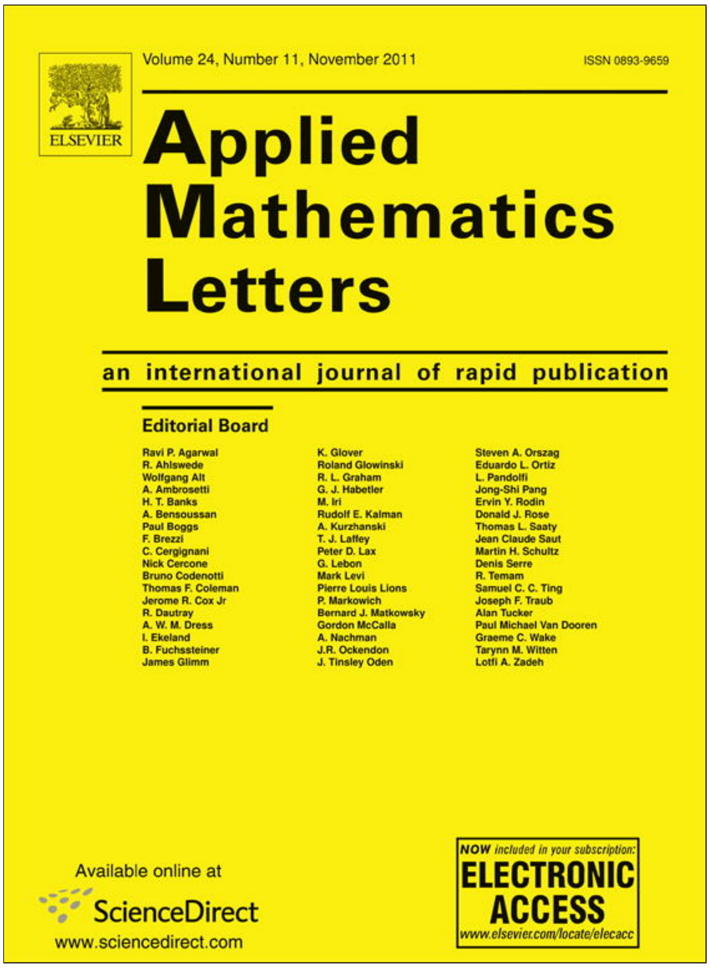
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Supereulerian graphs and matchings

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ABSTRACT

A graph G is called *supereulerian* if G has a spanning Eulerian subgraph. Let $\alpha'(G)$ be the maximum number of independent edges in the graph G. In this paper, we show that if G is a 2-edge-connected simple graph and $\alpha'(G) \leq 2$, then G is supereulerian if and only if G is not $K_{2,t}$ for some odd number t.

1. Introduction

We use [1] for terminology and notation not defined here, and consider simple finite graphs only. Let G be a graph and let $O(G)$ denote the set of all vertices in G with odd degrees. If $O(G) = \emptyset$, then G is called an *even* graph. An *Eulerian* graph is a connected graph G with $O(G) = \emptyset$, i.e., a connected even graph. The graph K_1 is an Eulerian graph. If a graph contains a spanning Eulerian subgraph, then it is called *superEulerian*. Let $\alpha'(G)$ be the maximum number of independent edges in the graph G. Obviously every graph G has one $\alpha'(G)$-matching.

A subgraph H of a graph G is *dominating* if $E(G - V(H)) = \emptyset$. So a closed trail is called a *dominating closed trail* if it is dominating. Note that a closed trail of a graph G is also an Eulerian subgraph of G. Hence we can prove a graph is superEulerian by showing that the graph has a spanning closed trail.

Motivated by the Chinese Postman Problem, Boesch et al. [2] proposed the superEulerian graph problem: determine when a graph has a spanning Eulerian subgraph. They indicated that this might be a difficult problem. Pulleyblank [3] showed that such a decision problem, even when restricted to planar graphs, is NP-complete. Jaeger [4] and Catlin [5] independently showed that every 4-edge-connected graph is superEulerian.

Let $F(G)$ denote the minimum number of edges that must be added to G in order to obtain a super-graph that has two edge-disjoint spanning trees. Catlin [5] defined the reduction of a graph.

Theorem 1 (*Catlin et al. [6]*). *Let G be a connected graph. If $F(G) \leq 2$, then exactly one of the following holds:*

(i) *G is superEulerian;*
(ii) *G has a cut edge (bridge);*
(iii) *The reduction of G is $K_{2,s}$ for some odd integer $s \geq 3$.*

Motivated by the above result, we obtain the following main result.

Theorem 2. *If G is a 2-edge-connected simple graph and $\alpha'(G) \leq 2$, then G is superEulerian if and only if G is not $K_{2,t}$ for some odd number t.*

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2. Proof of Theorem 2

Let $C = u_1u_2 \cdots u_k \cdots u_1$ be the longest closed trail of G, where C contains k vertices and some of the k vertices may be repeated, then $|E(C)| \geq 3$. Note that every edge in C must be in some cycle of C. Since $\alpha'(G) \leq 2$, it follows that $3 \leq c(G) \leq 5$, where $c(G)$ means the circumference of G. Suppose G is not $K_{2,t}$ for some odd number t, then we only need to show the following two claims to finish the proof.

Claim I. *C is dominating.*

Proof of Claim I. By way of contradiction, we assume that C is not dominating, then there exists at least one edge xy that is neither included in C nor incident with any vertex in C, i.e., $x \notin V(C)$ and $y \notin V(C)$. Since G is 2-edge-connected, xy must be in some cycle C_1 of G and $3 \leq |E(C_1)| \leq 5$. Now we consider the set $V(C_1) \cap V(C)$. If $V(C_1) \cap V(C) = \emptyset$, then there exists at least one path P to connect C and C_1 since G is connected. Pick one edge $e_1 \in P$, one edge $e_2 \in C_1$ that is not adjacent to e_1, and one edge $e_3 \in C$ that is not adjacent to e_1, then $\{e_1, e_2, e_3\}$ is an independent edge set with order 3, a contradiction with $\alpha'(G) \leq 2$. So $V(C_1) \cap V(C)$ is not empty, then we need to discuss the following cases.

Case 1: $|V(C_1) \cap V(C)| = 1$.

We assume $V(C_1) \cap V(C) = \{u\}$ and let $C' = C \cup C_1$, then C' is a longer closed trail than C, a contradiction. So Case 1 does not hold.

Case 2: $|V(C_1) \cap V(C)| \geq 2$.

If we give cycle C_1 an orientation with the direction from y to x, then we can assume that u is the first vertex in $V(C_1) \cap V(C)$ starting from x on C_1 and v is the last one. Since u and v are both in the closed trail C, there exists at least one path in C to connect u and v. For convenience, we can suppose that Q is the shortest path among all in C to connect u and v. If $|E(Q)| \geq 3$, then we can suppose that $Q = uw_1w_2 \cdots w_tv$ where $t \geq 2$. Let $Y = \{xy, uw_1, w_tv\}$, then Y is an independent edge set with order 3, a contradiction. So it follows that $|E(Q)| \leq 2$. We use P' to denote the path from u to v in C_1 that contains the edge xy. If $|E(Q)| = 1$, i.e., $uv \in E(C)$, then let $C' = (C - uv) \cup P'$, then C' is a longer closed trail than C, a contradiction. Otherwise, $|E(Q)| = 2$, i.e., there exists a vertex w such that $uw \in E(C)$ and $vw \in E(C)$, then let $C' = (C - uw - vw) \cup P'$. In fact, $P' = uxyv$ in this situation since $c(G) \leq 5$. If C' is still connected, then C' is a longer closed trail than C, a contradiction. If C' is disconnected, then w must be in a cycle C_2 of C that does not contain uw or vw. Assume $wz \in E(C_2)$ and let $Z = \{wz, ux, yv\}$, then Z is an independent edge set with order 3, a contradiction. So Case 2 does not hold.

Above all, Claim I is proved, i.e., C is dominating. □

Claim II. *C is spanning.*

Proof of Claim II. By way of contradiction, we assume that C is not spanning, then there exists at least one vertex x that is not included in C. Then x must be adjacent to at least two vertices u and v in C since C is dominating and G is 2-edge-connected. Let P be the shortest path in C to connect u and v. If $|E(P)| \geq 4$, then $P \cup \{ux, vx\}$ is a cycle with length at least 6, contradicting that $c(G) \leq 5$. So $1 \leq |E(P)| \leq 3$.

If $|E(P)| = 1$, i.e., $uv \in E(C)$, let $C' = (C - uv) \bigcup \{ux, vx\}$, then C' is longer closed trail than C, a contradiction.

If $|E(P)| = 3$, we may assume that $P = uw_1w_2v$. Since C is a closed trail, the degree of v in C is at least two, i.e., there exists one edge vw_3 in C such that w_3 is not from $\{u, w_1, w_2\}$ since P is the shortest path in C to connect u and v. Let $X = \{w_1w_2, vw_3, ux\}$, then X is an independent edge set with order 3, a contradiction.

So we only need to deal with the remaining case when $|E(P)| = 2$, i.e., $P = uwv$. Since every edge in C must be in some cycle in C, it suffices to consider the following two cases.

Case 1: uw and wv are in the same cycle D in C.

Since P is the shortest path in C to connect u and v and $c(G) \leq 5, 4 \leq |E(D)| \leq 5$. If $|E(D)| = 5$, then we assume $D = uw_1w_2vwu$. Let $X = \{w_1w_2, uw, xv\}$, then X is an independent edge set with order 3, a contradiction. So $|E(D)| = 4$, then $D \bigcup \{ux, xv\} = K_{2,3}$, in this situation either G is superEulerian or it forces G to be $K_{2,t}$ where t is odd since $\alpha'(G) \leq 2$ and G is 2-edge-connected.

Case 2: uw and wv are not in the same cycle.

Suppose $uw \in E(C_1)$ and $wv \in E(C_2)$, where C_1 and C_2 are two different cycles in C. We only need to discuss the following two subcases.

Subcase 2.1 $E(C_1 \bigcap C_2) = \emptyset$.

Since $3 \leq |E(C_1)| \leq 5$ and $3 \leq |E(C_2)| \leq 5$, we can choose some edge $e_1 \in E(C_1)$, some edge $e_2 \in E(C_2)$ and some edge $e_3 \in \{ux, xv\}$ to form an independent edge set $X = \{e_1, e_2, e_3\}$ with order 3, a contradiction.

Subcase 2.2 $E(C_1 \bigcap C_2) \neq \emptyset$.

Let C_0 be the symmetric difference of C_1 and C_2, i.e., $C_0 = C_1 \Delta C_2$, then C_0 is a union of cycles in C and $\{uw, wv\} \subseteq E(C_0)$. If uw and wv are in the same cycle of C_0, then we can go back to Case 1; otherwise, uw and wv are in two edge-disjoint cycles C_1' and C_2' of C_0, respectively. Then we can go back to Subcase 2.1.

Above all, Claim II is proved, i.e., C is spanning.

Therefore, we have finished the proof of Theorem 2. □

3. Concluding remark

Let m, n be two positive integers. Let $H_1 \cong K_{2,m}$ and $H_2 \cong K_{2,n}$ be two complete bipartite graphs. Let u_1, v_1 be two nonadjacent vertices of degree m in H_1, and u_2, v_2 be two nonadjacent vertices of degree n in H_2. Let $S_{n,m}$ denote the graph obtained from H_1 and H_2 by identifying v_1 and v_2, and by connecting u_1 and u_2 with a new edge u_1u_2. Note that $S_{1,1}$ is the same as C_5, the 5-cycle.

Define $K_{1,3}(1, 1, 1)$ to be the graph obtained from a 6-cycle $C = u_1u_2u_3u_4u_5u_6u_1$ by adding one vertex u and three edges uu_1, uu_3 and uu_5.

To extend our main result in this paper, we present the following two conjectures as further research.

Conjecture 3. *If G is a 2-edge-connected simple graph and $\alpha'(G) \leq 3$, then G is superEulerian if and only if G is not one of $\{K_{2,t}, S_{n,m}, K_{1,3}(1, 1, 1)\}$ where n, m are natural numbers and t is an odd number.*

Conjecture 4. *If G is a 3-edge-connected simple graph and $\alpha'(G) \leq 5$, then G is superEulerian if and only if G is not contractible to the Petersen graph.*

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