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Supereulerian graphs in the graph family $C_2(6, k)$

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ABSTRACT

For integers l and k with $l > 0$, and $k \geq 0$, $C_h(l, k)$ denotes the collection of h-edge-connected simple graphs G on n vertices such that for every edge-cut X with $2 \leq |X| \leq 3$, each component of $G - X$ has at least $(n - k)/l$ vertices. We prove that for any integer $k > 0$, there exists an integer $N = N(k)$ such that for any $n \geq N$, any graph $G \in C_2(6, k)$ on n vertices is supereulerian if and only if G cannot be contracted to a member in a well-characterized family of graphs. This extends former results in [J. Adv. Math. 28 (1999) 65–69] by Catlin and Li, in [Discrete Appl. Math. 120 (2002) 35–43] by Broersma and Xiong, in [Discrete Appl. Math. 145 (2005) 422–428] by D. Li, Lai and Zhan, and in [Discrete Math. 309 (2009) 2937–2942] by X. Li, D. Li and Lai.

1. Introduction

Graphs in this paper are finite, undirected and loopless. Graphs may have multiple edges. A graph G is nontrivial if it contains at least one edge. We follow Bondy and Murty [2] for undefined notations and terminologies. For a graph G, $\kappa(G)$ and $\kappa'(G)$ denote the connectivity and the edge-connectivity of graph G, respectively, and $O(G)$ denotes the set of all odd degree vertices of G. For $X \subset E(G)$, the **contraction** G/X is obtained from G by contracting each edge of X and deleting the resulting loops. If $H \subset G$, we use G/H instead of $G/E(H)$. A graph G is **Eulerian** if it is a connected graph with $O(G) = \varnothing$. A graph is **supereulerian** if it has a spanning Eulerian subgraph. In particular, K_1 is both Eulerian and supereulerian.

Throughout this paper, we denote by $\mathcal{S}$ the family of all supereulerian graphs. For integers h, l and k with $l > 0, 0 < h \leq 3$ and $k \geq 0$, let $C_h(l, k)$ denote the family of h-edge-connected graphs G such that for every bond X with two or three edges, each component of $G - X$ has at least $(|V(G)| - k)/l$ vertices.

The supereulerian problem of a graph G is to determine whether G is a supereulerian graph. This problem was first raised by Boesch et al. [1]. They pointed out in [1] that this problem is very difficult. Pulleyblank [17] showed that determining if a graph is supereulerian is NP-complete. For the literature concerning the problem, see Catlin's survey [4] and its complement [10]. Catlin and Li [9] are the first pioneers who considered the problem of characterizing supereulerian graphs in the family $C_h(l, k)$. Their study was followed by several researchers.

Definition 1.1. Let $K_{2,3}(e)$ denote the graph obtained from $K_{2,3}$ by replacing an edge $e \in E(K_{2,3})$ by a path of length 2. Let m, l, t be natural numbers with $t \geq 2$ and $m, l \geq 1$. Let $K_{2,t}(u, u')$ be $K_{2,t}$ with u, u' being the nonadjacent vertices of degree t. Let $K'_{2,t}(u, u', u'')$ be the graph obtained from $K_{2,t}(u, u')$ by adding a new vertex u'' that joins to u' only. Hence, u'' has degree 1 and u has degree t in $K'_{2,t}(u, u', u'')$. Let $K''_{2,t}(u, u', u'')$ be the graph obtained from $K_{2,t}(u, u')$ by adding a new vertex u'' that joins to a vertex of degree 2 of $K_{2,t}$. Hence, u'' has degree 1 and both u and u' have degree t in $K''_{2,t}(u, u', u'')$. Let $S(m, l)$ be the graph obtained from $K_{2,m}(u, u')$ and $K'_{2,l}(w, w', w'')$ by identifying u with w, and w'' with u'; let $J(m, l)$ denote the

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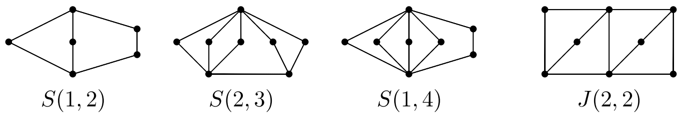

Fig. 1. The graphs in $\mathcal{F}'$.

graph obtained from $K_{2,m+1}$ and a $K'_{2,l}(w, w', w'')$ by identifying w with 2-vertex and w'' with an $(m+1)$-vertex in $K_{2,m+1}$, respectively.

Let $\mathcal{F}' = \{S(1, 2),\ S(2, 3),\ S(1, 4),\ J(2, 2),\ K_{2,3},\ K_{2,5}\}$ (see Fig. 1).

Theorem 1.2 (*Catlin and Li, Theorem 6 of [9]*). *If $G \in C_2(5, 0)$, then $G \in \mathcal{S}$ if and only if G cannot be contracted to $K_{2,3}$.*

Theorem 1.3 (*Broersma and Xiong, Theorem 7 of [3]*). *Suppose that $G \in C_2(5, 2)$ and $n \geq 13$. Then $G \in \mathcal{S}$ if and only if G cannot be contracted to $K_{2,3}$ or to $K_{2,5}$.*

Theorem 1.4 (*Li et al. Theorem 1.3 of [13]*). *Suppose that $G \in C_2(6, 0)$. Then $G \in \mathcal{S}$ if and only if G cannot be contracted to a member in $\{K_{2,3}, K_{2,5}$ or $K_{2,3}(e)\}$.*

Theorem 1.5 (*Li et al. Theorem 14 of [14]*). *Let $G \in C_2(6, 5)$ be a graph with $n = |V(G)| > 35$. Then $G \in \mathcal{S}$ if and only if G cannot be contracted to a member in $\mathcal{F}'$.*

Chen [10] and Xiong et al. [16] also studied the supereulerian problem for graphs in $C_3(l, k)$. Jeager [12] and Catlin [5] proved that every 4-edge-connected graph is supereulerian, and so the study is of interest only when $h < 4$.

The supereulerian problem for graphs in $C_2(6, k)$, for an arbitrary positive integer k, remains open [14]. The main purpose of this paper is to answer this question. The attempt to answer this question leads us to prove an associate result which is of interest on its own. We prove the following.

Theorem 1.6. *Let $k > 0$ be an integer. Then there exists an integer $N(k) \leq 7k$ such that, for any graph $G \in C_2(6, k)$ with $|V(G)| > N(k)$, $G \in \mathcal{S}$ if and only if G cannot be contracted to a member in $\mathcal{F}'$.*

2. Preliminaries

A graph G is **collapsible** if for any even subset $R \subseteq V(G)$, G has a spanning connected subgraph H such that $O(H) = R$. The **reduction** of G is the graph obtained from G by contracting each maximal collapsible subgraph of G to a distinct vertex. If G is the reduction of itself, then G is reduced.

By definition, the 3-cycle C_3 is collapsible, and any collapsible graph is supereulerian.

Define $F(G)$ to be the minimum number of edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. The **edge arboricity** $a(G)$ of a graph G is the minimum number of forests in G whose union contains G. Nash-Williams [15] proved

$$a(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil. \tag{1}$$

Theorem 2.1 (*Catlin*). *Let G be a graph.*

(i) *(Theorem* 2 *in* [5]*) If $F(G) = 0$, then G is collapsible.*
(ii) *(Theorem* 3 *in* [5]*) If H is a collapsible subgraph of G, then $G \in \mathcal{S}$ if and only if $G/H \in \mathcal{S}$.*
(iii) *(Theorem* 8(iv) *in* [5]*) If H is a collapsible subgraph of G, then G is collapsible if and only if G/H is collapsible.*
(iv) *(Theorems* 5 *and* 8(iii) *in* [5]*) If G is reduced, then any subgraph of G is reduced and $a(G) \leq 2$.*
(v) *(Theorem* 8(iv) *in* [5]*) If $a(G) \leq 2$, then $F(G) = 2|V(G)| - |E(G)| - 2$. In particular, if G is a reduced graph, then $F(G) = 2|V(G)| - |E(G)| - 2$.*
(vi) *(Lemma* 1 *in* [6]*) For any $e \in E(K_{3,3})$, $K_{3,3} - e$ is collapsible.*

Theorem 2.2 (*Catlin et al., Theorem 6 in [7]*). *For a graph G, if $\max_{K \subseteq G} \frac{|E(K)|}{|V(K)|-1} \geq 2$, then G has a nontrivial induced subgraph H that has two edge-disjoint spanning trees, i.e. $F(H) = 0$.*

The following corollary derives from the above two theorems directly.

Corollary 2.3. *If G is reduced, then $|E(H)|/(|V(H)| - 1) < 2$ for any nontrivial induced subgraph H of G.*

Proof. By Theorem 2.1(iv) and Eq. (1), $|E(H)|/(|V(H)| - 1) \leq 2$ for any nontrivial induced subgraph H of G. Assume there exists H such that $|E(H)|/(|V(H)| - 1) = 2$. Then by Theorems 2.2 and 2.1(i), G has a nontrivial collapsible subgraph, contrary to that G is reduced. Hence, $|E(H)|/(|V(H)| - 1) < 2$. $\square$

Theorem 2.4 (*Catlin, Theorem 7 in [5]*)**.** *If $F(G) \leq 1$, then G is collapsible if and only if $\kappa'(G) \geq 2$.*

Theorem 2.5 (*Catlin et al., Theorem 1.3 in [8]*)**.** *If G is connected and if $F(G) \leq 2$, then G is collapsible or the reduction of G is either K_2 or $K_{2,t}$ for some $t \geq 1$.*

Notation 2.6. *For a graph G and an integer i, $D_i(G)$ denotes the set of all vertices of degree i in G. Let $d_G(v)$ denote the degree of v in G and $d_i(G) = |D_i(G)|$. When the graph G is understood in the context, we use the following short-hand notations: $D_i = D_i(G)$, $d(v) = d_G(v)$ and $d_i = d_i(G)$. Moreover, for an integer $k \geq 0$, a vertex of degree k in a graph G is sometimes referred as a k-**vertex** of G.*

Theorem 2.7 (*Catlin, Theorem 8 and Lemma 5 of [5]*)**.** *If G is reduced, then G is simple and has no K_3. Moreover, if $\kappa'(G) \geq 2$, then $\sum_{i=2}^{3} |D_i(G)| \geq 4$, and when $\sum_{i=2}^{3} |D_i(G)| = 4$, G must be Eulerian.*

3. An associate result

The main purpose of this section is to prove the following associate result, which plays a key role in the proof of Theorem 1.6.

Theorem 3.1. *If G is a 2-edge-connected reduced graph which satisfies*

(i) $d_2 + d_3 \leq 6$,
(ii) $d_3 + d_5 \leq 2$,

then either $G \in \mathcal{S}$ or $G \in \mathcal{F}'$.

Definition 3.2. Let $\mathcal{A} = \{G : G$ is a 2-edge-connected reduced graph which satisfies $d_2 + d_3 \leq 6$ and $d_3 + d_5 \leq 2\}$ and $\mathcal{A}_3 = \{G \in \mathcal{A} : G \notin \mathcal{S}$ and $F(G) = 3\}$. Then by the following Lemma 3.3, for any $G \in \mathcal{A}_3$, we have $d_2 + d_3 = 6$, $d_3 + d_5 = 2$ and $d_j = 0$ for all $j \geq 6$.

We first prove some needed lemmas.

Lemma 3.3. *If $G \in \mathcal{A}$, then either G is Eulerian or $F(G) \leq 3$. Furthermore, if $F(G) = 3$, then either G is Eulerian or $d_2 + d_3 = 6$, $d_3 + d_5 = 2$ and $d_j = 0$ for all $j \geq 6$.*

Proof. Note that $F(G) \leq 4$ since

$$\begin{aligned} 2F(G) &= 4|V(G)| - 2|E(G)| - 4 = 4\sum_{i\geq 2} d_i - \sum_{i\geq 2} id_i - 4 \\ &= 2(d_2 + d_3) - (d_3 + d_5) - \sum_{i\geq 6}(i-4)d_i - 4 \\ &\leq 8 - (d_3 + d_5) - \sum_{i\geq 6}(i-4)d_i \leq 8. \end{aligned}$$

If $F(G) = 4$, then $d_3 + d_5 = 0$ and $d_j = 0$ for all $j \geq 6$. Since G has no odd-degree vertices, G is Eulerian.

Suppose $F(G) = 3$. If there exists some $j \geq 6$ such that $d_j > 0$, then $j = 6$, $d_6 = 1$ and $d_3 + d_5 = 0$. Therefore, G is Eulerian. If $d_j = 0$ for all $j \geq 6$, then $d_2 + d_3 = 6$, $d_3 + d_5 = 2$. □

Lemma 3.4. *If $G \in \mathcal{A}_3$, then we must have $(d_2, d_3, d_5) \in \{(4, 2, 0), (5, 1, 1), (6, 0, 2)\}$.*

Proof. If $d_3 = 2$, then $d_2 = 4$ and $d_5 = 0$. If $d_3 = 1$, then $d_2 = 5$ and $d_5 = 1$. If $d_3 = 0$, then $d_2 = 6$ and $d_5 = 2$. □

Lemma 3.5. *If a 2-edge-connected graph $G \notin \mathcal{S}$ and $|O(G)| = 2$, then $O(G)$ is an independent set.*

Proof. G has two odd vertices, say u and v. If u and v are adjacent, then $G - uv$ is Eulerian. Therefore, $G \in \mathcal{S}$, a contradiction. □

Lemma 3.6. *If G is reduced and $e = uv$ where $u, v \in D_2(G)$, then the following statements hold.*

(i) *If $G/e \in \mathcal{S}$, then $G \in \mathcal{S}$.*
(ii) $F(G/e) = F(G) - 1$.

Proof. Part (i) follows from Lemma 3 of [5]. To prove Part (ii), we first show that the $a(G/e) \leq 2$.

By Corollary 2.3, $\frac{|E(H)|}{|V(H)|-1} < 2$, for any nontrivial induced subgraph H of G. We now argue by contradiction to show that $a(G/e) \leq 2$, and assume that G/e has a nontrivial induced subgraph L' with $\frac{|E(L')|}{|V(L')|-1} > 2$. Let L be the induced subgraph of G such that either $L = L'$, or $e \in E(L)$ and $L/e = L'$. Since $\frac{|E(H)|}{|V(H)|-1} < 2$, for any nontrivial induced subgraph H of G, we must have $e \in E(L)$.

Since $e \in E(L)$, both $|E(L)| = |E(L')| + 1$ and $|V(L)| \leq |V(L')| + 1$ hold. Since $\frac{|E(L')|}{|V(L')|-1} > 2$, $|E(L')| \geq 2|V(L')| - 1$, which implies that

$$\frac{|E(L)|}{|V(L)|-1} \geq \frac{|E(L')|+1}{|V(L')|} \geq \frac{2|V(L')|}{|V(L')|} = 2,$$

contrary to $\frac{|E(L)|}{|V(L)|-1} < 2$.

Thus $a(G/e) \leq 2$. By Theorem 2.1(v),

$$\begin{aligned} 2F(G/e) &= 4|V(G/e)| - 2|E(G/e)| - 4 = 4(|V(G)|-1) - 2(|E(G)|-1) - 4 \\ &= 4|V(G)| - 2|E(G)| - 4 - 2 = 2F(G) - 2, \end{aligned}$$

and so Part (ii) holds. □

Notation 3.7. *Suppose that H is a subgraph of a graph L. Let $d_{i,L}(H)$ denote the number of vertices of H of degree i in L, and v_H the vertex in L/H onto which H is contracted.*

Lemma 3.8. *Let H be a subgraph of a graph L. Then each of the following statement holds:*

(i) $4|V(H)| - 2|E(H)| - 4 = \sum_{i>0}(4-i)d_{i,L}(H) + d(v_H) - 4$. *In particular, if $d_{i,L}(v) = 0$ for all $i \geq 6$, $i = 1$ and H is reduced, then* $2F(H) = 2d_{2,L}(H) + d_{3,L}(H) - d_{5,L}(H) + d(v_H) - 4$.

(ii) *For any H, $F(H-e) \leq F(H) + 1$.*

Proof. First notice that

$$2|E(H)| = \sum_{v \in H} d_L(v) - d(v_H) = \sum_{i>0} i d_{i,L}(H) - d(v_H).$$

Therefore,

$$\begin{aligned} 4|V(H)| - 2|E(H)| - 4 &= 4\sum_{i>0} d_{i,L}(H) - \left(\sum_{i>0} i d_{i,L}(H) - d(v_H)\right) - 4 \\ &= \sum_{i>0}(4-i)d_{i,L}(H) + d(v_H) - 4. \end{aligned}$$

So part (i) holds.

For any H, suppose X is a set of edges not in H, but adding X to H will result in a graph with 2 edge disjoint spanning trees. Then adding $X \bigcup e$ to $H - e$ will also result in a graph with 2 edge-disjoint spanning trees. Therefore, part (ii) holds. □

Lemma 3.9. *If $G \in \mathcal{A}_3$, then either $G \in \{S(1,2), S(1,4)\}$ or $D_2(G)$ is an independent set.*

Proof. Suppose there exist $u, v \in D_2(G)$ such that $e = uv \in E(G)$.

Let $G' = G/e$. By Lemma 3.6(i), $G' \notin \mathcal{S}$. By Lemmas 3.3 and 3.6(ii), $F(G') \leq F(G) - 1 \leq 3 - 1 = 2$. Since $\kappa'(G') \geq 2$, the reduction of G' is not K_2 or $K_{2,1}$. Since $G' \notin \mathcal{S}$, G' is not collapsible. Let G_0 denote the reduction of G'. By Theorem 2.1(ii) and Theorem 2.5,

$$G_0 = K_{2,t}, \quad \text{for some } t \geq 3, \text{ where } t \text{ is odd.} \tag{2}$$

Let v_e denote the new vertex obtained from contracting the edge e of G. Then G' has at most one nontrivial collapsible subgraph, as any nontrivial collapsible subgraph must contain v_e. Since $d_2(G) + d_3(G) = 6$, $d_3(G) + d_5(G) = 2$ and $d_j(G) = 0$ for all $j \geq 6$, we have $t = 3$ or 5, and so $G_0 \in \{K_{2,3}, K_{2,5}\}$ by (2). Let H' denote the collapsible subgraph of G' containing v_e, and H denote the preimage of H' from contraction.

Suppose $H = K_2$. Then H' contains only one vertex v_e. Therefore, $H = \{e\}$ and $G/e = G'$. If $G/e = K_{2,3}$, then $G = S(1,2)$. If $G/e = K_{2,5}$, then $G = S(1,4)$.

Next we will show that $H = K_2$. By contradiction, suppose that $H \neq K_2$. Then H' is a nontrivial collapsible subgraph of G'. Therefore, $\kappa'(H') \geq 2$. So $\kappa'(H) \geq 2$. By Theorem 2.4, since H is not a collapsible subgraph of G, $F(H) > 1$. Then $G/H = G'/H' = G_0 \in \{K_{2,3}, K_{2,5}\}$.

Suppose $G_0 = K_{2,3}$. Since $u, v \in H$, $d_{2,G}(H) \geq 2$. Note that $d_2(G) + d_3(G) = 6$ and $d_3(G) + d_5(G) = 2$. So there are two possibilities (see Table 1). Computing $F(H)$ by using Lemma 3.8(i), we have $F(H) = 1$, contrary to $F(H) > 1$.

Suppose $G_0 = K_{2,5}$. Note that $d_{2,G}(H) \geq 2$, $d_2(G) + d_3(G) = 6$ and $d_3(G) + d_5(G) = 2$. Then there is only one possibility (see Table 2). Computing $F(H)$ by using Lemma 3.8(i), $F(H) = 1$, contrary to $F(H) > 1$.

Thus, if $G \notin \mathcal{S}$, then either $G \in \{S(1,2), S(1,4)\}$ or $D_2(G)$ is an independent set. □

Lemma 3.10. *If K is an induced subgraph of a graph L, then each of the following holds:*

(i) *If $d_3(L) + d_5(L) \leq 2$, $d_2(L) + d_3(L) \leq 6$ and $L/K \in \mathcal{F}'$, then* $2|V(K)| - |E(K)| - 2 \leq 1$.

(ii) *If $L \in \mathcal{A}$ and $L/K \in \mathcal{F}'$, then we have $F(K) \leq 1$. Moreover, $F(K) = 1$ only if $L/K \in \{K_{2,3}, K_{2,5}\}$ and $d_2(L) + d_3(L) = 6$.*

Proof. First we prove part (i). Since $L/K \in \mathcal{F}'$, we have $d_3(L) + d_5(L) = 2$. If $d_2(L) + d_3(L) = 6$, then we have the following possibilities (see Table 3. The last column of Table 3 defines the Type of the subgraphs arising from contraction, which will be used in the proof of Lemma 3.13).

Table 1
The table for computing $F(H)$ when $G_0 = K_{2,3}$.

$d(v_H)$	$d_{2,G}(H)$	$d_{3,G}(H)$	$d_{5,G}(H)$	$F(H)$
2	2	0	0	1
3	2	0	1	1

Table 2
The table for computing $F(H)$ when $G_0 = K_{2,5}$.

$d(v_H)$	$d_{2,G}(H)$	$d_{3,G}(H)$	$d_{5,G}(H)$	$F(H)$
2	2	0	0	1

Table 3
The table in the proof of Lemma 3.10.

L/K	$d(v_K)$	$d_{2,L}(K)$	$d_{3,L}(K)$	$d_{5,L}(K)$	$2\|V(K)\| - \|E(K)\| - 2 \le$	Type
$K_{2,3}$	2	2	0	0	1	A
	3	1	1	0	1	B
		2	0	1	1	C
$K_{2,5}$	2	2	0	0	1	D
	5	0	1	0	1	E
		1	0	1	1	F
$S(1,2)$	2	1	0	0	0	G
	3	0	1	0	0	H
		1	0	1	0	I
$S(1,4)$	2	1	0	0	0	J
	5	0	0	1	0	K
$S(2,3)$	2	1	0	0	0	L
	3	0	1	0	0	M
		1	0	1	0	N
	4	0	0	0	0	O
	5	0	0	1	0	P
$J(2,2)$	2	1	0	0	0	Q
	3	0	1	0	0	R
		1	0	1	0	S
	4	0	0	0	0	T

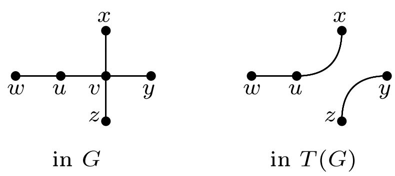

Fig. 2. The operator T on a graph G.

If $d_2(L) + d_3(L) < 6$, then $d_{2,L}(K)$ decreases at least by one. Computing $2|V(K)| - |E(K)| - 2$ by using Lemma 3.8(i), $2|V(K)| - |E(K)| - 2$ decreases at least by one. So $2|V(K)| - |E(K)| - 2 \le 0$. Hence part (i) holds.

If $L \in \mathcal{A}$, then K is reduced. So $F(K) = 2|V(K)| - |E(K)| - 2 \le 1$. From the proof of part (i), the equality holds only if $L/K \in \{K_{2,3}, K_{2,5}\}$ and $d_2(L) + d_3(L) = 6$. □

Definition 3.11. Let $u \in D_2(G)$ and $v \in D_4(G)$. Suppose $N(u) = \{v, w\}$ and $N(v) = \{u, x, y, z\}$. Define $T(G) = (G - v) + \{yz, ux\}$ (see Fig. 2).

Lemma 3.12. *Let G be a 2-edge-connected reduced graph, and let $e = uv \in E(G)$ such that $u \in D_2(G)$ and $v \in D_4(G)$. Let $N(u) = \{v, w\}$ and $N(v) = \{u, x, y, z\}$. Then*

(i) *$T(G)$ is 2-edge-connected (relabelling the vertices if needed).*
(ii) *$a(T(G)) \le 2$. Therefore, $F(K) = 2|V(K)| - |E(K)| - 2$ for any induced subgraph K of $T(G)$.*
(iii) *If $T(G) \in \mathcal{S}$, then $G \in \mathcal{S}$.*
(iv) *$T(G)$ has at most two nontrivial collapsible subgraphs which must contain yz or ux.*
(v) *Any two vertices in $N(v) = \{u, x, y, z\}$ are not adjacent.*
(vi) *If $G \in \mathcal{A}$, then the reduction of $T(G)$ is also in $\mathcal{A}$.*

Proof. Part (i) follows from the Splitting Lemma (see [11], on page III. 29).

By contradiction, assume there exists an induced subgraph K of $T(G)$ such that $|E(K)|/(|V(K)|-1) > 2$, i.e. $|E(K)| \geq 2|V(K)|-1$. Suppose H is the subgraph of G corresponding to K. By Corollary 2.3, $|E(H)|/(|V(H)|-1) < 2$. So $v \in H$. If both ux and yz are in K, then $|E(H)|/(|V(H)|-1) = (|E(K)|+2)/|V(K)| \geq (2|V(K)|+1)/|V(K)| > 2$, contrary to $|E(H)|/(|V(H)|-1) < 2$. If exactly one of ux and yz is in K, then $|E(H)|/(|V(H)|-1) = (|E(K)|+1)/|V(K)| \geq 2|V(K)|/|V(K)| = 2$, a contradiction. Thus $a(T(G)) \leq 2$ by (1). Hence, by Theorem 2.1(v), $F(K) = 2|V(K)| - |E(K)| - 2$, for any induced subgraph K of $T(G)$. Part (ii) holds.

If $T(G) \in \mathcal{S}$, suppose H is an spanning Eulerian subgraph of $T(G)$. Then H must contain ux since $d_{T(G)}(u) = 2$. If $yz \notin H$, then $H - ux + uv + vx$ is an Eulerian subgraph of G. If $yz \in H$, then $H - ux - yz + uv + vx + vy + vz$ is an Eulerian subgraph of G. Thus $G \in \mathcal{S}$. Part (iii) holds.

Any collapsible subgraph of $T(G)$ must contain the edge yz or ux. Otherwise, it is also a collapsible subgraph of G, contrary to that G is reduced. So $T(G)$ has at most two nontrivial collapsible subgraphs. Part (iv) holds.

Note that G is reduced, so there is no C_3 in G. It implies that part (v) holds.

Now we prove part (vi). Suppose H' is a maximum collapsible subgraph of $T(G)$. It suffices to prove that $T(G)/H'$, denoted by G_1, still satisfies $d_2(G_1) + d_3(G_1) \leq 6$ and $d_3(G_1) + d_5(G_1) \leq 2$. First, note that the number of odd degree vertices will not increase by contracting a subgraph. Otherwise, if after the contraction, the number of odd degree vertices increases by 1, then the number of odd vertices of the new graph obtained by contraction will be odd, contrary to that the number of odd vertices of a graph must be even. And since $G \in \mathcal{A}$, by Lemma 3.3, either G has no odd vertices or $F(G) \leq 3$. If $F(G) \leq 2$, then either G has no odd vertices or $G = K_{2,t}$ by Theorem 2.5. Since $d_2(G) + d_3(G) \leq 6$, $t \leq 6$. Hence the odd degree of G is at most 5. If $F(G) = 3$, by Lemma 3.3, either G has no odd vertices or $d_j = 0$ for all $j \geq 6$. Thus if $G \in \mathcal{A}$, then the odd degree vertices of G must be of degree 3 or 5. After the contraction, we still have $d_3(G_1) + d_5(G_1) \leq 2$.

If $d_2(G_1) + d_3(G_1) > d_2(G) + d_3(G)$, then $d(v_{H'}) = 2$ or 3. In each case, we will prove $H' - yz$ is a collapsible subgraph of G, contrary to that G is reduced.

Case 1. $d(v_{H'}) = 3$.

Since $d_2(G_1) + d_3(G_1) > d_2(G) + d_3(G)$, H' contains a 5-vertex of G and no 2 or 3-vertices of G. Therefore, $u \notin H'$ and $yz \in H'$. By part (ii) and computing $F(H')$ by using Lemma 3.8(i), $2F(H') = 2d_{2,G}(H') + d_{3,G}(H') - d_{5,G}(H') + 3 - 4 = -2$. By Lemma 3.8(ii), $F(H' - yz) \leq F(H') + 1 = 0$. Thus $H' - yz$ is a collapsible subgraph of G, contrary to that G is reduced.

Case 2. $d(v_{H'}) = 2$.

Then H' contains no vertex of degree 2 or 3 in G. Since the number of odd degree vertices of $T(G)/H'$ must be even, H' contains no 5-vertex of G. Therefore, $2F(H') = 2d_{2,G}(H') + d_{3,G}(H') - d_{5,G}(H') + 2 - 4 = -2$. So again, $F(H' - yz) = 0$, a contradiction.

Hence, $d_2(G_1) + d_3(G_1) \leq d_2(G) + d_3(G) \leq 6$. □

Lemma 3.13. *If G is a counterexample of Theorem 3.1 with $|V(G)|$ minimized, then no vertex in $D_2(G)$ is adjacent to a vertex in $D_4(G)$.*

Proof. By the hypothesis, G is a 2-edge-connected reduced graph which satisfies $d_2(G) + d_3(G) \leq 6$ and $d_3(G) + d_5(G) \leq 2$, and G is neither supereulerian nor in $\mathcal{F}'$. Since G is reduced,

$$G \text{ has no nontrivial collapsible subgraphs.} \tag{3}$$

Therefore, by Lemma 2.1(vi),

$$G \text{ has no } K_{3,3} - e. \tag{4}$$

By contradiction, we assume that there exist $u \in D_2(G)$ and $v \in D_4(G)$ such that $uv \in E(G)$. We use notations in Lemma 3.12, and denote $G' = T(G)$. Then $G' \notin \mathcal{S}$ by Lemma 3.12(iii) and $a(G') \leq 2$ by Lemma 3.12(ii). We will prove that either $G \in \mathcal{S}$ or $G \in \mathcal{F}'$.

Suppose G_1 is the reduction of G'. Then $G_1 \notin \mathcal{S}$, and by Lemma 3.12(vi) $G_1 \in \mathcal{A}$. Since G is minimized and $|V(G_1)| \leq |V(G')| = |V(G)| - 1$, $G_1 \in \mathcal{F}'$. There are three cases, depending on the number of nontrivial collapsible subgraphs in G' by Lemma 3.12(iv).

Case 1. G' does not have a nontrivial collapsible subgraph, i.e. $G_1 = G'$.

If $G' \in \{K_{2,3}, K_{2,5}, S(1,2), S(2,3), S(1,4)\}$, no matter how we choose y and z, the vertices u, x, y, z will be in a C_4 or C_5 in G'. Then in G, at least two of them are adjacent, contrary to Lemma 3.12(v).

Suppose $G' = J(2,2)$. A trail in G' with first edge e_1 and last edge e_2 is called an (e_1, e_2)-trail. Note that the cycle of G' is of length 4 or 6. If the shortest (ux, yz)-trail in G' is of length 3 or less, then at least two of u, x, y, z are adjacent in G, contrary to Lemma 3.12(v). So the shortest (ux, yz)-trail is of length 4. Therefore, ux and yz are in a C_6. By symmetry, there are two possibilities (see Fig. 3(a) and (b)). But both of them are supereulerian, contrary to $G \notin \mathcal{S}$.

The proof for the cases when G' has one or two nontrivial collapsible subgraphs are similar but more complicated. Details can be found in the Appendix. □

Proof of Theorem 3.1. By contradiction, suppose G satisfies (i) and (ii), but $G \notin \mathcal{S}$ and $G \notin \mathcal{F}'$ with $|V(G)|$ minimized. By Lemma 3.3, Theorem 2.5 and $G \notin \{K_{2,3}, K_{2,5}\}$, $F(G) = 3$. Therefore, $G \in \mathcal{A}_3$. By Lemma 3.4, $(d_2, d_3, d_5) \in \{(4,2,0), (5,1,1), (6,0,2)\}$. By Lemmas 3.5, 3.9 and 3.13, each vertex in $D_2(G)$ must be adjacent to two odd degree vertices which are not adjacent. But this is impossible when $(d_2, d_3, d_5) \in \{(4,2,0), (5,1,1), (6,0,2)\}$.

Thus the theorem holds.

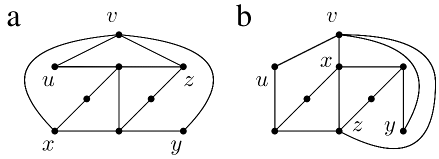

Fig. 3. The graphs in the proof of Case 1.

4. Proof of the main result

In this section, we are now ready to prove our main result Theorem 1.6.

Proof. Let $G \in C_2(6, k)$ be a graph with $n = |V(G)| > 7k$. Then we will prove that $G \in \mathcal{S}$ if and only if G cannot be contracted to a member in $\mathcal{F}'$. Clearly, if G can be contracted to a member in $\mathcal{F}'$, then $G \notin \mathcal{S}$.

Let G' be the reduction of G. By Theorem 2.1(ii), it suffices to show if $G' \notin \mathcal{S}$, then $G' \in \mathcal{F}'$, which implies that G can be contracted to a member in $\mathcal{F}'$. As $G' = K_1$ implies that $G \in \mathcal{S}$, we may assume that G' is 2-edge-connected and nontrivial. Let $d'_i = |d_i(G')|$.

By Theorem 2.7, if $d'_2 + d'_3 = 4$, then $G' \in \mathcal{S}$. Therefore, we only consider the case when $d'_2 + d'_3 \geq 5$. We shall assume that $G' \notin \mathcal{S}$ to find a contradiction or to get $G' \in \mathcal{F}'$.

Case 1. $d'_2 + d'_3 = 5$.

Subcase 1.1. $F(G') \leq 2$.

By Theorem 2.5, since $\kappa'(G') \geq 2$ and $G' \notin \mathcal{S}$, $G' = K_{2,t}$ with t odd. Since $d'_2 + d'_3 = 5$, we have $t = 3$ or $t = 5$ and so $G' \in \{K_{2,3},\ K_{2,5}\} \subset \mathcal{F}'$.

Subcase 1.2. $F(G') \geq 3$.

By Theorem 2.1(v), we have

$$\begin{aligned}
6 &\leq 2F(G') = 4|V(G')| - 2|E(G')| - 4 \\
&= 4\sum_{j\geq 2} d'_j - \sum_{j\geq 2} jd'_j - 4 \\
&= (d'_2 + d'_3) + d'_2 + \sum_{j\geq 5}(4-j)d'_j - 4 \\
&= 1 + d'_2 + \sum_{j\geq 5}(4-j)d'_j.
\end{aligned}$$

Note that $d'_2 + d'_3 = 5$ and $(4-j)d'_j \leq 0$ for any $j \geq 5$. It follows that $d'_2 = 5$, $d'_3 = 0$, and $d'_j = 0 (j \geq 5)$. Thus G' is Eulerian contrary to that $G' \notin \mathcal{S}$.

Case 2. $d'_2 + d'_3 = 6$.

If $F(G') \leq 2$, then by $\kappa'(G') \geq 2$ and by Theorem 2.5, $G' = K_{2,t}$ with $t \geq 3$ odd since G' is not supereulerian. As $d'_2 + d'_3 = 6$, this is impossible. Therefore, we must have $F(G') \geq 3$.

Subcase 2.1. $F(G') = 3$.

$$\begin{aligned}
6 &= 2F(G') = 4|V(G')| - 2|E(G')| - 4 \\
&= 4\sum_{j\geq 2} d'_j - \sum_{j\geq 2} d'_j - 4 \\
&= 2(d'_2 + d'_3) - (d'_3 + d'_5) + \sum_{j\geq 6}(4-j)d'_j - 4 \\
&= 8 - (d'_3 + d'_5) + \sum_{j\geq 6}(4-j)d'_j.
\end{aligned}$$

It follows that $(d'_3 + d'_5) \leq 2$. By Theorem 3.1, since $G' \notin \mathcal{S}$, we have $G' \in \mathcal{F}'$.

Subcase 2.2. $F(G') \geq 4$.

Since $d'_2 + d'_3 = 6$,

$$\begin{aligned}
8 &\leq 2F(G') = (d'_2 + d'_3) + d'_2 + \sum_{j\geq 5}(4-j)d'_j - 4 \\
&= 2 + d'_2 + \sum_{j\geq 5}(4-j)d'_j.
\end{aligned}$$

It follows that $d'_2 = 6$, $d'_3 = 0$ and $d'_j = 0$ $(j \geq 5)$. Hence G' is Eulerian, contrary to $G' \notin \mathcal{S}$.

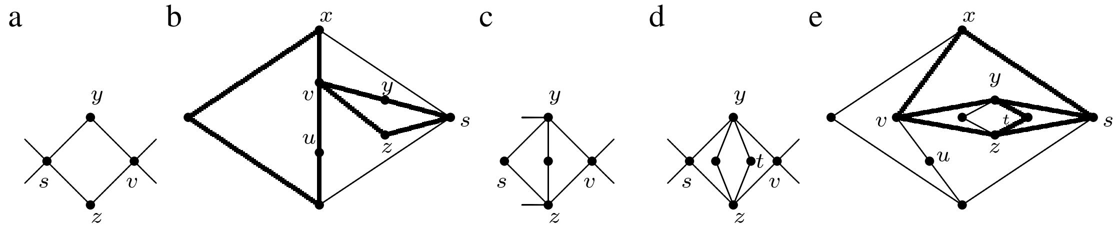

Fig. 4. The graphs in the proof of Subcases 2.1.1 and 2.1.4.

Case 3. $d'_2 + d'_3 \geq 7$.

Let $c = d'_2 + d'_3$, and $H_1, H_2, \ldots, H_c$ denote the subgraphs of G whose contraction images in G' are the vertices of degree at most 3 in G'. Since $G \in C_2(6, k)$, for each i with $1 \leq i \leq c$, $|V(H_i)| \geq (n-k)/6$. It follows any $c \geq 7$ that

$$n = |V(G)| \geq \sum_{i=1}^{7} |V(H_i)| \geq \frac{7(n-k)}{6}.$$

Therefore, $n \leq 7k$, a contradiction.

This completes the proof of Theorem 1.6. □

Appendix. Proof for the other two cases of Lemma 3.13

Case 2. G' has only one collapsible subgraph, say H'.

Let H be the subgraph of G corresponding to H', i.e. $T(G[E(H)]) = H'$. Then ux and yz are not both in H'. Otherwise, $G/H \in \mathcal{F}'$. By Lemma 3.10(ii), $F(H) \leq 1$. Since $\kappa'(H) \geq 2$, by Theorem 2.4, H is collapsible, contrary to (3).

Since $G_1 = G'/H' \in \mathcal{F}'$, by Lemma 3.10(i) and Lemma 3.12(ii), $F(H') = 2|V(H')| - |E(H')| - 2 \leq 1$. We consider two subcases.

Subcase 2.1. $yz \in H'$.

Then ux is not in H'. Since $\kappa'(H') \geq 2$ and $d(u) = 2$, u is not in H'. But x may or may not be in H'. If x is in H', then $|V(H)| = |V(H')| + 1$ and $|E(H)| = |E(H')| + 2$. So $F(H) = 2|V(H)| - |E(H)| - 2 = 2|V(H')| - |E(H')| - 2 \leq 1$. As $\kappa'(H) \geq 2$, by Theorem 2.4, H is collapsible, contrary to (3).

Then x is not in H'. Then $|V(H)| = |V(H')| + 1$ and $|E(H)| = |E(H')| + 1$. So $F(H) = (2|V(H')| - |E(H')| - 2) + 1 \leq 2$. Since H is not collapsible and $\kappa'(H) \geq 2$, $F(H) = 2$. It implies that $H = K_{2,t}$ for some t. Therefore, $H' = H - \{yv, vz\} + yz$. By the definition of $F(H')$, $F(H') = 1$. By Lemma 3.10(i), H' must be of Type A, B, C, D, E or F (see Table 3) and $G_1 \in \{K_{2,3}, K_{2,5}\}$. Since x and u are not in H, v is of degree 2 in H. Then both y and z are t-vertices in H with $2 \leq t \leq 5$.

Notice that $t \neq 5$. Otherwise, $d_G(y) = d_G(z) = 5$ since $d_G = 0$ for all $j \geq 6$. That $G'/H' \in \{K_{2,3}, K_{2,5}\}$ and $y, z \in H'$ implies that there is at least another 3 or 5-vertex except y and z, contrary to $d_3(G) + d_5(G) \leq 2$. Hence, $2 \leq t \leq 4$.

Subcase 2.1.1. H' is of Type A.

Notice that $d_{2,G}(H) = d_{2,G'}(H')$, $d_{3,G}(H) = d_{3,G'}(H')$ and $d_{5,G}(H) = d_{5,G'}(H')$, so H has two vertices of degree 2 in G and other vertices of H are of degree 4 in G. Since $d_G(u) = 2$ and $ux \in G_1$, x is a vertex of degree 3 in both G_1 and G. If $H = K_{2,2}$, then by Lemma 3.12(v), $d_G(y) = d_G(z) = 2$, so $G \in \mathcal{S}$ (see Fig. 4(a) and (b)). If $H = K_{2,3}$, then one of y and z is adjacent to x (see Fig. 4(c)), contrary to Lemma 3.12(v). If $H = K_{2,4}$, then $G[s, t, v, x, y, z]$ is $K_{3,3} - e$ (see Fig. 4(d) and (e)), contrary to (4).

Subcase 2.1.2. H' is of Type B.

Then H has one 2-vertex, one 3-vertex and other vertices are of degree 4 in G. If $H = K_{2,2}$, then $G \in \mathcal{S}$ (see Fig. 5(a) and (b)) or $G = J(2, 2)$ (see Fig. 5(c) and (d)). If $H = K_{2,3}$ (see Fig. 5(e) and (f)), since H' is of type B, H' has a 2-vertex in G. Let this vertex be t. Then t is adjacent to y, z. So t is not adjacent to u. Without loss of generality, assume y is the 3-vertex in G, and so z is a 4-vertex in G. Let $s \in N(y) \cap N(z)$ be another 2-vertex in H'. By Lemma 3.12(v), y, z, are not adjacent to u. Since $v_{H'}$ is adjacent to u, but y, z, t are not adjacent to u, we have that s is adjacent to u. Moreover, v is also adjacent to u in G. Therefore, $G[s, t, u, v, y, z]$ is $K_{3,3} - e$, contrary to (4). If $H = K_{2,4}$ (see Fig. 5(g)), then exactly one of s and t is adjacent to u. So $G[s, t, u, v, y, z]$ is $K_{3,3} - e$, contrary to (4).

Subcase 2.1.3. H' is of Type C.

Then H has two 2-vertices, one 5-vertex and other vertices are of degree 4 in G. If $H = K_{2,2}$, by Lemma 3.12(v), $d_G(s) = 5$ (see Fig. 6(a)), then $G = S(2, 3)$ (see Fig. 6(b)). If $H = K_{2,3}$ (see Fig. 6(c)), then y or z is adjacent to u, contrary to Lemma 3.12(v). If $H = K_{2,4}$ (see Fig. 6(d)), since s is adjacent to u, $G[s, t, u, v, y, z]$ is $K_{3,3} - e$, contrary to (4).

Subcase 2.1.4. H' is of Type D.

Similar to Subcase 2.1.1, if $H = K_{2,2}$ (see Fig. 4(a)), then $G \in \mathcal{S}$ (see Fig. 7(a)). If $H = K_{2,3}$, then one of y and z is adjacent to x (see Fig. 4(c)), contrary to Lemma 3.12(v). If $H = K_{2,4}$ (see Fig. 4(d)), then $G[s, t, v, x, y, z]$ is $K_{3,3} - e$, contrary to (4).

Subcase 2.1.5. H' is of Type E.

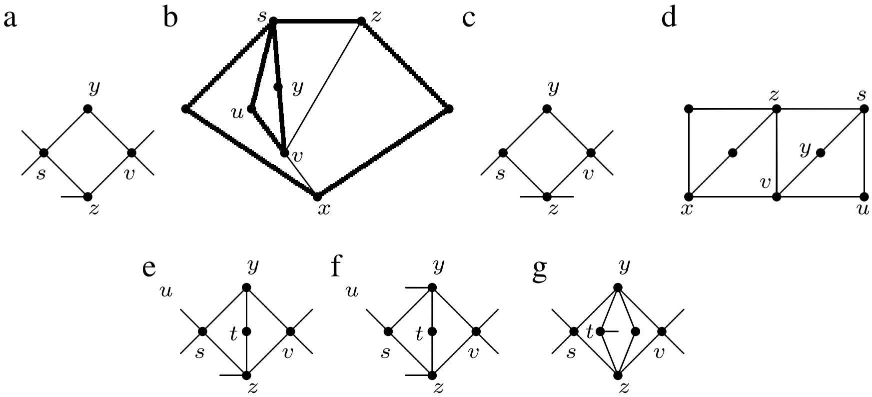

Fig. 5. The graphs in the proof of Subcase 2.1.2.

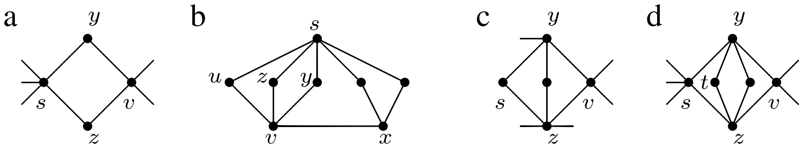

Fig. 6. The graphs in the proof of Subcase 2.1.3.

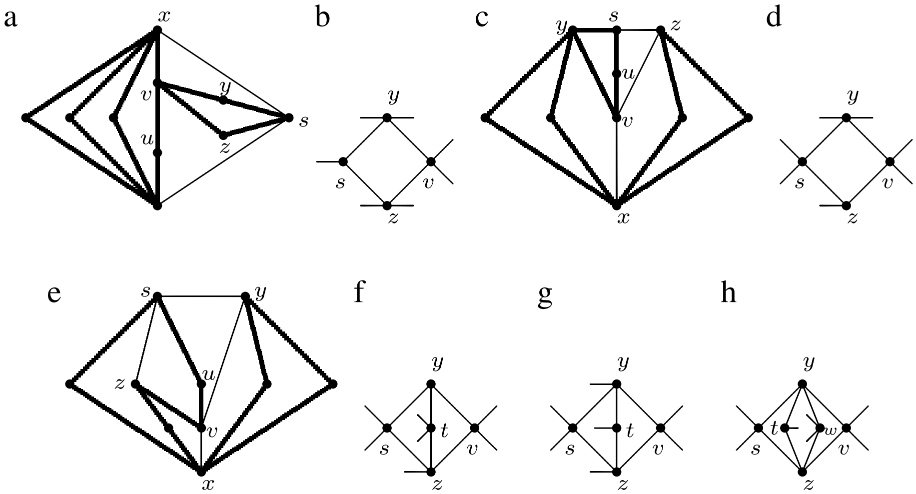

Fig. 7. The graphs in the proof of Subcases 2.1.4 and 2.1.5.

If $H = K_{2,2}$, then there are two possibilities (see Fig. 7(b) and (d)). In either case, $G \in \mathcal{S}$ (see Fig. 7(c) and (e)). If $H = K_{2,3}$ (see Fig. 7(f) and (g)), then u is adjacent to exactly one of s and t. Therefore, $G[s, t, u, v, y, z]$ is $K_{3,3} - e$, contrary to (4). If $H = K_{2,4}$ (see Fig. 7(h)), then u is adjacent to exactly one of s, t and w. Assume that u is adjacent to s. Then $G[s, t, u, v, y, z]$ is $K_{3,3} - e$, contrary to (4).

Subcase 2.1.6. H' is of Type F.

If $H = K_{2,2}$ (see Fig. 8(a) and (c)), then $G \in \mathcal{S}$ (see Fig. 8(b) and (d)). If $H = K_{2,3}$ (see Fig. 8(e)) or $H = K_{2,4}$ (see Fig. 8(f)), then $G[s, t, u, v, y, z]$ is $K_{3,3} - e$, contrary to (4).

Subcase 2.2. $ux \in H'$.

Similar to Subcase 2.1, if y or z is in H', then $|V(H)| = |V(H')| + 1$ and $|E(H)| = |E(H')| + 2$. Therefore, $F(H) \le 1$. So y and z are not in H', $F(H') = 1$ and $H = K_{2,t}$. Since u is a 2-vertex, $d_2(H') > 1$ and $t = 2$. So H' must be of Type A, B, C, D or F and H is $K_{2,2}$. Use the same argument to conclude that $G \in \mathcal{S}$ or $G \in \{J(2, 2), S(2, 3)\}$.

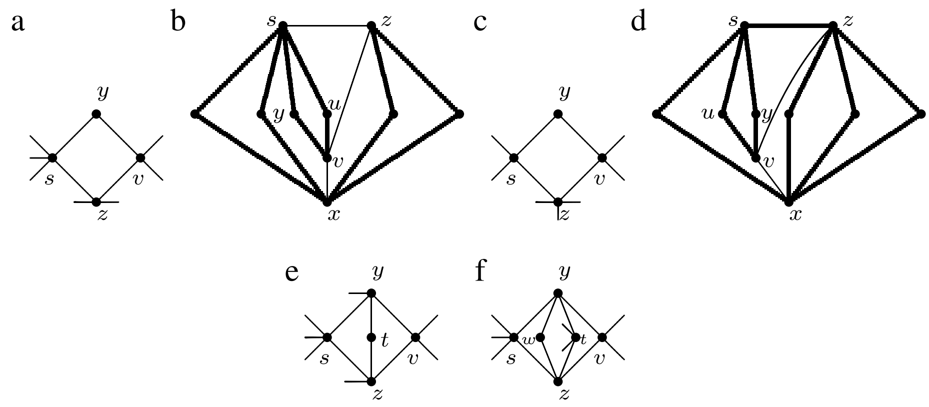

Fig. 8. The graphs in the proof of Subcase 2.1.6.

Table 4
The table in the proof of Case 3.

G_1	$\{d(v_{H_1'}), d(v_{H_2'})\}$	d_2'	d_3'	d_5'	$F(H_1)+F(H_2)$
$K_{2,3}$	{2, 2}	3	0	0	3
	{2, 3}	2	1	0	3
		3	0	1	3
	{3, 3}	1	2	0	3
		2	1	1	3
		3	0	2	3
$K_{2,5}$	{2, 2}	3	0	0	3
	{2, 5}	1	1	0	3
		2	0	1	3
	{5, 5}	1	0	2	3
$S(2, 3)$	{2, 2}	2	0	0	2
	{2, 3}	1	1	0	2
		2	0	1	2
	{2, 4}	1	0	0	2
	{2, 5}	1	0	1	2
	{3, 4}	1	0	1	2
	{3, 5}	1	0	2	2
$J(2, 2)$	{2, 2}	2	0	0	2
	{2, 3}	1	1	0	2
		2	0	1	2
	{2, 4}	1	0	0	2
	{3, 3}	1	1	1	2
		2	0	2	2
	{3, 4}	1	0	1	2

Case 3. G' has two nontrivial maximal collapsible subgraphs, say H_1' and H_2', such that $yz \in H_1'$ and $ux \in H_2'$.

Let H_1 and H_2 be the subgraphs of G corresponding to H_1' and H_2', respectively, i.e. $T(G[E(H_1)]) = H_1'$ and $T(G[E(H_2)]) = H_2'$. Then $G'/(H_1' \cup H_2')$ is in $\mathcal{F}'$. Notice that $v_{H_1'} \neq v_{H_2'}$. Otherwise, there exists a vertex t such that $t \in V(H_1') \bigcap V(H_2')$. Then $H_1' \cup H_2'$ is a connected collapsible subgraph of G', contrary to that H_1' and H_2' are maximal.

Let n' denote the number of vertices of $H_1 \cup H_2$, d_i' denote the number of vertices of $H_1 \cup H_2$ of degree i in G. Then $2|E(H_1 \cup H_2)| = \sum id_i' - d(v_{H_1'}) - d(v_{H_2'})$. Since v is in both H_1 and H_2, $|V(H_1)| + |V(H_2)| = n' + 1$.

$$\begin{aligned} 2F(H_1) + 2F(H_2) &= 4|V(H_1)| - 2|E(H_1)| - 4 + 4|V(H_2)| - 2|E(H_2)| - 4 \\ &= 4(|V(H_1)| + |V(H_2)|) - 2(|E(H_1)| + |E(H_2)|) - 8 \\ &= 4(n' + 1) - 2|E(H_1 \cup H_2)| - 8 \\ &= 4\left(\sum d_i' + 1\right) - \left(\sum id_i' - d(v_{H_1'}) - d(v_{H_2'})\right) - 8 \\ &\leq 2d_2' + d_3' - d_5' + d(v_{H_1'}) + d(v_{H_2'}) - 4. \end{aligned}$$

By Lemma 3.3 and $G \notin \mathcal{S}$, $F(G) = 2$ or $F(G) = 3$ with $d_2(G) + d_3(G) = 6$ and $d_3(G) + d_5(G) = 2$. If $F(G) = 2$, by Theorem 2.5, since $d_2(G) + d_3(G) \leq 6$, $\kappa'(G) \geq 2$ and $G \notin \mathcal{S}$, $G = K_{2,3}$ or $K_{2,5}$, contrary to $G \notin \mathcal{F}'$. Thus $F(G) = 3$ with $d_2(G) + d_3(G) = 6$ and $d_3(G) + d_5(G) = 2$. We have the following Table 4, where $\{d(H_1'), d(H_2')\}$ is a multi-set and $G_1 = G'/(H_1' \bigcup H_2') \in \mathcal{F}'$. Note that H_2' contains a 2-vertex u, so $d_2' \geq 1$. It helps us get rid of some cases.

If $G_1 = S(1, 2)$, then since $S(1, 2)$ has one more 2-vertex than $K_{2,3}$, the number of 2-vertices in $H_1 \cup H_2$ will decrease by 1 comparing to the case $G_1 = K_{2,3}$. Therefore, $F(H_1) + F(H_2)$ decreases by 1. Hence, $F(H_1) + F(H_2) \leq 3 - 1 = 2$. If $G_1 = S(1, 4)$, then since $S(1, 4)$ has one more 2-vertex than $K_{2,5}$, $F(H_1) + F(H_2)$ will decrease by 1 compared to the case $G_1 = K_{2,5}$. Thus, $F(H_1) + F(H_2) \leq 3 - 1 = 2$. So we have $F(H_1) + F(H_2) \leq 3$ (see Table 4). Thus $F(H_1) \leq 1$ or $F(H_2) \leq 1$. Since $\kappa'(H_i) \geq 2$ for $i = 1, 2$, then H_1 or H_2 is a collapsible subgraph of G, contrary to (3).

This completes the proof of Lemma 3.13.

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