph G has a unique edge subset $X_k \subseteq E(G)$, such that every component H of $G[X_k]$ is a $\mathcal{T}_k$-maximal subgraph. In particular, $G \notin \mathcal{T}_k$ if and only if $E(G) \ne X_k$.*
(ii) *If $G \notin \mathcal{T}_k$, then G/X_k contains no nontrivial subgraph H' with $\tau(H') \ge k$. (G/X_k is called the $(\tau \ge k)$-reduction of G.)*
(iii) *If $G \notin \mathcal{T}_k$, then $d(H') < k$ for any nontrivial subgraph H' of G/X_k.*

Proof. If $G \in \mathcal{T}_k$, then $X_k = E(G)$. Hence we assume that $G \notin \mathcal{T}_k$. Since $\tau(G) < k \le \overline{\tau}(G)$, there exists an integer j such that $i_{j-1} < k \le i_j$ by Theorem 2.4(a). Let $X_k = E_{i_j}$. Then each component H of $G[X_k]$ is a $\mathcal{T}_k$-maximal subgraph. By Theorem 2.4(c), X_k is unique. Thus part (i) holds.

To prove part (ii), we argue by contradiction. We assume G/X_k contains nontrivial subgraph H' with $\tau(H') \geq k$ and $V(H') = \{v_1, v_2, \ldots, v_h\}$ with $h \geq 2$. Without loss of generality, suppose the pre-image of v_i in G is H_i, and H_i is nontrivial for $1 \leq i \leq t$ and is trivial for $t+1 \leq i \leq h$. We will prove that $\tau(G') \geq k$, where $G' = G[\cup_{i=1}^{h} V(H_i)]$. By induction, if $t = 1$, then $G'/H_1 = H'$, and $H', H_1 \in \mathcal{T}_k$. Therefore, $G' \in \mathcal{T}_k$ by Proposition 2.1(C3). Assume it's true for all $t \leq s$. For $t = s+1$, consider G'/H_{s+1}. Then $G'/H_{s+1} \in \mathcal{T}_k$ by induction hypothesis. Thus $G' \in \mathcal{T}_k$ by Proposition 2.1(C3), and so part (ii) holds.

We argue by contradiction to prove (iii). Assume that $d(H') \geq k$. Then $|E(H')| \geq k(|V(H')| - 1)$. By Theorem 2.2, H' has a nontrivial subgraph H'' such that $H'' \in \mathcal{T}_k$. Note that H'' is also a nontrivial subgraph of G/X_k, contrary to part (ii). □

Notice that $d(G) \geq k$ implies $\overline{\tau}(G) \geq k$ by Theorem 2.2. Therefore if $d(G) \geq k$, then the unique edge subset X_k defined in Lemma 2.5(i) exists.

Lemma 2.6. *Let G be a graph satisfying $d(G) \geq k$ and let $X_k \subset E(G)$ be the edge subset defined in Lemma 2.5 (i). If $G[X_k]$ has at least two components, then for any nontrivial component H of $G[X_k]$, both $d(H) \geq k$, and $G[X_k]$ has at least one component H with $d(H) > k$.*

Proof. For any nontrivial component H of $G[X_k]$, by Lemma 2.5(i), $H \in \mathcal{T}_k$. Thus $|E(H)| \geq k(|V(H)| - 1)$, and so $d(H) \geq k$.

Suppose $G[X_k]$ has c components $H_1, H_2, \ldots, H_c$ with $c \geq 2$. By contradiction, assume $d(H) = k$ for any nontrivial component H of $G[X_k]$. Let $x = |E(G) - X_k|$. Then $|E(H_i)| = k(|V(H_i)| - 1)$ for any $1 \leq i \leq c$ and

$$|E(G)| = \sum_{i=1}^{c} |E(H_i)| + x = \sum_{i=1}^{c} (k|V(H_i)| - k) + x = k \sum_{i=1}^{c} |V(H_i)| - kc + x = k|V(G)| - kc + x.$$

Therefore, $x = |E(G)| - k|V(G)| + kc \geq k(|V(G)| - 1) - k|V(G)| + kc = k(c-1)$.

Let $G' = G/G[X_k]$. Then G' is a multigraph with $|V(G')| = c > 1$ and $|E(G')| = x$. Therefore, $d(G') \geq k$, contrary to Lemma 2.5 (iii). Hence $G[X_k]$ has at least one component H_i such that $d(H_i) > k$. □

Let H_1, H_2 be two subgraphs of a graph G. Define

$$E(H_1, H_2) = \{e = uv \in E(G) \ : \ u \in V(H_1), v \in V(H_2)\}.$$

Let $\alpha'(G)$ denote the size of a maximum matching of G and $\chi'(G)$ the edge chromatic number of G. Then we have the well-known Vizing Theorem.

Theorem 2.7 (*Theorem 17.4 of [2]*). *For any simple graph G on n vertices, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1 \leq n$.*

Since the set of edges of each color is a matching of G, we have the following observation.

Observation 2.8. *For any graph G, $|E(G)| \leq \chi'(G)\alpha'(G)$.*

Lemma 2.9. *For any simple graph G with $|E(G)| \geq 1$, $\alpha'(G) \geq \left\lceil \frac{\tau(G)}{2} \right\rceil$.*

Proof. We argue by induction on $n = |V(G)|$. It is trivial if $n = 2$. Assume that lemma holds for smaller n and $n \geq 3$.

Suppose $\tau(G) = k > 0$. Then for any $v \in V(G)$, $d(v) \geq k$. Assume first that G has a vertex v_0 of degree k. Let $G' = G - v_0$. Since $d_G(v_0) = k$ and $\tau(G) = k$, v_0 is not a cut-vertex of G. Therefore, G' is connected and $\tau(G') \geq \tau(G) = k$. By induction,

$$\alpha'(G) \geq \alpha'(G') \geq \left\lceil \frac{k}{2} \right\rceil = \left\lceil \frac{\tau(G)}{2} \right\rceil.$$

Hence now we assume that $\delta(G) \geq k+1$. Then by Observation 2.8 and Theorem 2.7,

$$n\alpha'(G) \geq \chi'(G')\alpha'(G) \geq |E(G)| \geq \frac{n}{2}(k+1).$$

Therefore, $\alpha'(G) \geq \frac{k+1}{2} \geq \left\lceil \frac{k}{2} \right\rceil$. □

Following the terminology in [3], the *strength* $\eta(G)$ is defined as

$$\eta(G) = \min\{d(G/X) : |V(X)| < |V(G)|\}.$$

As indicated in Corollary 5 of [3], $\tau(G) = \lfloor \eta(G) \rfloor$.

A subgraph H of G is *η-maximal* if for any subgraph H' of G that properly contains H, $\eta(H') < \eta(H)$.

Theorem 2.10 (*Theorem 6 in [3], Corollary 3.6 in [9]*). *For any integer k with $d(G) \geq k$, either $E(G)$ is the union of k edge-disjoint spanning trees, or G has a unique edge subset X such that $H = G[X]$ is η-maximal with $\eta(H) > k$.*

For a connected graph G with $\tau(G) \geq k$, define $E_k(G) = \{e \in E(G) : \tau(G - e) \geq k\}$.

Theorem 2.11 (*Theorem 4.2 in [9]*). *Let G be a connected graph with $\tau(G) \geq k$. Then $E_k(G) = E(G)$ if and only if $\eta(G) > k$.*

Lemma 2.12. *Let G be a simple graph and let $X_k \subset E(G)$ be the edge subset defined in Lemma 2.5 (i). If H' and H'' are two components of $G(X_k)$, then each of the following holds.*

(i) $|E(H', H'')| < k$.
(ii) *If $d(H') > k$, then there exists $K \subseteq H'$ such that $d(K) > k$ and $\tau(K - e) \geq k$ for any $e \in E(K)$.*
(iii) *If $d(H') > k$, then there exists $e' \in E(H')$ such that $\tau(H' - e') \geq k$, and $E(G) - X_k$ has at most one edge joining the ends of e' to H''.*

Proof. By Lemma 2.5(i), both H' and H'' are $\mathcal{T}_k$-maximal subgraphs of G.

Let v', v'' denote the two vertices in $G/(H' \cup H'')$ onto which H' and H'' are contracted, respectively. Let $G' = G[V(H') \cup V(H'')]$. If $|E(H', H'')| = h \geq k$, then $L' = G'/(H' \cup H'')[\{v', v''\}] \cong hK_2 \in \mathcal{T}_k$. As $H', L' \in \mathcal{T}_k$, it follows by Proposition 2.1(C3) that $G'/H'' \in \mathcal{T}_k$. Note that $H'' \in \mathcal{T}_k$, it follows by Proposition 2.1(C3) again that $G' = G[V(H') \cup V(H'')] \in \mathcal{T}_k$, contrary to the assumption that H' and H'' are $\mathcal{T}_k$-maximal subgraphs of G. Hence we must have $|E(H', H'')| < k$, and so (i) follows.

Part (ii) follows from Theorems 2.10 and 2.11 directly.

By Lemma 2.9 and part (ii), $\alpha'(K) \geq \lceil \frac{k}{2} \rceil$. Let M be a matching of K of size $\lceil \frac{k}{2} \rceil$. Then for any $e' \in M$, $K - e' \in \mathcal{T}_k$ by (ii). Since $e' \in E(K)$, $(H' - e')/(K - e') = H'/K$. By Proposition 2.1(C2), $(H' - e')/(K - e') \in \mathcal{T}_k$. Therefore, $H' - e' \in \mathcal{T}_k$ by Proposition 2.1(C3). If for any $e' \in M \subset E(H')$ there are at least two edges joining the ends of e' to H'', then $|E(H', H'')| \geq |E(K, H'')| \geq 2 \lceil \frac{k}{2} \rceil \geq k$, contrary to (i). Hence this proves (iii). □

Lemma 2.13. *Let G be a nontrivial graph with $\tau(G) \geq k$. If $d(G) = k$, then for any nontrivial subgraph H of G, $d(H) \leq k$. Moreover, if $\tau(H) \geq k$, then $d(H) = k$.*

Proof. Since $\tau(G) \geq k$ and $|E(G)| = k(|V(G)| - 1)$, $\tau(G) = k$ and $E(G)$ is a union of k edge-disjoint spanning trees. Let $T_1, T_2, \ldots, T_k$ be edge-disjoint spanning trees of G. Then for any nontrivial subgraph H of G, $|E(H) \cap E(T_i)| \leq |V(H)| - 1$, $1 \leq i \leq k$. Therefore,

$$|E(H)| = |E(H) \cap (\cup_{i=1}^{k} E(T_i))| = \sum_{i=1}^{k} |E(H) \cap E(T_i)| \leq k(|V(H)| - 1).$$

Thus $d(H) \leq k$. If $\tau(H) \geq k$, then $|E(H)| \geq k(|V(H)| - 1)$ and so $d(H) \geq k$. This, together with $d(H) \leq k$, implies $d(H) = k$. □

3. Characterizations of graphic sequences with realizations having k edge-disjoint spanning trees

We present the main result of the paper in this section, which is Theorem 1.1 restated here.

Theorem 3.1. *Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing graphic sequence. Then d has a realization G in $\mathcal{T}_k$ if and only if either $n = 1$ and $d_1 = 0$, or $n > 1$ and each of the following statements holds.*

(i) $d_n \geq k$,
(ii) $\sum_{i=1}^{n} d_i \geq 2k(n - 1)$.

Proof. The case when $n = 1$ is trivial and so we shall assume that $n > 1$. If $G \in \mathcal{T}_k$, $2k(|V(G)| - 1) \leq 2|E(G)| = \sum_{i=1}^{n} d_i$ and each vertex has degree at least k. This proves the necessity.

We now prove the sufficiency. Assume d is a nonincreasing graphic sequence satisfying both Theorem 3.1 (i) and (ii). We argue by contradiction and assume that

$$\text{every } d\text{-realization } G \text{ is not in } \mathcal{T}_k. \tag{1}$$

Suppose G is a d-realization. By (1), $G \notin \mathcal{T}_k$, and so by Lemma 2.5 (i), G has a unique edge subset $X_k \subseteq E(G)$ such that each component of $G[X_k]$ is a $\mathcal{T}_k$-maximal subgraph. Let $X = E(G) - X_k$. Since $G \notin \mathcal{T}_k$, $X \neq \emptyset$. Suppose $G - X$ has c components, $H_1, H_2, \ldots, H_c$, which are so labeled that $d(H_1) \geq d(H_2) \geq \cdots \geq d(H_t) \geq k$, and that $H_j = K_1$ for $j = t + 1, \ldots, c$. Define

$$\mathcal{F}_1(G) = \{H_i : d(H_i) > k\} \quad \text{and} \quad \mathcal{F}_2(G) = \{H_i : d(H_i) = k\}.$$

Then $|\mathcal{F}_1(G)| + |\mathcal{F}_2(G)| = t$.

Claim 1: If every d-realization is not in $\mathcal{T}_k$, then there exists a d-realization G such that $|\mathcal{F}_1(G)| = 1$.

By contradiction, suppose that for any d-realization G, $|\mathcal{F}_1(G)| \geq 2$. Choose a d-realization G such that

$$\omega(G - X) \text{ is minimized,} \tag{2}$$

and among all the d-realizations G satisfying (2), we further choose G so that

$$|X| \text{ is maximized.} \tag{3}$$

As $|\mathcal{F}_1(G)| \geq 2$, we have $d(H_1), d(H_2) > k$. By Lemma 2.12(iii), there exist $e_1 = u_1v_1 \in E(H_1)$ and $e_2 = u_2v_2 \in E(H_2)$ such that $H_1 - e_1, H_2 - e_2 \in \mathcal{T}_k$, and there exists at most one edge in X joining the ends of e_1 and e_2. Without loss of generality, assume $u_1u_2, v_1v_2 \notin E(G)$ and let

$$G_1 = (G - \{u_1v_1, u_2v_2\}) \cup \{u_1u_2, v_1v_2\} \quad \text{and} \quad X_1 = X \cup \{u_1u_2, v_1v_2\}. \tag{4}$$

Then by the choice of these edges u_1u_2, v_1v_2, G_1 is also a d-realization. By assumption, $G_1 \notin \mathcal{T}_k$ and $|\mathcal{F}_1(G_1)| \geq 2$. Since $G_1 - X_1 = (H_1 - u_1v_1) \cup (H_2 - u_2v_2) \cup H_3 \cup \cdots \cup H_c$ and since each component of $G_1 - X_1$ is in $\mathcal{T}_k$, it follows by (2) that X_1 is the unique subset of $E(G_1)$ such that $\omega(G_1 - X_1) = \omega(G - X) = c$ with each component of $G_1 - X_1$ being a $\mathcal{T}_k$-maximal subgraph. Now we have $|X_1| = |X| + 2$, contrary to (3). Thus Claim 1 holds.

By Lemma 2.6, for any graph G', either $G' \in \mathcal{T}_k$ or $|\mathcal{F}_1(G')| \geq 1$. Now we prove the theorem by contradiction. Suppose for every d-realization G, $G \notin \mathcal{T}_k$. Then by Claim 1, there exists G such that $|\mathcal{F}_1(G)| = 1$. Thus we can choose a d-realization G satisfying

$$|\mathcal{F}_1(G)| = 1 \quad \text{with } |V(H_1)| \text{ maximized.} \tag{5}$$

And subject to (5), we further choose G such that

$$|X| \text{ is maximized.} \tag{6}$$

We consider the following cases.

Case 1: $t \geq 2$. Thus $H_2 \neq K_1$.

By Lemma 2.12 (iii), there exist $e_1 \in E(H_1)$, $e_2 \in E(H_2)$ such that there is at most one edge in G joining e_1 and e_2 and $H_1 - e_1 \in \mathcal{T}_k$. Define G_1 and X_1 as in (4).

Since $d(H_2 - e_2) < k$, $H_2 - e_2$ is no longer in $\mathcal{T}_k$. Let $\mathcal{T}_k$-maximal subgraphs of $G_1[(H_1 - e_1) \cup (H_2 - e_2)]$ be $H_{1,2}, H_{2,1}, \ldots, H_{2,t_2}$ where $H_1 - e_1 \subseteq H_{1,2}$ and $H_{2,1} \cdots H_{2,t_2} \subseteq H_2 - e_2$. For each $H_{2,i}$, since $d(H_2) = k$ and $H_{2,i} \subseteq H_2$, by Lemma 2.13 either $d(H_{2,i}) = k$ or $H_{2,i} = K_1$. Notice that $G/(H_1 \cup H_2) = G_1/[(H_1 - e_1) \cup (H_2 - e_2)]$. Therefore, $H_{1,2}, H_{2,1}, \ldots, H_{2,t_2}, H_3, \ldots, H_c$ are $\mathcal{T}_k$-maximal subgraphs of G_1. By (5) and $\mathcal{F}_1(G_1) = \{H_{1,2}\}$, $H_{1,2} = H_1 - e_1$.

Let X' be the edge subset of G_1 such that $G_1 - X' = H_{1,2} \cup H_{2,1} \cup \cdots \cup H_{2,t_2} \cup H_3 \cup \cdots H_c$. Then $X \neq X_1$ and $X \subset X_1 \subset X'$, contrary to (6).

Case 2: $t = 1$, and so $H_2 = K_1$.

In this case, if $c = 2$, then by Theorem 3.1(i), there must be at least k edges between H_1 and H_2. Since $H_1 \in \mathcal{T}_k$, it follows that $G \in \mathcal{T}_k$, contrary to (1). Hence we must have $c \geq 3$.

For $i \geq 2$, denote $V(H_i) = \{x_i\}$. Note that for any $H_i = K_1$, there exists an $H_j = K_1$ such that $e = x_ix_j \in X$. For otherwise, x_i must only be adjacent to the vertices in H_1. By Theorem 3.1 (i), $|E(H_i, H_1)| \geq k$, contrary to Lemma 2.12 (i). Without loss of generality, we assume $x_2x_3 \in X$. By Lemma 2.12 (ii), there exists a nontrivial subgraph $K \subseteq H_1$ such that $K - e \in \mathcal{T}_k$ for any $e \in E(K)$.

Claim 2: There exists $e' = uv \in E(K)$ such that $ux_2, vx_3 \notin E(G)$.

In order to present the proof, we define

$$B_1 = \{v \in V(K) : vx_2, vx_3 \notin E(G)\}, \qquad B_2 = \{v \in V(K) : vx_2 \in E(G),\ vx_3 \notin E(G)\},$$
$$B_3 = \{v \in V(K) : vx_2 \notin E(G),\ vx_3 \in E(G)\}, \qquad B_4 = \{v \in V(K) : vx_2, vx_3 \in E(G)\}$$

and let $N(B_1) = \{v \in V(K) : \exists u \in B_1 \text{ such that } uv \in E(K)\}$. Note that by definition, we have

$$V(K) = B_1 \cup B_2 \cup B_3 \cup B_4. \tag{7}$$

If $B_1 = \emptyset$, then $N(B_2) \cup N(B_3) \subseteq B_4$, forcing $|B_4| \geq k - 1$, and so x_2 will have at least k edges joining K, contrary to $x_2 \notin V(H_1)$. Hence $B_1 \neq \emptyset$. If $E(G[B_1]) \neq \emptyset$, then Claim 2 holds. Thus we may assume that $E(G[B_1]) = \emptyset$. It follows that $N(B_1) \cap B_1 = \emptyset$.

Firstly, we shall show that

$$N(B_1) \cap [B_2 \cup B_3] \neq \emptyset. \tag{8}$$

If (8) fails, then by (7), $N(B_1) \subseteq B_4$. Since $K \in \mathcal{T}_k$, for any vertex $v \in B_1$, $d_K(v) \geq k$. Therefore, $|B_4| \geq |N(B_1)| \geq k$. But then by definition of B_4, $|E(H_1, H_2)| \geq |E(B_4, x_2)| = |B_4| \geq k$, contrary to Lemma 2.12 (i). This verifies (8).

By (8), we first assume that there exists $v \in N(B_1) \cap B_2$. Thus there exists $u \in B_1$ such that $uv \in E(K)$. By the definitions of B_2 and B_1, both $vx_3 \notin E(G)$ and $ux_2 \notin E(G)$, and so Claim 2 follows.

Next, we assume that there exists $u \in N(B_1) \cap B_3$. Thus there exists $v \in B_1$ such that $uv \in E(K)$. By the definitions of B_3 and B_1, $ux_2 \notin E(G)$ and $vx_3 \notin E(G)$. Thus, Claim 2 must hold. This completes the proof for Claim 2.

By Claim 2, define

$$G_2 = (G - x_2x_3 - uv) \cup \{ux_2, vx_3\} \quad \text{and} \quad X_2 = X - x_2x_3 \cup \{ux_2, vx_3\}.$$

Then by the choice of u, v, x_2 and x_3, G_2 is also a d-realization. We shall show that $|\mathcal{F}_1(G_2)| = 1$. Assume, on the contrary, that $|\mathcal{F}_1(G_2)| \geq 2$. Then there exists $S \in \mathcal{F}_1(G_2)$ and $S \neq H_1 - uv$. By Proposition 2.1(C4), $V(S) \cap V(H_1) = \emptyset$. But then S is a subgraph of G other than H_1, contrary to the assumption that $|\mathcal{F}_1(G)| = 1$.

By (5), $H_1 - uv$ is a $\mathcal{T}_k$-maximal subgraph of G_2. Since $G_2[H_2 \cup \cdots \cup H_c] = G[H_2 \cup \cdots \cup H_c] - x_2x_3, H_2, \ldots, H_c$ are $\mathcal{T}_k$-maximal subgraphs of G_2. But now $|X_2| = |X_1| + 1$, contrary to (6).

This completes the proof of the theorem. □

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