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Group Connectivity and Group Colorings of Graphs — A Survey

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Abstract In 1950s, Tutte introduced the theory of nowhere-zero flows as a tool to investigate the coloring problem of maps, together with his most fascinating conjectures on nowhere-zero flows. These have been extended by Jaeger et al. in 1992 to group connectivity, the nonhomogeneous form of nowhere-zero flows. Let G be a 2-edge-connected undirected graph, A be an (additive) abelian group and $A^* = A - \{0\}$. The graph G is A-connected if G has an orientation $D(G)$ such that for every map $b : V(G) \mapsto A$ satisfying $\sum_{v \in V(G)} b(v) = 0$, there is a function $f : E(G) \mapsto A^*$ such that for each vertex $v \in V(G)$, the total amount of f-values on the edges directed out from v minus the total amount of f-values on the edges directed into v is equal to $b(v)$. The group coloring of a graph arises from the dual concept of group connectivity. There have been lots of investigations on these subjects. This survey provides a summary of researches on group connectivity and group colorings of graphs. It contains the following sections.
1. Nowhere-zero Flows and Group Connectivity of Graphs
2. Complete Families and A-reductions
3. Reductions with Edge-deletions, Vertex-deletions and Vertex-splitting
4. Group Colorings as a Dual Concept of Group Connectivity

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1 Nowhere-zero Flows and Group Connectivity of Graphs

We consider finite graphs which may contain loops or multiple edges. Notations undefined in this survey will follow [1]. In particular, $\delta(G), \Delta(G), \kappa(G), \kappa'(G)$ and $\chi(G)$ denote the minimum degree, maximum degree, connectivity, edge-connectivity, and chromatic number of a graph G, respectively. When we emphasize the fact that a graph G may have multiple edges, we also refer to G as a multigraph. Following the terminology in matroid theory, a nontrivial 2-regular connected graph is called a *circuit*. A circuit with k edges is a *k-circuit*, and denoted by C_k. A *wheel* W_k is the graph obtained from C_k by adding a new vertex x and k edges xv_i ($i = 1, 2, \ldots, k$), where $V(C_i) = \{v_1, v_2, \ldots, v_k\}$. The groups considered in this paper are finite abelian (additive) groups. For a finite abelian group A, the additive identity of A will be denoted by 0 (zero) throughout this paper. Let G and H be graphs. If H is a subgraph of G, we write $H \subseteq G$. Throughout this paper, if $X \subseteq E(G)$ is an edge subset of a graph G, then we also use X to denote the subgraph of G induced by the edge set X.

Let G be a digraph. For a vertex $v \in V(G)$, let

$$E^-_G(v) = \{(u, v) \in E(G) | u \in V(G)\} \quad \text{and} \quad E^+_G(v) = \{(v, u) \in E(G) | u \in V(G)\}.$$

The subscript G may be omitted when G is understood from the context.

Let A be a nontrivial (additive) abelian group with the additive identity 0, and let $A^* = A - \{0\}$ denote the set of nonzero elements in A. For subsets $X \subseteq E(G)$ and $A' \subseteq A$, define $F(X, A') = \{f | f : X \to A'\}$ as the set of all functions from X into A'. In particular, we define

$$F^*(X, A) = F(X, A^*) = \{f | f : X \to A^*\}.$$

When H is a subgraph of G, we define $F(H, A) = F(E(H), A)$. For each $f \in F(G, A)$, the *boundary* of f is a function $\partial f : V(G) \to A$ defined by

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where "$\sum$" refers to the addition in A. If $\partial f = 0$, then f is called an *A-flow* of G. Define

$$F_0(G, A) = \{f \in F(G, A) | \partial f = 0\}$$

to be the set of all A-flows of G.

We shall adopt the following notational convenience: For subsets $X \subseteq E(G)$ and $A' \subseteq A$, any map $f : X \to A'$ is also viewed as a map $f : E(G) \to A$ such that $f(e) = 0$ for all $e \in E(G) - X$. With this agreement, we have

$$F(X, A') \subseteq F(G, A) \quad \text{and} \quad F_0(X, A') \subseteq F_0(G, A).$$

The *support* of a map f, denoted by $S(f)$, is the set of all elements x such that $f(x) \neq 0$ in the domain of f.

Let G be an undirected graph and A be an abelian group. A function $b : V(G) \mapsto A$ is called an *A-valued zero-sum function on G* if $\sum_{v \in V(G)} b(v) = 0$ in G. The set of all A-valued zero-sum functions on G is denoted by $Z(G, A)$. A graph G is *A-connected* if G has an orientation G' such that for every function $b \in Z(G, A)$, there is a function $f \in F^*(G', A)$ such that $b = \partial f$. Following the terminology for network flows with nonhomogeneous demands in combinatorial optimization (see, for example, Chapter 3 of [2]), we call such a function f a *nowhere-zero (A, b)-flow* (abbreviated as (A, b)-NZF). For an abelian group A, let $\langle A \rangle$ denote the family of graphs that are A-connected. It was observed in [3] that $G \in \langle A \rangle$ is independent of the orientation of G.

An *A-nowhere-zero flow* (abbreviated as A-NZF) in G is a function $f \in F^*(G, A)$ such that $\partial f = 0$. If $A = \mathbf{Z}_k$, an A-nowhere-zero flow is also called a *nowhere-zero k-flow* (abbreviated as k-NZF). The nowhere-zero-flow problems were first introduced by Tutte [4], and surveyed by Jaeger in [5]. For a 2-edge-connected graph G, define the *flow number* of G by

$$\Lambda(G) = \min\{k \mid G \text{ has a } k\text{-NZF}\}. \tag{1}$$

The following fascinating conjectures of Tutte and Jaeger on nowhere-zero flows remain open up to the present.

Conjecture 1.1 (Tutte [4, 6], see also [5]) *Let G be a graph.*

(i) *If $\kappa'(G) \geq 2$, then $\Lambda(G) \leq 5$.*

(ii) *If $\kappa'(G) \geq 2$ and if G does not have a subgraph contractible to the Petersen graph, then $\Lambda(G) \leq 4$.*

(iii) *If $\kappa'(G) \geq 4$, then $\Lambda(G) \leq 3$.*

(iv) *There exists an integer $k \geq 4$ such that for any graph G, if $\kappa'(G) \geq k$, then $\Lambda(G) \leq 3$.*

The concept of A-connectivity was introduced by Jaeger et al. in [3], where A-NZF's were successfully generalized to A-connectivity. A concept similar to the group connectivity was independently introduced in [7], with a different motivation from [3].

Proposition 1.2 *Let G be a connected graph and let A be a nontrivial abelian group. Then $\forall f \in F(G, A)$, $\partial f \in Z(G, A)$.*

Proof It suffices to show that the following sum is 0 in A:

$$\sum_{v \in V(G)} \partial f(v) = \sum_{v \in V(G)} \left(\sum_{e \in E_G^+(v)} f(e) - \sum_{e \in E_G^-(v)} f(e) \right).$$

For every edge $e \in E(G)$ oriented from a vertex u to a vertex v, $f(e)$ occurs once with a positive sign when e is viewed as an edge in $E_G^+(u)$, and once with a negative sign when e is viewed as an edge in $E_G^-(v)$. Therefore, the sum must be zero. □

Theorem 1.3 *Each of the following statements holds:*

(i) (Jeager et al. [3, Proposition 2.1]) *A graph G is connected if and only if every zero sum function $b \in Z(G, A)$ is the boundary of some function $f \in F(G, A)$.*

(ii) *If G has a spanning tree T, then $\forall\, b \in Z(G, A)$, $\exists\, f \in F(T, A)$ such that $\partial f = b$.*

Proof (i) If G has two components H_1 and H_2, then pick $v_i \in V(H_i)$ for $i \in \{1, 2\}$, and define

$$b(z) = \begin{cases} 1, & \text{if } z = v_1, \\ -1, & \text{if } z = v_2, \\ 0, & \text{otherwise.} \end{cases}$$

Then $b \in Z(G, A)$. But there is no $f \in F(G, A)$ with $\partial f = b$.

(ii) Argue by induction on $|V(T)|$, and the result is trivial if $|V(T)| = 1$. Assume that $|V(T)| \geq 2$ and u is a vertex of G with exactly one neighbor $v \in V(T)$. Let $T' = T - u$. For any $b \in Z(G, A)$, define

$$b'(z) = \begin{cases} b(z), & \text{if } z \in V(G) - \{u, v\}, \\ b(u) + b(v), & \text{if } z = v. \end{cases}$$

Then $b' \in Z(G, A)$. By induction, $\exists\, f' \in F(T', A) \subseteq F(T, A)$ such that $\partial f' = b'$. Assume that in G, the edge $e = uv$ is directed from u to v. Let $f(e) = b(u)$ and $f(e') = f'(e')$, for $e' \in E(T) - e$. Then $\partial f = b$. □

For a 2-edge-connected graph G, define the *group connectivity number* of G as

$$\Lambda_g(G) = \min\{k \mid \text{if } A \text{ is an abelian group with } |A| \geq k, \text{ then } G \in \langle A \rangle\}.$$

Alon et al. [8] showed that whether or not a graph G is $\mathbf{Z}_3$-connected is not related to the existence of additive bases in some vector spaces over the 3-element field. As the nowhere-zero-flow problem is the corresponding homogeneous case of the group connectivity problem, Jaeger et al. [3] proposed the following conjectures, which, as suggested by a result of Kochol [9], are stronger than the corresponding conjectures above.

Conjecture 1.4 (Jeager et al. [3]) *Let G be a graph.*

(i) *If $\kappa'(G) \geq 3$, then $\Lambda_g(G) \leq 5$.*

(ii) *If $\kappa'(G) \geq 5$, then $\Lambda_g(G) \leq 3$.*

(iii) *There exists an integer $k \geq 5$ such that if $\kappa'(G) \geq k$, then $\Lambda_g(G) \leq 3$.*

The group connectivity numbers of certain families of graphs are known.

Theorem 1.5 *Let $m, n \geq 2$ be integers. Each of the following statements holds:*

(i) ([3] and [10, Lemma 3.3]) $\Lambda_g(C_n) = n + 1$.

(ii) ([10, Corollary 3.5]) $\Lambda_g(K_n) = 3$ *if* $n \geq 5$.

(iii) ([11, Theorem 4.6]) *Let $m \geq n \geq 2$ be integers. Then* $\Lambda_g(K_{m,n}) = \begin{cases} 5, & \text{if } n = 2, \\ 4, & \text{if } n = 3, \\ 3, & \text{if } n \geq 4. \end{cases}$

(iv) ([11, 12]) *For* $n \geq 1$, $W_{2n} \in \langle \mathbf{Z}_3 \rangle$.

(v) ([13, 14]) *Let* $G \cong W_{2n+1}$ *for some integer* $n \geq 1$ *and* $b \in Z(G, \mathbf{Z}_3)$. *Then for any orientation* D *of* G, *there exists an* $f \in F^*(G, \mathbf{Z}_3)$ *with* $\partial f = b$ *under* D *if and only if* $b \neq 0$.

Yao and Gong [15] also investigated the group connectivity of Kneser graphs.

For an abelian group A, the A-connectedness of a graph G can be verified in several equivalent ways.

Lemma 1.6 *Let* A *be an abelian group, and let* G *be a graph. Let* C *be a circuit of* G *with a fixed orientation* D *making* C *a directed circuit. Then* $\forall\, a \in A$ *and* $\forall\, e \in E(C)$, *there is exactly one flow of* G *with support* $E(C)$ *and* $f(e) = a$.

Proof We may assume that C is oriented in D as a directed circuit. Then the constant map $f \equiv a$ is in $F_0(C, A)$. □

Theorem 1.7 (Parts (i)–(iii) are from Proposition 2.2 of [3]) *Let* G *be a connected graph and* A *an abelian group. The following statements are equivalent*:

(i) $G \in \langle A \rangle$. *That is*, $\forall\, b \in Z(G, A)$, $\exists\, f \in F^*(G, A)$ *such that* $\partial f = b$.

(ii) $\forall\, \overline{f} \in F(G, A)$, $\exists\, f \in F_0(G, A)$ *such that* $\forall\, e \in E(G)$, $f(e) \neq \overline{f}(e)$.

(iii) $\forall\, b \in Z(G, A)$, *and* $\forall\, \overline{f} \in F(G, A)$, $\exists\, f \in F(G, A)$ *such that* $\partial f = b$ *and* $f(e) \neq \overline{f}(e)$, $\forall\, e \in E(G)$.

(iv) *For any spanning tree* T *of* G, *for any* $\overline{f} \in F(T, A)$, *there exists* $f \in F_0(G, A)$ *such that* $f(e) \neq \overline{f}(e)$ *for every* $e \in T$ *and* $f(e) \neq 0$ *for every* $e \notin T$. *Therefore*, $f - \overline{f} \in F^*(G, A)$.

(v) *For a fixed spanning tree* T *of* G, *for any* $\overline{f} \in F(T, A)$, *there exists* $f \in F_0(G, A)$ *such that* $f(e) \neq \overline{f}(e)$ *for every* $e \in T$ *and* $f(e) \neq 0$ *for every* $e \notin T$. *Therefore*, $f - \overline{f} \in F^*(G, A)$.

Proof (iv) ⇒ (ii) Suppose that $\overline{f} \in F(G, A)$ and that T is a spanning tree of G. For any edge $e \in E(G) - E(T)$, there is a unique circuit C_e such that $E(C_e) \subseteq E(T) \cup \{e\}$. By Lemma 1.6, there is a flow $f_e \in F_0(G, A)$ with support $E(C_e)$ and $f_e(e) = \overline{f}(e)$. Then $f_{E(G)-E(T)} = \sum_{e \in E(G) \setminus E(T)} f_e \in F_0(G, A)$ with $f_{E(G)-E(T)}(e) = \overline{f}(e)$, $\forall\, e \in E(G) - E(T)$. Define $\overline{f}_T = \overline{f} - f_{E(G)-E(T)}$. Then, for any $e \notin E(T)$, $\overline{f}_T(e) = 0$, and so $\overline{f}_T \in F(T, A)$. By (iv), $\exists\, f_T \in F_0(G, A)$ such that $f_T - \overline{f}_T \in F^*(T, A)$ and $f_T(e) \neq 0$, $\forall\, e \notin E(T)$. Let $f = f_T + f_{E(G)-E(T)}$. Since $f_{E(G)-E(T)} \in F_0(G, A)$ and since $f_T \in F_0(G, A)$, $\partial f = \partial f_{E(G)-E(T)} + \partial f_T = 0 + 0 = 0$, and so $f \in F_0(G, A)$. Moreover, let $e \in E(G)$. If $e \in E(T)$, since $f_T - \overline{f}_T \in F^*(T, A)$, we have $f(e) - \overline{f}(e) = (f_T + f_{E(G)-E(T)})(e) - (\overline{f}_T + f_{E(G)-E(T)})(e) = f_T(e) - \overline{f}_T(e) \neq 0$. If $e \in E(G) - E(T)$, then $f(e) - \overline{f}(e) = f_T(e) - 0 \neq 0$. Therefore, $f - \overline{f} \in F^*(G, A)$.

(ii) ⇒ (iv) $\forall\, \overline{f} \in F(T, A)$, for any spanning tree T of G, define $\overline{f}_1(e) = \overline{f}(e)$ if $e \in E(T)$ and $\overline{f}_1(e) = 0$ if $e \in E(G) - E(T)$. By (ii), $\exists\, f \in F_0(G, A)$ such that $f - \overline{f}_1 \in F^*(G, A)$. In particular, $\forall\, e \in E(T)$, $f(e) \neq \overline{f}(e)$, and $\forall\, e \in E(G) - E(T)$, $f(e) \neq \overline{f}(e) = 0$.

(iv) ⇒ (v) This is a special case of (iv).

(v) ⇒ (i) Let T be a given spanning tree. $\forall\, b \in Z(G, A)$, by Theorem 1.3 (ii), $\exists\, \overline{f} \in F(T, A)$ such that $\partial \overline{f} = b$. By (v), $\exists\, f_0 \in F_0(G, A)$ satisfies (v). Let $f = \overline{f} - f_0$. Then $\partial f = b$ and $f \in F^*(G, A)$. □

While the property for a graph G to be A-connected is independent of the choice of the orientation of G, whether this property depends on the structure of the group is not clear.

Problem 1.8 Let $\{\kappa' \geq 3\}$ denote the family of all 3-edge-connected graphs. Is it true that for two abelian groups A_1 and A_2, if $|A_1| = |A_2|$, then

$$\langle A_1 \rangle \cap \{\kappa' \geq 3\} = \langle A_2 \rangle \cap \{\kappa' \geq 3\}?$$

Problem 1.9 For two abelian groups A_1 and A_2, if

$$\langle A_1 \rangle \cap \{\kappa' \geq 3\} = \langle A_2 \rangle \cap \{\kappa' \geq 3\},$$

does it imply that $|A_1| = |A_2|$?

2 Complete Families and A-Reductions

For a subset $X \subseteq E(G)$, the contraction G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the edges in X. Note that even if G is simple, G/X may have loops or multiple edges. For convenience, we write G/e for $G/\{e\}$, where $e \in E(G)$. If H is a subgraph of G, then G/H denotes $G/E(H)$.

Proposition 2.1 ([10, Proposition 3.2]) *For any abelian group A, $\langle A \rangle$ is a family of connected graphs satisfying*:

(C1) $K_1 \in \langle A \rangle$,

(C2) *If $e \in E(G)$ and if $G \in \langle A \rangle$, then $G/e \in \langle A \rangle$, and*

(C3) *If $H \subseteq G$ and if $H, G/H \in \langle A \rangle$, then $G \in \langle A \rangle$.*

Following Catlin [16], a family of connected graphs satisfying (C1), (C2) and (C3) is called a *complete family*. As indicated in [16] and [17], complete families are useful as they are associated with certain type of reduction methods. General discussions on complete families can be found in [17, 18] and [19], among others. The next two results are often applied in the investigations of group connectivity of graphs.

Theorem 2.2 *Let H be a connected subgraph of G, $b \in Z(G, A)$, and $G' = G/H$ with v_H denoting the vertex in G' onto which H is contracted. Let $b' \in Z(G', A)$ be defined as follows*:

$$b'(v) = \begin{cases} b(v), & \text{if } v \in V(G') - \{v_H\} = V(G) - V(H), \\ b(H), & \text{if } v = v_H. \end{cases}$$

If $f' \in F(G', A)$ is such that $\partial f' = b'$, then there exists $f \in F(G, A)$ satisfying $\partial f = b$ such that $S(f') \subseteq S(f)$.

Proof Extend f' to be a function $f' \in F(G, A)$ by defining $f'(e) = 0$ for each $e \in E(H)$. Let

$$b_1(v) = b(v) - \partial f'(v), \quad \forall\, v \in V(H).$$

Then $b_1 \in Z(H, A)$ since

$$b_1(H) = b(H) - \sum_{v \in V(H)} \partial f'(v) = b(H) - \partial f'(v_H) = 0.$$

Since H is connected, by Proposition 1.2, there exists a function $f_1 \in F(H, A)$ such that $\partial f_1 = b_1$. Extend f_1 to a function in $F(G, A)$ by defining $f_1(e) = 0$, for $e \in E(G) - E(H)$. Then $f = f_1 + f' \in F^*(G, A)$ is the desirable map satisfying $\partial f = b$ such that $S(f') \subseteq S(f)$. $\square$

Lemma 2.3 ([20, Lemma 2.1]) *Let T be a connected spanning subgraph of G. If for each edge $e \in E(T)$, G has a subgraph $H_e \in \langle A \rangle$ with $e \in E(H_e)$, then $G \in \langle A \rangle$.*

Since the family of all A-connected graphs forms a complete family, one can derive the following properties from Proposition 2.1.

Lemma 2.4 *Let G be a graph, let A be an abelian group with $|A| \geq 3$.*

(i) ([20, Lemma 2.2]) *Let H_1, H_2 be subgraphs of G such that $H_1, H_2 \in \langle A \rangle$. If $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2 \in \langle A \rangle$.*

(ii) ([20, Lemma 2.3]) *For every vertex $v \in V(G)$, there is a unique maximal subgraph H_v of G that is A-connected and contains v.*

(iii) ([20, Corollary 2.2]) *Let H be a subgraph of G such that $H \in \langle A \rangle$, and let v_H denote the vertex in G/H onto which H is contracted. If $L' \in \langle A \rangle$ is a subgraph of G/H such that $v_H \in V(L')$, then $L = G[E(L') \cup E(H)]$, an edge induced subgraph, is also in $\langle A \rangle$.*

As a consequence of Lemma 2.4, for each graph G and for a fixed abelian group A, G has a unique subgraph $M_A(G)$ such that each component of $M_A(G)$ is a maximal subgraph of G that is in $\langle A \rangle$. With these notations, we present the following definition.

Definition 2.5 *The contraction $R_A(G) = G/M_A(G)$ is called the* A-reduction *of G. If $G = R_A(G)$, then G is* A-reduced.

Corollary 2.6 ([20, Corollary 2.3]) *Let G be a graph. Each of the following statements holds:*

(i) $R_A(R_A(G)) = R_A(G)$.

(ii) $G \in \langle A \rangle$ *if and only if* $R_A(G) \cong K_1$.

Definition 2.7 *Let H be a subgraph of G and k an integer. The* k-closure*, written as $\mathrm{cl}_k(H)$, is the transitive closure of the operator $H \to H \cup C$, where C is a circuit with $|E(C) - E(H)| \leq k$. In other words, $\mathrm{cl}_k(H)$ is the (unique) maximal subgraph of G of the form $H \cup C_1 \cup \cdots \cup C_n$ where for each i, $1 \leq i \leq n$, C_i is a circuit of G such that $|E(C_i) - [E(H \cup C_1 \cup \cdots \cup C_{i-1})]| \leq k$.*

The k-closure of a subgraph was first introduced by Seymour in [21]. It was proved that the k-closure preserves A-connectivity for large enough A, which also follows from Proposition 2.1 and Theorem 1.5 (ii).

Theorem 2.8 (Jeager et al. [3, Corollary 2.4]) *Let $k \geq 3$ be an integer, and A be an abelian group with $|A| > k$. Let H be a subgraph of a graph G such that $H \in \langle A \rangle$.*

(i) *The k-closure of H in G is a subgraph of $M_A(G)$.*

(ii) *The k-closure of H in G is in $\langle A \rangle$.*

3 Reductions with Edge-deletions, Vertex-deletions and Vertex-splitting

Definition 3.1 *Let G be a graph and let $v \in V(G)$ be a vertex of degree $m \geq 4$. Let $N(v) = \{v_1, v_2, \ldots, v_m\}$ denote the set of vertices adjacent to the vertex v, and $X = \{vv_1, vv_2\}$. The graph $G_{[v,X]}$ is obtained from $G - \{vv_1, vv_2\}$ by adding a new edge that joins v_1 and v_2. If $m = 2k$ is even and if*

$$M = \langle \{v_1, v_{k+1}\}, \{v_2, v_{k+2}\}, \ldots, \{v_k, v_{2k}\} \rangle$$

is a way to pair the vertices in $N(v)$, then $G_{(v,M)}$ denotes the graph obtained from $G - v$ by adding k new edges e_i joining v_i and v_{k+i} $(1 \leq i \leq k)$.

The next theorem shows that edge-deletion and vertex-splitting can be used as a way of reduction.

Theorem 3.2 *Let A be an abelian group. Let G be a graph and let $v \in V(G)$ be a vertex of degree $m \geq 4$. Each of the following statements holds:*

(i) (Jeager et al. [3, Corollary 2.3]) *Let $e = v_1v_2$ be an edge in G. If $G - e \in \langle A \rangle$, then $G \in \langle A \rangle$.*

(ii) (Lai [10, Lemma 3.1]) *If for some X of two edges incident with v in G, $G_{[v,X]} \in \langle A \rangle$, then $G \in \langle A \rangle$.*

(iii) (Lai [10, Lemma 3.1]) *Let m be even, M be a way to pair the vertices of $N(v)$ such that $G_{(v,M)} \in \langle A \rangle$, and let $b \in Z(G, A)$ be given. If $b(v) = 0 \in A$, then there is a function $f \in F^*(G, A)$ such that $\partial f = b$.*

Definition 3.3 *Let $v \in V(G)$ be a vertex. Partition the edges of G incident with v into two nonempty sets E' and E''. Split v into two vertices v' and v'', each incident with edges in E' and E'', respectively. This yields the graph G_v, called an* elementary detachment *of G (at v). A* detachment *of G is a graph obtained from G by applying a finite number of elementary detachments on G.*

With the next lemma, one can argue by induction to show that if a detachment of a graph G is in $\langle A \rangle$, then $G \in \langle A \rangle$.

Lemma 3.4 (Lai [20, Lemma 2.4]) *Let G_v be an elementary detachment of G and let $b \in Z(G, A)$. Define $b' : V(G_v) \mapsto A$ by*

$$b'(z) = \begin{cases} b(v), & \text{if } z = v', \\ 0, & \text{if } z = v'', \\ b(z), & \text{otherwise.} \end{cases} \tag{2}$$

If G_v has an (A, b')-NZF, then G has an (A, b)-NZF.

Corollary 3.5 (Lai [20, Lemma 2.4]) *Let G be a graph, and $v \in V(G)$ be a vertex of degree at least 4 in G. Let e_1, e_2 be two edges in G incident with v. Let G_v be the graph obtained from G by splitting v into v' and v'' such that v'' is incident with e_1 and e_2, while v' is incident with all the other edges formerly incident with v in G. If $G_v/e_1 \in \langle A \rangle$, then $G \in \langle A \rangle$.*

Lemma 3.6 *Let G be a graph and $v \in V(G)$ with $d_G(v) = 2$. Then $G \in \langle \mathbf{Z}_3 \rangle$ if and only if $G - v \in \langle \mathbf{Z}_3 \rangle$.*

Proof If $G - v \in \langle \mathbf{Z}_3 \rangle$, then $G/(G - v)$, as a 2-circuit, is also in $\langle \mathbf{Z}_3 \rangle$. It follows by Proposition 2.1 (C3) that $G \in \langle \mathbf{Z}_3 \rangle$.

Conversely, assume that $G \in \langle \mathbf{Z}_3 \rangle$. Suppose that the two neighbors of v in G are u and w such that the orientation D of G makes $d^-_D(v) = 2$ and $d^+_D(v) = 0$. Let $b' \in Z(G - v, \mathbf{Z}_3)$. Define $b : V(G) \mapsto \mathbf{Z}_3$ as follows:

$$b(z) = \begin{cases} b'(z), & \text{if } z \notin \{u, v, w\}, \\ 1, & \text{if } z = v, \\ b'(z) + 1, & \text{if } z \in \{u, w\}. \end{cases}$$

Since $b' \in Z(G-v, \mathbf{Z}_3)$, $b \in Z(G, \mathbf{Z}_3)$. By the assumption that $G \in \langle \mathbf{Z}_3 \rangle$, $\exists f \in F^*(G, \mathbf{Z}_3)$ such that $\partial f = b$. As $d_D^-(v) = 2$ and $d_D^+(v) = 0$, $f(uv) = f(wv) = 1$. Let f' denote the restriction of f on $G - v$. Then by $f(uv) = f(wv) = 1$, $\partial f' = b'$, and so $G - v \in \langle \mathbf{Z}_3 \rangle$. □

Corollary 3.7 *Let G be a graph with an edge cut $X \subseteq E(G)$ such that $|X| = 2$, and let G_1 and G_2 be the two components of $G - X$. Then $G \in \langle \mathbf{Z}_3 \rangle$ if and only if $G_1, G_2 \in \langle \mathbf{Z}_3 \rangle$.*

Proof If both G_1 and G_2 are $\mathbf{Z}_3$-connected, then $G/(G_1 \cup G_2)$ is a 2-circuit, which is also $\mathbf{Z}_3$-connected. By Proposition 2.1 (C3), G is $\mathbf{Z}_3$-connected.

Conversely, let $i \in \{1, 2\}$ and suppose that $G \in \langle \mathbf{Z}_3 \rangle$. Then by Proposition 2.1 (C2), $G/G_i \in \langle \mathbf{Z}_3 \rangle$, and so by Lemma 3.6, G_{3-i} must be $\mathbf{Z}_3$-connected. □

Definition 3.8 *Let $f : E(G) \to A$ and $X \in E(G)$. The function from X to A given by $e \mapsto f(e)$, for all $e \in X$, is called the* restriction *of f to X and denoted by $f|_X$. For $b \in Z(G, A)$, a graph G is* (A, b)-extensible from v, *if $\forall f' : E(v) \mapsto A^*$ such that $\partial f'(v) = b(v)$, there exists an $f \in F^*(G, A)$ with $\partial f = b$ such that $f|_{E(v)} = f'$. If for any $b \in Z(G, A)$, G is (A, b)-extensible from v, then G is called* A-extensible from v.

By definition, if G is A-extensible from v, then G is A-connected.

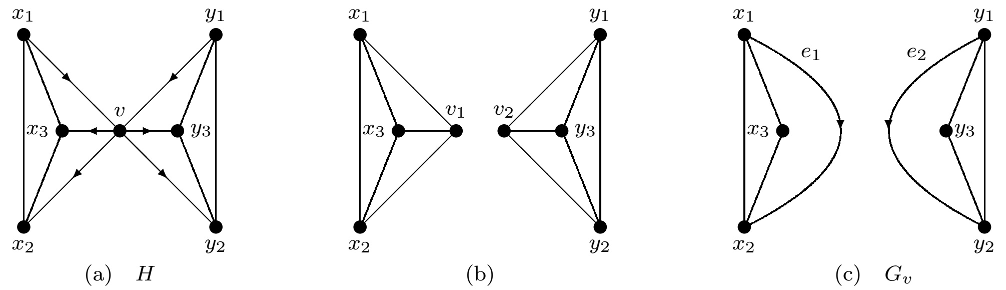

Figure 3.1 Reduction in Lemma 3.9 if $d_G(v) = 6$

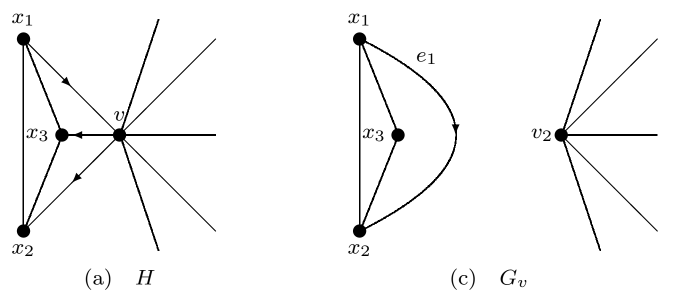

Figure 3.2 Reduction in Lemma 3.9 if $f_G(v) > 6$

Lemma 3.9 (Lai et al. [22, Lemma 2.3]) *Let G be a graph and $H \cong K_4$ be a subgraph of G and $v \in V(H)$ (see Figure 3.1 (a) and Figure 3.2 (a)). If $d_G(v) = 6$ and G has another subgraph $H' \cong K_4$ with $V(H) \cap V(H') = \{v\}$, and v has neighbors $\{x_1, x_2, x_3\}$ in H, v has neighbors $\{y_1, y_2, y_3\}$ in H', then let G_v be the graph obtained from G by splitting the vertex $v \in V(G)$ into v_1, v_2 (see Figure 3.1 (b)), and by the first deleting x_3v_1, y_3v_2 and then contracting v_1x_1, v_2y_1 (see Figure 3.1 (c)); if $d_G(v) > 6$, let G_v be the graph obtained from G by splitting the vertex $v \in V(G)$ into v_1, v_2, deleting the edge v_1x_3, and then contracting v_1x_1 (see Figure 3.2 (c)).*

(i) *If* $G_v \in \langle \mathbf{Z}_3 \rangle$, *then* $G \in \langle \mathbf{Z}_3 \rangle$.

(ii) *If for some* $u \in V(G) - v$, G_v *is* $\mathbf{Z}_3$*-extensible from* u, *then* G *is also* $\mathbf{Z}_3$*-extensible from* u.

Proof The proof of (i) is given in [22]. The proof of (ii) is similar to that of (i) and so it is omitted. □

4 Group Colorings as a Dual Concept of Group Connectivity

In this section, we assume the basic knowledge of matroid theory. Both [23] and [24] can be used as resourceful references. In particular, we assume the knowledge of cycle matroids of graphs. Unless otherwise stated, in this section, A denotes a (multiplicative) group that is not necessarily abelian, and we again use $F(G, A)$ to denote the set of all maps from $E(G)$ into A. While the definition of the group colorings is given for any arbitrary group, all the results in this section are restricted to abelian groups.

The dual relationship between graph colorings and nowhere zero flows has been observed by Tutte [6]. Steinberg [25] particularly surveyed the 3-coloring problems. Thus the dual version of the 3-coloring theorem on planar graphs (see [26–29] and [30], among others) indicates that every 4-edge-connected planar graph has a 3-NZF. Jaeger et al. [3] also indicates that a similar duality between group connectivity and group colorings exists. The following is an extended version of the discussion on this duality in [31].

Let G be a graph. Denote by D an orientation of $E(G)$. An *arc* is an oriented edge uv of G (assumed to be directed from u to v), and is denoted by (u, v).

For $f \in F(G, A)$, an (A, f)*-coloring* of G under the orientation D is a function c: $V(G) \to A$ such that for every arc $e = (u, v) \in E(G)$, $c(u)c(v)^{-1} \neq f(e)$. G is A*-colorable* under the orientation D if and only if for every $f \in F(G, A)$, G has an (A, f)-coloring. For a group A, let $\langle A \rangle^*$ denote the family of all graphs that are A-colorable.

We first indicate that the property for a graph G to be A-colorable is independent of the choice of the orientation of the graph.

Proposition 4.1 *Let* D *be an orientation of* $E(G)$ *and* E_0 *be a subset of* $E(G)$. *Let* D' *be the orientation of* $E(G)$ *obtained from* D *by reversing the direction of every arc in* E_0. *Assume that* A *is a non-trivial abelian group. If* G *is* A*-colorable under the orientation* D, *then* G *is also* A*-colorable under the orientation* D'.

Proof Let $f' \in F(G, A)$. We consider the ordered pair (D, f), where f is defined as follows:

$$f(e) = \begin{cases} f'(e), & \text{if } e \notin E_0, \\ -f'(e), & \text{if } e \in E_0. \end{cases} \tag{3}$$

Since G is A-colorable under the orientation D, there exists a function c: $V(G) \to A$ such that for every arc $e = xy \in E[D(G)]$, $c(x) - c(y) \neq f(e)$. If $e \notin E_0$, then $e \in E[D'(G)]$ and $c(x) - c(y) \neq f(e) = f'(e)$. If $e \in E_0$, then $yx \in E[D'(G)]$ and $c(x) - c(y) \neq f(e)$, namely, $c(y) - c(x) \neq -f(e) = f'(e)$. Hence, G is A-colorable under the orientation D'. □

For a group A, let $\langle A \rangle^*$ denote the set of all graphs that are A-colorable. The *group* *chromatic number* of a graph G is defined by

$$\chi_g(G) = \min\{\ m \mid G \in \langle A \rangle^* \text{ for any group } A \text{ with } |A| \geq m\}. \tag{4}$$

Accordingly, the *abelian group chromatic number* of G is defined by

$$\chi_a(G) = \min\{\ m \mid G \in \langle A \rangle^* \text{ for any abelian group } A \text{ with } |A| \geq m\}. \tag{5}$$

Let $\chi(G)$ denote the chromatic number of a graph G. Then it follows immediately from the definitions that

$$\chi(G) \leq \chi_a(G) \leq \chi_g(G). \tag{6}$$

As noted in [32], the gap between $\chi(G)$ and $\chi_a(G)$ can in general be arbitrarily large. However, the difference between $\chi_a(G)$ and $\chi_g(G)$ is not well understood.

Tutte first indicated the duality between the chromatic number and the flow number among planar graphs.

Theorem 4.2 (Tutte [4]) *Let G be a (loopless) plane graph and G^* be its plane dual. Then $\chi(G) = \Lambda(G^*)$.*

To understand the duality between group connectivity and group colorings, we need the following definition.

Definition 4.3 *Let M be a connected regular matroid and A be an abelian group. Define the* circuit matrix D_M *of M to be the incident matrix of circuits against elements. Thus d_{ij}, the (i,j)-entry of D_M, is one if the circuit labelled i contains the element labelled j, and is zero otherwise. Denote by $D_M(C)$ the row of D_M corresponding to the circuit C of M. Define the* cocircuit matrix $D_M^* = D_{M^*}$. *When M is understood from the context, we shall write D and D^* for D_M and D_M^*, respectively.*

An orientation (referred to as a pair of orthogonal signing *on Page* 456 *of* [24]) *of a regular matroid M is a pair of matrices $(w(D_M), w(D_M^*))$ obtained from (D_M, D_M^*) by multiplying each entry of each of D_M and D_M^* by a factor in $\{1, -1\}$ such that $w(D) \cdot (w(D^*))^T = 0$. Let $F_0(M, A)$ denote the subset of $F(M, A)$ such that $f \in F_0(M, A)$ if and only if the matrix product $f \cdot w(D^*)^T = 0$, where $w(D^*)^T$ is the transpose of $w(D^*)$, and where each function $f \in F(M, A)$ is viewed as a row vector indexed by the elements in $E(M)$ in the same way that the columns of D are indexed.*

Notice that a cocircuit of M^* is a circuit of M. By the definition of $F_0(M, A)$, we have the following lemma.

Lemma 4.4 *Fix an orthogonal signing $(w(D_M), w(D_M^*))$ of a regular matroid M. A function $f \in F_0(M^*, A)$ if and only if, for any circuit C of M, the dot product $w(D_M(C)) \cdot f = 0$.*

Following Theorem 1.7 (ii), we give the following definition of group connectivity of a regular matroid.

Definition 4.5 *Let A be a nontrivial abelian group, and M be a connected regular matroid with a pair of orthogonal signing $(w(D_M), w(D_M^*))$. For any $\bar{f} \in F(M, A)$, an $f \in F_0(M, A)$ is an $(A, \bar{f})$-*NZF *if $f - \bar{f} \in F^*(M, A)$. The matroid M is A-*connected *if for any $\bar{f} \in F(M, A)$, M has an $(A, \bar{f})$-*NZF.

In the next theorem, we shall show the duality between A-coloring and A-connectivity. Before presenting the theorem and its proof, we shall first show an example explaining the key points.

Let G be a graph, $M = M(G)$ be the cycle matroid of G and $M^* = M^*(G)$ be the dual matroid of M. An *arborescence rooted at* v, T, is an orientation of a tree such that the indegree of every vertex other than v is exactly one, while the indegree of v is zero. Therefore, if G is a connected graph with a distinguished vertex v, then G has an orientation D such that $D(G)$ has a spanning arborescence rooted at v.

Example 4.6 Let $G = K_4$ be a plane graph and let G^* denote its geometric dual. We first show how the A-colorability of G will imply the A-connectedness of G^*.

We assume that both G and G^* are orientated, as shown in Figure 4.1.

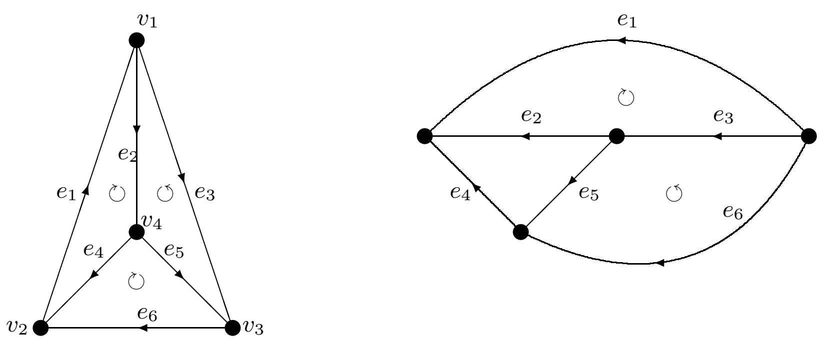

Figure 4.1 The Graph G and its geometric dual G^*

We first choose a base $B = \{e_1, e_4, e_5\} \in \mathcal{B}(M(G))$. Then the corresponding cobase $B^* = E - B = \{e_2, e_3, e_5\}$. Once B and B^* are determined, we can compute the corresponding element-fundamental circuit incidence matrices in the follows way. For D, give each circuit of G an orientation. For fundamental circuit C, and for an element $e_i \in C$, the e_i-th component in the incidence vector of C is 1 if the orientation of e_i agrees with the orientation of C, and is -1 if the orientation of e_i disagrees with the orientation of C. (Note that the orientation for the fundamental circuit $\{e_1, e_3, e_4, e_5\}$ is not indicated in Figure 4.1). The entries of D^* can be computed similarly. In this way, both D and D^* in this example can be found as follows:

$D =$

elements	fundamental circuits with respect to B	e_1	e_2	e_3	e_4	e_5	e_6
e_2	e_1 e_2 e_4	1	1	0	1	0	0
e_6	e_4 e_5 e_6	0	0	0	-1	1	1
e_3	e_1 e_3 e_4 e_5	1	0	1	1	-1	0

and the corresponding cocircuit-element incidence matrix

$D^* =$

elements	fundamental circuits with respect to B^*	e_1	e_2	e_3	e_4	e_5	e_6
e_1	e_1 e_2 e_3	-1	1	1	0	0	0
e_4	e_2 e_3 e_4 e_6	0	-1	-1	1	0	1
e_5	e_3 e_5 e_6	0	0	1	0	1	-1

.

As $D \cdot (D^*)^T = 0$, the pair (D, D^*) is a signing of $M(G)$. Let $A = \mathbf{Z}_4 = \{0, 1, 2, 3\}$ denote the cyclic group of order 4. We shall first show that G^* is A-connected, assuming G is A-colorable.

For a graph Γ with n vertices and m edges, any $f \in F(\Gamma, A)$ can be denoted as $f = (f(e_1), \dots f(e_m))$, and so it can be viewed as an m-dimensional vector with each component being in A. Similarly, any map $c : V(\Gamma) \mapsto A$ can also be written as an n-dimensional vector $c = (c(v_1), \dots, c(v_n))$, with each component being in A.

As an example, let $\overline{f} = (1, 2, 3, 0, 2, 1) \in F(G^*, A)$, we shall find an $f_0 \in F_0(G, A)$ such that $f_0 - \overline{f}$ is nowhere-zero. Since $F(G^*, A) = F(G, A)$, we can view $\overline{f} \in F(G, A)$. Since G is A-colorable, there exists an $(A, \overline{f})$-coloring $c = (0, 2, 2, 1)$.

Now we shall construct $f_0 \in F_0(G^*, A)$ such that $f_0 - \overline{f}$ is nowhere-zero. Recall that $B = \{e_1, e_4, e_5\} \in \mathcal{B}(M(G))$ has been chosen. Let $T = G^*[B]$ as a spanning tree. For each $e = (v_i, v_j) \in B$, define $f_0(e) = c(v_i) - c(v_j)$, and let $x_i = f(e_i)$ for each $e_i \notin B$. Therefore, $f_0 = (2, x_2, x_3, 3, 3, x_6)$. Thus we can determine the values of the unknown x_i's by solving

$$D^T \cdot f_0 = 0.$$

As examples, to determine x_2, let $C_{e_2} = \{e_1, e_2, e_4\}$ denote the fundamental circuit of e_2 with respect to B, and let $D(e_2)$ denote the row of D corresponding to C_{e_2}. Then $D(e_2) \cdot f_0 = 0$ implies that $2 + x_2 - 1 = 0$, and so $x_2 \equiv -1 \equiv 3 \pmod 4$. Similarly, let $C_{e_3} = \{e_1, e_3, e_4, e_5\}$ and $C_{e_6} = \{e_4, e_5, e_6\}$ denote the fundamental circuits of e_3 and e_6 with respect to B, respectively, and let $D(e_3)$, $D(e_6)$ denote the rows of D corresponding to C_{e_3} and C_{e_6}, respectively. Then $D(e_3) \cdot f_0 = 0$ and $D(e_6) \cdot f_0 = 0$ respectively imply that $2 + x_3 - 1 + 1 = 0$ and $1 - 1 + x_6 = 0$, and so $x_3 \equiv -2 \equiv 2 \pmod 4$ and $x_6 = 0 \pmod 4$. It follows that $f_0 = (2, 3, 2, 3, 3, 0)$.

We continue this example and show that G^* is A-colorable if G is A-connected. As an example, let $f' = (1, 2, 3, 0, 2, 1) \in F(G, A)$. We want to find a map $c : V(G) \mapsto A$ such that for any oriented edge $e = (u, v)$, $c(u) - c(v) \neq f'(e)$. We again choose $B = \{e_1, e_4, e_5\} \in \mathcal{B}(M(G))$. Then in G^*, $B^* = E - B = \{e_2, e_3, e_6\} \in \mathcal{B}(M(G)^*) = \mathcal{B}(M^*(G))$. Note that G is A-connected. Given f', there exists an $f_0 \in F_0(G, A)$ such that $f' - f_0$ is nowhere-zero. In order to have a better understanding of how f_0, f' and c are related, we choose $f_0 = (2, 3, 2, 3, 3, 0)$, and start the construction of c. Firstly, we orient G^* from the orientation of G (for example, we can turn the arrows of all edges in G 90 degrees). Choose a vertex as a root r of the tree B^* in G^*.

We first set $c(r) = 0$ and $S := \{r\}$. While $\overline{S} = V - S \neq \emptyset$, we examine the edge subset in B^* with head in S and tail in $\overline{S}$:

$$[S, \overline{S}]_{B^*} = \{(v_{i_1}, v_{j_1}), (v_{i_2}, v_{j_2}), \dots, (v_{i_k}, v_{j_k})\}.$$

For each such edge $e_t = (v_{i_t}, v_{j_t})$, define

$$c(v_{j_t}) = c(v_{i_t}) - f_0(e_t), \quad 1 \le t \le k,$$

and update $S := S \cup \{v_{j_1}, v_{j_2}, \dots, v_{j_k}\}$. Thus S denotes the set of all the vertices on which c is defined in the process.

In this example, we choose $r = v_4$. Then $S = \{v_4\}$, and $[S, \overline{S}]_{B^*} = \{(v_4, v_2), (v_4, v_3)\}$. By definition of c, we extend $c(v_2) = c(v_4) - f_0(e_4) \equiv 0 - 3 \equiv 1 \pmod 4$, and $c(v_3) = c(v_4) - f_0(e_5) \equiv 0 - 3 \equiv 1 \pmod 4$. Thus we update $S = \{v_2, v_3, v_4\}$, and examine $[S, \overline{S}]_{B^*} = \{(v_2, v_1)\}$. By definition of c again, $c(v_1) = c(v_2) - f_0(e_1) \equiv 1 - 2 \equiv 3 \pmod 4$. Hence $c = (3, 1, 1, 0)$.

Theorem 4.7 *Let G be a 2-connected graph and A be an abelian group. Then G is A-colorable if and only if $M^*(G)$ is A-connected.*

Proof Let $V(G) = \{v_1, v_2, \dots, v_n\}$. We shall fix an orientation $D = D(G)$ of G so that $D(G)$ has a spanning arborescense T rooted at v_1. Since $E(M^*) = E(M) = E(G)$, $F(M^*, A) = F(M, A) = F(G, A)$.

First we assume that G is A-colorable. Given $f' \in F(M^*, A) = F(G, A)$, there is an (A, f')-coloring $c : V(G) \mapsto A$ such that $c(x) - c(y) \neq f'(x, y)$, for each directed edge (x, y) in $D(G)$.

Define $f \in F(T, A)$ by

$$f(v_i, v_j) = c(v_i) - c(v_j), \quad \text{whenever } (v_i, v_j) \in E(T). \tag{7}$$

Extend f to a function in $F(G, A)$ as follows: For each edge $e = (x, y) \in E(G) - E(T)$, the undirected underlying graph of $T + (x, y)$ has a unique circuit C_e, the fundamental circuit of e with respect to T. The orientation D of G gives rise to an orientation $w(D(C_e))$ of C_e. Define $f(e)$ to be the unique value which makes the dot product $(w(D(C_e))) \cdot f = 0$. Now for each fundamental circuit C with respect to T, $(w(D(C))) \cdot f = 0$, and so the matrix product $(w(D)) \cdot f = 0$, since any circuit, when viewed as a vector in the cycle space, is a linear combination of the fundamental circuits. Therefore, by Lemma 4.4, $f \in F_0(M^*, A)$.

By (7) and by the way f is defined on $E(G) - E(T)$, for each directed edge (x, y) in $D(G)$, $f(x, y) = c(x) - x(y) \neq f'(x, y)$, and so M^* is A-connected.

Conversely, assume that M^* is A-connected. Let $f' \in F(G, A) = F(M^*, A)$. Then there is a function $f \in F_0(M^*, A)$ such that $f(e) \neq f'(e)$ for each directed edge e in $D(G)$. We define a map $c : V(T) \mapsto A$ inductively as shown by the algorithm below.

Algorithm: Definition of c:

Step 1 Set $c(v_1) := 0$ and $S := \{v_1\}$.

Step 2 While $\bar{S} = V(G) - S \neq \emptyset$, Do if $[S, \bar{S}]_T = \{(v_{i_1}, v_{j_1}), (v_{i_2}, v_{j_2}), \dots, (v_{i_k}, v_{j_k})\}$, then $c(v_{j_t}) := c(v_{i_t}) - f(v_{i_t}, v_{j_t})$, where $1 \le t \le k$. Set $S := S \cup \{v_{j_1}, v_{j_2}, \dots v_{j_k}\}$.

Since $V(G) = V(T)$, c defined above is also a map from $V(G)$ to A. For each directed edge $e = (x, y)$ in $D(G)$, if $e \in E(T)$, then $f(x, y) = c(x) - c(y) \neq f'(x, y)$. Assume that $e \in E(G) - E(T)$. Then the undirected underlying graph of $T + e$ has a unique circuit C_e. Since $f \in F_0(G, A)$, the dot product $f \cdot (D(C_e))^T = 0$, and so we have both $f(x, y) = c(x) - c(y)$ and $f(x, y) \neq f'(x, y)$. Thus G is A-colorable. □

Corollary 4.8 *Let G be a connected plane graph, G^* the geometric dual of G, and A an abelian group. Then each of the following statements holds*:

(i) *G is A-connected if and only if G^* is A-colorable.*

(ii) *$\Lambda_g(G) = \chi_a(G)$.*

Proof (i) follows from the fact that $M(G^*) \cong (M(G))^*$, and (ii) follows from (i). □

Conjecture 4.9 *Let $\chi_l(G)$ denote the choice number of a graph G. Then $\chi_l(G) \le \chi_g(G)$.* (*Evidence can be found in* [32] *and* [33]*, especially in* [33]).

Conjecture 4.10 *Let $k > 1$ be an integer. If G does not have a K_k-minor, then $\chi_g(G) \le k$.*

Remark This may be the group coloring version of the Hadwiger conjecture. It holds when $k \leq 5$ (see [32] and [33]).

While the property for a graph G to be A-colorable is independent of the choice of the orientation of G, whether this property depends on the structure of the group is not clear. We have the following problems.

Problem 4.11 Let $\mathcal{S}$ denote the family of all simple graphs. Is it true that for two abelian groups A_1 and A_2, if $|A_1| = |A_2|$, then

$$\langle A_1 \rangle^* \cap \mathcal{S} = \langle A_2 \rangle^* \cap \mathcal{S}?$$

Problem 4.12 For two abelian groups A_1 and A_2, if

$$\langle A_1 \rangle^* \cap \mathcal{S} = \langle A_2 \rangle^* \cap \mathcal{S},$$

does it imply that $|A_1| = |A_2|$?

5 Brooks Theorem, Its Variations and Dual Forms

The classical Brooks coloring theorem states that for every connected simple graph G, $\chi(G) \leq \Delta(G) + 1$, where equality holds if and only if G is an odd circuit or a complete graph. Similar results can also be obtained for group chromatic numbers. As $\chi(G) \leq \chi_a(G) \leq \chi_g(G)$, the following theorem extends the classic Brooks Coloring Theorem.

Theorem 5.1 *Let G be a connected simple graph with spectral radius $\rho(G)$ (the maximum eigenvalue of the adjacency matrix of G).*

(i) (Lai and Zhang [32, Theorem 4.2]) $\chi_a(G) \leq \Delta(G) + 1$, *where equality holds if and only if G is a circuit* ($\Delta(G) = 2$), *or G is complete* ($\Delta(G) \geq 3$).

(ii) (Liu and Lai [34, Theorem 4.8.2]) $\chi_a(G) \leq \rho(G) + 1$, *where equality holds if and only if G is a complete graph or a circuit.*

Unlike the classic vertex coloring, the group chromatic number of a multigraph G can be quite different from the group chromatic number of its simplification. Li [35] investigated the group chromatic number of multigrahgs.

Definition 5.2 *Let G denote a multigraph graph. Define a relation on $E(G)$: two edges have the relation if and only if they are identical or they form a 2-circuit in G. Then this relation is an equivalence relation on $E(G)$. The* multiplicity *of G, denoted by $M(G)$, is the maximal size of an equivalence class. The* simplification *of G, denoted by $\tilde{G}$, is a simple graph obtained by replacing every equivalence class by a single edge. For a graph H and a positive integer k, we define kH to be the graph obtained by replacing each edge of H by a class of k parallel edges.*

The following theorem extends Theorem 5.1 not only from simple graphs to multigraphs, but also from abelian groups to groups.

Theorem 5.3 (Li [35]) *For any connected multi-graph G,*

$$\chi_g(G) \leq \Delta(G) + 1, \tag{8}$$

where equality holds if and only if for some positive integer k, $G = kC_n$ or kK_n.

The following fact has been observed, which motivates the definition of semi-critical graphs.

Lemma 5.4 (Lai et al. [37]) *Let G be a graph and $v \in V(G)$, and $H = G - v$. If $d_G(v) < \chi_a(H)$, then $\chi_a(G) = \chi_a(H)$.*

The Szekeres and Wilf inequality [36] indicates that the classical Brooks coloring inequality can be extended so that $\Delta(G)$ can be replaced by a more generic graphical function. Investigation to find the same kind of inequality for group chromatic number has also been conducted [37].

Definition 5.5 *A graph G is $\chi_a(G)$-*semi-critical *if $\chi_a(G - v) < \chi_a(G)$, for every vertex $v \in G$ with $d(v) = \delta(G)$.*

Lemma 5.6 (Lai et al. [37]) *If G is $\chi_a(G)$-semicritical with $\chi_a(G) = k$, then $d_G(v) \geq k - 1$ for all $v \in V(G)$.*

Let $\gamma(G)$ be a real-valued graphical function with the following two properties:

(P1) If H is an induced subgraph of G, then $\gamma(H) \leq \gamma(G)$;

(P2) $\gamma(G) \geq \delta(G)$ with equality if and only if G is regular.

Theorem 5.7 (Lai et al. [37]) *Let γ be a graphical function satisfying* (P1) *and* (P2). *For any connected simple graph G,*

$$\chi_a(G) \leq \gamma(G) + 1.$$

If G is $\chi_a(G)$-semi-critical, then $\chi_a(G) = \gamma(G) + 1$ if and only if G is a circuit or a complete graph.

In view of Theorem 5.7, a question arises: "What is the dual version of Brooks Coloring Theorem?" In an attempt to answer this question, the following definitions are needed.

Definition 5.8 *Let G be a 2-edge-connected graph with $|V(G)| \geq 2$, and let $\mathcal{C}(G)$ denote the set of all circuits of G. Define, $\forall\, e \in E(G)$,*

$$d_G(e) = \min\{|C| \mid e \in E(C) \text{ and } C \in \mathcal{C}(G)\},$$

and $\Delta'(G) = \max\{d_G(e) \mid e \in E(G)\}$.

Problem 5.9 It can be proved that $\Lambda_g(G) \leq \Delta'(G) + 1$. Determine the graphs with equality.

Problem 5.9 seems to be somewhat difficult. There have been some efforts in solving this problem.

Definition 5.10 *Let $m, n \geq 1$ and $t \geq 2$ be integers. Let tK_2 denote the loopless connected graph with two vertices and t edges. Define $K_{2,t}(m)$ by replacing each edge of tK_2 by a path of length exactly m, and define $K_n(m)$ to be the graph obtained from the complete graph K_n by replacing each edge of K_n by a path of length exactly m.*

With these definitions, we have the following dual form of Brooks Coloring Theorem.

Theorem 5.11 (Yao [38]) *Let G be a 2-edge-connected graph and let $c(G)$ denote the circumference of G (the length of a longest circuit of G). Then*

$$\Lambda_g(G) \leq c(G) + 1, \tag{9}$$

where equality holds if and only if each of the following statements holds:

(i) *G has at least one block B such that either $c(G)$ is odd and B is an odd circuit of length $c(G)$, or $c(G)$ is even and B is isomorphic to a $K_{2,t}(c(G)/2)$, for some integer $t \geq 2$.*

(ii) *Every block H of G is either a subgraph with $\Lambda_g(H) \le c(G)$, or $c(G)$ is odd and H is a circuit of length $c(G)$, or $c(G)$ is even and H is isomorphic to a $K_{2,t}(c(G)/2)$, for some integer $t \ge 2$.*

This theorem can be improved since for many graphs, $c(G) \ge \Delta'(G)$. This observation motivates the next definition.

Definition 5.12 *Let G be a 2-edge-connected graph with $|V(G)| \ge 2$, and let P_2 denote a path of 2 edges. For each $P_2 \subseteq G$, define $g_2(P_2) = \min\{|C| \mid P_2 \subseteq C \in \mathcal{C}(G)\}$ and $g_2(G) = \max\{g_2(P_2) \,|P_2 \subset G\}$.*

It follows from this definition that, for a plane graph G with its geometric dual G^*, we have $g_2(G) \le \Delta(G^*)$.

Theorem 5.13 (Yao [38]) *Let G be a 2-edge-connected graph. Then*

$$\Lambda_g(G) \le g_2(G) + 1, \tag{10}$$

where equality holds if and only if $G \in \{C_k | k \ge 2\} \cup \{K_{2,t}(m) | m \ge 1, t \ge 3\} \cup \{K_4(k) | k \ge 1\}$.

Other results attacking Problem 5.9 when $\Delta'(G) = 3$ are summarized in Subsection 7.7.

6 Planar Graphs

Let G be a plane graph which has no loops or multiple edges. Call G a *near-triangulation* if every face of G other than the exterior face is a 3-circuit.

Definition 6.1 *Let ϕ_1 and ϕ_2 be two maps with domains D_1 and D_2, respectively. Assume that $\phi_1(x) = \phi_2(x)$ for any $x \in D_1 \cap D_2$. If $D_1 \subseteq D_2$, then ϕ_2 is an* extension *of ϕ_1 and we write $\phi_2|_{D_1} = \phi_1$. Note that if $\phi_1(x) = \phi_2(x)$ for every $x \in D_1 \cap D_2$, then the map*

$$\phi(x) = \begin{cases} \phi_1(x), & \text{if } x \in D_1, \\ \phi_2(x), & \text{if } x \in D_2 \end{cases}$$

is a well-defined map with domain $D_1 \cup D_2$. In this case, we say that ϕ is obtained by combining ϕ_1 and ϕ_2.

Theorem 6.2 (Lai and Zhang [33, Theorem 2.1]) *Assume that G is a near-triangulation with an orientation D and an exterior directed circuit $C = v_1v_2\cdots v_pv_1$. Let A be an abelian group with $|A| \ge 5$ and $f \in F(G, A)$. Let $a_1, a_2 \in A$ such that $a_1 - a_2 \ne f(v_1v_2)$, and let $A_3, A_4, \ldots, A_p$ be subsets of A such that $|A_i| = 2$ for $4 \le i \le p$, and $|A_3| = |A| - 3$. Then there is an (A, f)-coloring $c : V(G) \mapsto A$ such that $c(v_1) = a_1$, $c(v_2) = a_2$, and $c(v_i) \notin A_i$, $3 \le i \le p$.*

Definition 6.3 *Let $H \subseteq G$ be graphs, and A be a group. Given an $f \in F(G, A)$, if for an $(A, f|_{E(H)})$-coloring c_0 of H, there is an (A, f)-coloring c of G such that c is an extension of c_0, then we say that c_0 is* extended to *c. If any $(A, f|_{E(H)})$-coloring c_0 of H can be extended to an (A, f)-coloring c, then we say that (G, H) is (A, f)*-extensible. *If for any $f \in F(G, A)$, (G, H) is (A, f)-extensible, then (G, H) is A*-extensible.

Corollary 6.4 *Let G be a simple planar graph and let $H \subseteq G$ be a subgraph. Each of the following statements holds*:

(i) (Lai and Zhang [33, Corollary 2.2]) *If H is isomorphic to a K_2 or a K_3, then for any group A with $|A| \ge 5$, (G, H) is A-extensible.*

(ii) (Lai and Zhang [33, Corollary 2.3]) *If H is isomorphic to a K_4, then for any group A with $|A| \geq 5$, (G, H) is A-extensible.*

Theorem 6.5 *Let G be a simple planar graph. Each of the following statements holds:*

(i) (Lai and Li [39, Theorem 1.1]) *If the girth of G is at least 5, then $\chi_a(G) \leq 3$.*

(ii) (Li [40]) *If all the 3 circuits of G are vertex disjoint, then $\chi_a(G) \leq 4$.*

Král et al. in [41] constructed the families of simple planar graphs showing the sharpness of some of these results.

Theorem 6.6 (Král et al. [41]) (i) ([41, Theorem 3]) *There exist simple bipartite planar graphs G with $\chi_a(G) = 4$.*

(ii) ([41, Theorem 4]) *There exist simple planar graphs G with $\chi_a(G) = 5$.*

Using the duality, we have the following two results for planar graphs in terms of group connectivity.

Theorem 6.7 *Let G be a simple planar graph. Each of the following statements holds:*

(i) *If $\kappa'(G) \geq 3$, then $\Lambda_g(G) \leq 5$.*

(ii) *If $\kappa'(G) \geq 5$, then $\Lambda_g(G) \leq 3$.*

Using the decomposition of K_5-minor free graphs and of $K_{3,3}$-minor free graphs by Wagner [42], the following results are also proved.

Theorem 6.8 *Let G be a simple graph. Each of the following statements holds:*

(i) (Lai and Zhang [33, Theorem 3.3]) *If G does not have a K_5-minor, then $\chi_a(G) \leq 5$.*

(ii) (Lai and Li [43, Theorem 2.8]) *If G does not have a $K_{3,3}$-minor, then $\chi_a(G) \leq 5$.*

(iii) (Lai and Li [39, Theorem 3.4]) *If G does not have a $K_{3,3}$-minor and if the girth of G is at least 5, then $\chi_a(G) \leq 3$.*

7 Group Connectivity of Graphs

Lots of attentions have been paid to the investigation of bounds of group connectivity number of graphs. We survey the results in this area in different categories.

7.1 Highly Connected Graphs and Collapsible Graphs

When the edge connectivity of a graph G is high, then $\Lambda_g(G)$ will not be too large.

Theorem 7.1 (Jeager et al. [3]) *Let G be a graph.*

(i) *If $\kappa'(G) \geq 3$, then $\Lambda_g(G) \leq 6$.*

(ii) *If $\kappa'(G) \geq 4$, then $\Lambda_g(G) \leq 4$.*

In fact, a stronger version of Part (ii) in Theorem 7.1 was also proved in [3]. We here present a different proof.

Theorem 7.2 (Jeager et al. [3]) *Every graph which contains 2 edge-disjoint spanning trees (in particular every 4-edge connected graph) is A-connected for every abelian group A of order $|A| \geq 4$.*

Proof Throughout the proof, $E = E(G)$. Suppose that G has 2 edge disjoint spanning trees T_1 and T_2. Let A be an abelian group with $|A| \geq 4$. Let $\overline{f} \in F(T_2, A)$, we are to find $f \in F_0(G, A)$ such that $f - \overline{f} \in F^*(G, A)$, and so by Theorem 1.7, G is A-connected. For a spanning tree T

of G and for an edge $e \in E - T$, let $C(e, T)$ denote the unique circuit in $T \cup e$, known as the fundamental circuit of e with respect to T.

Let $C_1 = \triangle_{e \in E(G) - T_1} C(e, T_1)$ denote the binary sum of these fundamental circuits with respect to T_1. Then by Lemma 1.6, $\exists f_1 \in F_0(G, A)$ such that $f_1(e) = \pm x \neq 0$ for any $e \in C_1$, and $f_1(e) = 0$ for any $e \notin C_1$. Note that $E - T_1 \subseteq C_1$.

Define $X = \{e \in T_1 | f_1(e) = \overline{f}(e)\}$ and $Y = T_1 - X$. Let $C_2 = \triangle_{e \in X} C(e, T_2)$. Since $|A| \geq 4$, $\exists y \in A - \{0, x, -x\}$, and so by Lemma 1.6, $\exists f_2 \in F_0(G, A)$ such that $f_2(e) = \pm y \neq 0$ for any $e \in C_2$, and $f_2(e) = 0$ for any $e \notin C_2$.

Let $f = f_1 + f_2$. Since $f_1, f_2 \in F_0(G, A)$, $f \in F_0(G, A)$. Note that $E = (E - T_1) \cup X \cup Y$ is a disjoint union. By $E - T_1 \subseteq C_1$,

$$E(G) = (C_1 \cap C_2) \cup [(E - T_1) - C_2] \cup (X - C_1) \cup Y \tag{11}$$

is also a disjoint union. As $\overline{f}(e) = 0$, $\forall\, e \notin T_2$, it follows that

$$(f - \overline{f})(e) = f_1(e) + f_2(e) - \overline{f}(e) = \begin{cases} \pm x \pm y, & \text{if } e \in C_1 \cap C_2, \\ \pm x, & \text{if } e \in (E - T_1) - C_2, \\ \pm y, & \text{if } e \in X - C_1, \\ \neq 0, & \text{if } e \in Y. \end{cases}$$

Thus, $f(e) - \overline{f}(e) \in F^*(G, A)$, and so by Theorem 1.7, $G \in \langle A \rangle$. □

Theorem 7.3 *Let G be a simple graph on $n \geq 3$ vertices. Each of the following statements holds:*

(i) (Lai and Zhang [44]) *If $\kappa'(G) \geq 4 \log_2(n)$, then $\Lambda(G) \leq 3$.*

(ii) (Lai et al. [45]) *If $\kappa'(G) \geq 3 \log_2(n)$, then $\Lambda_g(G) \leq 3$.*

For a graph G, let $O(G)$ denote the set of all odd degree vertices of G. For a subset $T \subseteq V(G)$ with $|T| \equiv 0 \pmod 2$, a subgraph H is a *T-join* if $O(H) = T$. A graph G is *collapsible* if for any even subset $T \subseteq V(G)$, G always has a spanning connected T-join.

Theorem 7.4 (Lai [46, (1.6)]) *Let G be a collapsible graph and let A be an abelian group with $|A| = 4$. Then $G \in \langle A \rangle$.*

The relationship between collapsible graphs and A-connected graphs is not clear. Many questions are waiting to be answered. We believe that Conjecture 7.5 would hold. There are also two open problems in this direction that are worth investigating.

Conjecture 7.5 *If G is collapsible, then $\Lambda_g(G) \leq 4$.*

The key to show Conjecture 7.5 is that if G is collapsible, then $\Lambda_g(G) \leq 5$. It is routine to show that if G is collapsible, then $\Lambda_g(G) \leq 6$. This is known but not yet published.

Problem 7.6 It is well known that every graph with a spanning Eulerian subgraph (called a supereulerian graph) has a nowhere-zero 4-flow. Is it true that if G is a 3-edge-connected supereulerian graph, then $\Lambda_g(G) \leq 4$? (or $\Lambda_g(G) \leq 5$?)

Problem 7.7 Let F_k^o denote the family of graphs H such that for any graph G, if G has H as a subgraph, then G has a k-NZF if and only if G/H has a k-NZF. Let $\mathcal{CL}$ denote the family of collapsible graphs. Catlin [16] showed that $\mathcal{CL} \subset F_4^o$. Theorem 7.4 indicates that

$\mathcal{CL} \subseteq \langle A\rangle \subset F_4^o$ if $A \in \{\mathbf{Z}_4, \mathbf{Z}_2 \times \mathbf{Z}_2\}$. However, it is not known that for any other abelian group A with $|A| \geq 3$, whether $\mathcal{CL} \subseteq \langle A\rangle$. In particular, we do not know any counterexample to the following conjectures:

(i) If A is an abelian group with $|A| \geq 5$, then $\mathcal{CL} \subseteq \langle A\rangle$.

(ii) $\langle \mathbf{Z}_3\rangle \subset \mathcal{CL}$.

7.2 Degree Conditions

The earliest sufficient degree condition for $\mathbf{Z}_3$-connectivity was first proved by Barat and Thomassen in an implicit form.

Theorem 7.8 (Barat and Thomassen [47, Theorem 5.2]) *There exists a positive integer n_1 such that every simple graph on $n \geq n_1$ vertices with minimum degree at least $n/2$ is $\mathbf{Z}_3$-connected.*

A much sharper result is claimed without proofs in [47].

Theorem 7.9 (Barat and Thomassen [47, Theorem 5.3]) *There exists a positive integer n_2 such that every simple 2-edge-connected graph on $n \geq n_2$ vertices with minimum degree at least $n/4$ is $\mathbf{Z}_3$-connected.*

This theorem is, unfortunately, incorrect. An explicit counterexample (see [48]) can be given as follows. Let n be an integer with $n \equiv 0 \pmod 3$. Obtain $G(n)$ from K_3 by replacing every vertex of K_3 by a complete graph $K_{\frac{n}{3}}$. Then $G(n)$ is a 2-edge-connected simple graph with $\delta(G(n)) = \frac{n}{3} - 1 > \frac{n}{4}$. However, as $G(n)$ can be contracted to K_3, by Proposition 2.1 (C2), $G(n)$ cannot be $\mathbf{Z}_3$-connected.

In terms of explicit degree sufficient conditions for nowhere-zero flows, Fan and Zhou were the first to conduct such investigations. They published two papers in 2008 on sufficient degree conditions for a graph to have a 3-NZF. Their results are later improved by Luo et al. to sufficient degree conditions of $\mathbf{Z}_3$-connectivity.

Theorem 7.10 (Fan and Zhou [49]) *Let G be a simple graph on $n \geq 3$ vertices. If for every pair of nonadjacent vertices u and v in G, $d_G(u) + d_G(v) \geq n$, then either G has a nowhere-zero 3-flow or G is isomorphic to one of the six well classified exceptional graphs.*

For positive integers q and m, let $K^+_{2,q}$ be the graph obtained from $K_{2,q}$ be adding an edge joining two vertices of degree q, and let $K^+_{3,m}$ be the graph obtained from $K_{3,m}$ be adding an edge joining two vertices of degree m.

Theorem 7.11 (Fan and Zhou [50]) *Let G be a 2-edge-connected simple graph on n vertices. If $d_G(u) + d_G(v) \geq n$ for each $uv \in E(G)$, then G has no nowhere-zero 3-flow if and only if G is $K^+_{3,n-3}$ or G is one of the 5 exceptional graphs on at most 6 vertices.*

Theorem 7.12 (Lou et al. [51]) *Let G be a simple graph on n vertices. Suppose that $d_G(u) + d_G(v) \geq n$ for every pair of nonadjacent vertices. Each of the following statements holds:*

(i) *If $n \geq 3$, then either $\Lambda_G(G) \leq 3$, or G is one of the 12 well classified exceptional graphs.*

(ii) *If $n \geq 8$, then $\Lambda_G(G) \leq 3$.*

There are other results concerning degree conditions for group connectivity in general.

Theorem 7.13 (Sun et al. [52]) *Let G be a simple graph on $n \geq 3$ vertices. If $d_G(u) + d_G(v)$*

$\geq n$ for every pair of nonadjacent vertices, then either $\Lambda_g(G) \leq 4$, or G can be contracted to a 4-cycle.

Recently, Yao et al. [53], Zhang et al. [54], and Li et al. [55] improved some of the results above.

Theorem 7.14 (Yao et al. [53]) *Let A be an abelian group with $|A| \geq 4$, and let G be a 2-edge-connected simple graph on $n \geq 13$ vertices. If for every $uv \notin E(G)$, $\max\{d_G(u), d_G(v)\} \geq n/4$, then either G is A-connected, or $|A| = 4$ and G', the A-reduction of G obtained from G by contracting all nontrivial maximal A-connected subgraphs of G, satisfies $G' \in \{K_{2,3}, C_4, C_5\}$. Moreover, if $G' \in \{K_{2,3}, C_4\}$, then $\Lambda_g(G) = 5$; and if $G' = C_5$, then $\Lambda_g(G) = 6$.*

Theorem 7.15 (Zhang et al. [54]) *Let G be a simple graph on n vertices with $\delta(G) \geq 2$. If $d_G(x) + d_G(y) \geq n$ for each $xy \in E(G)$, then G is $\mathbf{Z}_3$-connected if and only if G is not one of the 13 exceptional graphs, and G does not belong to a family of $K_{2,p}$, $K_{2,q}^+$, $K_{3,l}$ and $K_{3,m}^+$, where $p \geq 2, q \geq 1, l \geq 3$ and $m \geq 2$.*

Theorem 7.16 (Li et al. [55]) *Let G be a simple 2-edge-connected graph on $n \geq 3$ vertices. If for every $uv \notin E(G)$, $\max\{d(u), d(v)\} \geq \frac{n}{2}$, then G is $\mathbf{Z}_3$-connected if and only if G is not isomorphic to one of 22 exceptional graphs or the $\mathbf{Z}_3$-reduction of G is not one of 4 specified graphs.*

Luo et al. considered the problem where the degree sequence of G has a realization in $\mathbf{Z}_3$. Two results are obtained.

Theorem 7.17 (Luo et al. [56, Theorems 3.1]) *Let $(d_1, d_2, \ldots, d_n)$ be a graphical sequence with $d_1 \geq d_2 > \cdots > d_n$. Then there exists a $\mathbf{Z}_3$-connected graph with the degree sequence $(d_1, d_2, \ldots, d_n)$ if $d_n \geq 3$ and $d_{n-3} \geq 4$.*

Theorem 7.18 (Luo et al. [56, Theorems 3.2]) *Let $(d_1, d_2, \ldots, d_n)$ be a graphical sequence with $n-1 = d_1 \geq d_2 \cdots d_n \geq 3$. Then there exists a $\mathbf{Z}_3$-connected graph with the degree sequence $(d_1, \ldots, d_n)$ if and only if $(d_1, \ldots, d_n) \neq (k, 3, 3, \ldots, 3), (k, k, 3, 3, \ldots, 3)$, where $k \geq 3$ is odd.*

A complete characterization for degree sequences that have realizations in $\mathbf{Z}_3$ seems to be an interesting result to be found. In general, characterizations of degree sequence that have realizations in $\langle A \rangle$ are needed to be investigated. We complete this subsection by the following problems and conjectures.

Problem 7.19 We consider a simple graph G on n vertices. If G satisfies a degree condition above, then $|E(G)| = O(n^2)$. Therefore, these degree condition results are in fact sufficient conditions for a graph G with edge density $O(n^2)$ to be in $\langle A \rangle$. Theorems 7.2 and 7.3, on the other hand, are sufficient conditions for a graph with $O(n)$ edges to be in $\langle A \rangle$ for all $|A| \geq 4$, and with $O(n \log(n))$ edges to be in $\langle \mathbf{Z}_3 \rangle$, respectively. Therefore, finding sufficient conditions for a graph with fewer than $O(n \log(n))$ edges or with $O(n)$ edges to be in $\langle \mathbf{Z}_3 \rangle$ will be of interest.

In particular, we propose the following conjecture.

Conjecture 7.20 *Let $k \geq 4$ and $l \geq 5$ be integers. Let G be a simple k-edge-connected graph. Define*

$$\sigma_2(G) = \min\{d(u) + d(v) \mid uv \notin E(G)\},$$

$$\sigma_2'(G) = \min\{d(u) + d(v) \mid uv \in E(G)\}.$$

(i) *There exists an integer* n_k *such that every simple* k*-edge-connected graph on* $n \geq n_k$ *vertices with* $\sigma_2(G) \geq \log(n)$ *has a* 3-NZF.

(ii) *There exists an integer* n_k' *such that every simple* k*-edge-connected graph on* $n \geq n_k'$ *vertices with* $\sigma_2'(G) \geq \log(n)$ *has a* 3-NZF.

(iii) *There exists an integer* m_l *such that every simple* l*-edge-connected graph on* $n \geq m_l$ *vertices with* $\sigma_2(G) \geq \log(n)$ *is* $\mathbf{Z}_3$*-connected.*

(iv) *There exists an integer* m_l' *such that every simple* l*-edge-connected graph on* $n \geq m_l'$ *vertices with* $\sigma_2'(G) \geq \log(n)$ *is* $\mathbf{Z}_3$*-connected.*

While higher density of edges in a graph would imply small group connectivity number, it has been observed that graphs with small group connectivity number cannot have too few edges. This motivates the next problem.

Problem 7.21 For fixed integers $k \geq 3$ and sufficiently large n, there exists a minimum number $h(n, k)$ such that every graph G with n vertices and with $\Lambda_g(G) \leq k$ must satisfy $|E(G)| \geq h(n, k)$. For example, $h(n, k) \geq n$ is a trivial lower bound. Determine $h(n, k)$.

7.3 Complementary Graphs

The famous Ramsey theorem [57] states that for any fixed number k, when the order of a simple graph G is sufficiently large, either G or G^c contains K_k as a subgraph. There have been several Ramsey type of results in the literature. Among them are the following results.

Theorem 7.22 *Let* G *be a simple graph on* n *vertices.*

(i) (Nebeský [58]) *If* $n \geq 5$*, then either* $L(G)$ *or* $L(G^c)$ *is hamiltonian.*

(ii) (Nebeský [59]) *If* $n \geq 4$*, then either* G *or* G^c *has an Eulerian subgraph of order at least* $n - 1$*, with an explicitly described class of exceptional graphs.*

(iii) ([60, Theorem 1 and Corollary 3]) *If* $n \geq 61$*, and if* $\min\{\kappa'(G), \kappa'(G^c)\} \geq 2$*, then either* G *or* G^c *has a spanning Eulerian subgraph, and so either* G *or* G^c *has a* 4-NZF.

Theorem 7.23 (Li [61], Hou et al. [62]) *Let* G *be a simple graph of order* $n \geq 8$*. If* $\min\{\delta(G), \delta(G^c)\} \geq 2$*, then either* $\Lambda_g(G) \leq 4$*, or* $\Lambda_g(G^c) \leq 4$*.*

Theorem 7.24 (Li [61], Hou et al. [62]) *Let* G *be a simple graph on* $n \geq 44$ *vertices. If* $\min\{\delta(G), \delta(G^c)\} \geq 4$*, then either* G *or* G^c *is* $\mathbf{Z}_3$*-connected.*

7.4 Products of Graphs

For graph products, we adopt the notation in [63]. Let G_1, G_2 be two graphs. The *cartesian product* graph $G = G_1 \times G_2$ is a graph with vertex set $V(G) = V(G_1) \times V(G_2)$ and edge set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)\}$. The *strong product* graph $G = G_1 \otimes G_2$ is a graph with vertex set $V(G) = V(G_1) \times V(G_2)$ and edge set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1$ and $u_2v_2 \in E(G_2)$, or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$, or both $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)\}$. The *lexicographic product* (sometimes called composition, tensor or wreath product) $G = G_1[G_2]$ is a graph with vertex set $V(G) = V(G_1) \times V(G_2)$ and edge set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G_1)$, or $u_1 = v_1$ and $u_2v_2 \in E(G_2)\}$.

Theorem 7.25 (Yao [38]) *Let G and H be two connected simple graphs. Then $\Lambda_g(G\otimes H) \le 4$, where equality holds if and only if both G and H are trees and* $\min\{|V(G)|, |V(H)|\} = 2$.

Corollary 7.26 (Yao [38]) *Let G and H be two connected simple graphs. Then $G \otimes H$ has a nowhere-zero 3-flow if and only if either one of G and H is not a tree, or both G and H are trees with* $\min\{|V(G)|, |V(H)|\} \ge 3$.

Theorem 7.27 (Yao [38]) *Let G and H be two connected simple graphs. Then $\Lambda_g(G[H]) \le 4$, where equality holds if and only if both G and H are trees and* $\min\{|V(G)|, |V(H)|\} = 2$.

Corollary 7.28 (Yao [38]) *Let G and H be two connected simple graphs. Then $G[H]$ has a nowhere-zero 3-flow if and only if either one of G and H is not a tree, or both G and H are trees with* $\min\{|V(G)|, |V(H)|\} \ge 3$.

Theorem 7.29 (Yao [38]) *Let G and H be two connected simple graphs. Then $\Lambda_g(G\times H) \le 5$, where equality holds if and only if both G and H are trees such that one of the following statements holds*:

(i) $\min\{|V(G)|, |V(H)|\} = 2$, *or*

(ii) $\min\{|V(G)|, |V(H)|\} \ge 3$ *and both G and H are of diameter at most 2.*

7.5 Graphs with Diameter at Most 2

In 1990, Lai investigated the problem whether Tutte's 4-flow conjecture (Conjecture 1.1 (ii)) holds for graphs with small diameters. He proved [64] that if a graph G with $\kappa'(G) \ge 2$ has diameter at most 2, then G has a 4-NZF if and only if G is not the Petersen graph. To explain this result, we need the graph $S_{m,n}$ defined below:

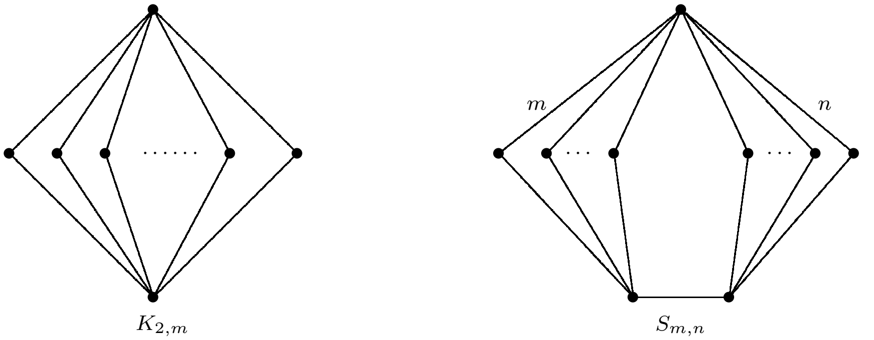

Figure 7.1 $K_{2,m}$ and $S_{m,n}$

Theorem 7.30 (Lai and Yao [65]) *Let G be a 2-edge-connected loopless graph with diameter at most 2. Then each of the following statements holds*:

(i) $\Lambda_g(G) \le 6$, *where $\Lambda_g(G) = 6$ if and only if $G = C_5$.*

(ii) *If $G \ne C_5$, then $\Lambda_g(G) = 5$ if and only if $G \cong P_{10}$, the Petersen graph, or for some integers m, n with $m > n > 0$, $G \cong S_{m,n}$, or G has a collapsible subgraph H with $|V(H)| \le 2$ such that $G/H \cong K_{2,m}$ for some integer $m > 1$.*

7.6 Line Graphs and Claw-free Graphs

We shall follow [66] to define a line graph. The *line graph* of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where for an integer $k \in \{0, 1, 2\}$, two vertices in $L(G)$ are joined by k

edges in $L(G)$ if and only if the corresponding edges in G are sharing k common vertices in G. In other words, if e_1 and e_2 are adjacent but not parallel in G, then e_1 and e_2 are joined by one edge in $L(G)$; if e_1 and e_2 are parallel edges in G, then e_1 and e_2 are joined by two (parallel) edges in $L(G)$. Note that our definition for line is slightly different from the one defined in [1] (called an edge graph there). But when G is a simple graph, both definitions are consistent with each other. The main reason for us to adopt this definition in [66] instead of the traditional definition of a line graph is explained in the introduction section of [22].

For a vertex $v \in V(G)$, $N_G(v) = \{v' \in V(G) : vv' \in E(G)\}$ is the *neighborhood* of v in G, and $N_G[v] = N_G(v) \cup \{v\}$ is the *closed neighborhood* of v in G. A vertex v is *locally connected* if $G[N_G(v)]$ is connected. A graph G is *claw-free* if G does not have an induced subgraph isomorphic to $K_{1,3}$. It has been well known (see [67, 68]) that every line graph is a claw-free graph.

Following the definition of Ryjáček [69], a graph H is the *closure* of a claw-free graph G, denoted by $H = \mathrm{cl}(G)$, if

(CL1) There is a sequence of graphs $G_1, \ldots, G_t$ such that $G_1 = G, G_t = H, V(G_{i+1}) = V(G_i)$ and $E(G_{i+1}) = E(G_i) \cup \{uv : u, v \in N_{G_i}(x_i), uv \notin E(G_i)\}$ for some $x_i \in V(G_i)$ with connected non-complete $G_i[N_{G_i}(x_i)]$, for $i = 1, \ldots, t-1$, and

(CL2) No vertex of H has a connected non-complete neighborhood.

In their dissertations, Shao and Yao proved the following results.

Theorem 7.31 (Shao [70], Lai et al. [22]) *Let G be a graph with $\kappa'(G) \geq 4$. Then $L(G) \in \langle \mathbf{Z}_3 \rangle$.*

Theorem 7.32 (Yao [38]) *Let A be an abelian group with $|A| \geq 4$ and G be a claw-free graph with $\delta(G) \geq 3$. Each of the following statements holds:*

(i) *Suppose that a vertex $v \in V(G)$ is locally connected, and $x, y \in N_G(v)$ are not adjacent. If $G + xy$ is A-connected, then G is A-connected.*

(ii) *If $\mathrm{cl}(G)$ is A-connected, then G is A-connected.*

Theorem 7.33 (Yao [38]) *Let G be a claw-free graph with $\delta(G) \geq 7$. If $\mathrm{cl}(G) \in \langle \mathbf{Z}_3 \rangle$, then $G \in \langle \mathbf{Z}_3 \rangle$.*

Definition 7.34 *For an integer $m \geq 3$, if every edge of a graph G lies in a circuit of length at most m in G, then we say that G is a J_m* graph.

Theorem 7.35 (Yao [38]) *Each of the following statements holds:*

(i) *Every 6-edge-connected J_3 line graph is $\mathbf{Z}_3$-connected.*

(ii) *Every 7-edge-connected J_3 claw-free graph is $\mathbf{Z}_3$-connected.*

The results above are proved via the investigation on the graphs with a clique partition, as defined below.

Definition 7.36 *For a connected graph G, a partition $(E_1, E_2, \ldots, E_k)$ of $E(G)$ is a* clique partition *of G if $G[E_i]$ is spanned by a maximal complete subgraph of G for each $i \in \{1, 2, \ldots, k\}$. Furthermore, for an integer $k \geq 3$, $(E_1, E_2, \ldots, E_k)$ is a $(\geq k)$-*clique partition *of G if for each $i \in \{1, 2, \ldots, k\}$, $G[E_i]$ is spanned by a K_{n_i} with $n_i \geq k$.*

Theorem 7.37 *Each of the following statements holds:*

(i) (Shao [70], Lai et al. [22]) *Every* 4*-edge-connected graph with a* (≥ 4)*-clique partition is* $\mathbf{Z}_3$*-connected. In particular, every line graph of a* 4*-edge-connected graph is* $\mathbf{Z}_3$*-connected.*

(ii) (Yao [38]) *Every* 6*-edge-connected graph with a* (≥ 3)*-clique partition is* $\mathbf{Z}_3$*-connected.*

7.7 Triangular Graphs

We consider in this section the special case of Problem 5.9 when $\Delta'(G) = 3$. A graph G with $\Delta'(G) \leq 3$ is also referred to as a J_3 *graph*. Even in this special case, Problem 5.9 has not been completely solved. Only some partial results were obtained.

Broersma and Veldman introduced the concept of k-triangular graphs in [71]. A graph G is k*-triangular* if each edge of G is in at least k triangles. A 1-triangular graph is just a triangular graph.

Conjecture 7.38 (Xu and Zhang [72]) *Every* 4*-edge-connected triangular graph has a* $\mathbf{Z}_3$*-NZF.*

It was further asked (see [73, Problem 1]) whether every 4-edge-connected triangular graph is $\mathbf{Z}_3$-connected. This was shown in the negative in [73]. This motivates the next conjecture.

Conjecture 7.39 ([73, Conjecture 5]) *Let* G *be a* 5*-edge-connected graph with each edge contained in a circuit of length at most* 3*. Then* G *is* $\mathbf{Z}_3$*-connected.*

As a supporting evidence, the following result was proved in [73].

Theorem 7.40 (Lai et al. [73, Theorem 2]) *If* G *is a disjoint union of* K_3*'s and if* $\kappa'(G) \geq 6$*, then* $\Lambda_g(G) \leq 3$*.*

As shown by Lemma 4.6 of [10], there exist infinitely many 3-edge-connected 2-triangular graphs that are not $\mathbf{Z}_3$-connected. These graphs motivate the authors to consider the $\mathbf{Z}_3$-connectivity of 4-edge-connected 2-triangular graphs.

Theorem 7.41 (Hou et al. [74]) *Every* 4*-edge-connected* 2*-triangular graph is* $\mathbf{Z}_3$*-connected.*

Corollary 7.42 (Hou et al. [74]) *Let* G *be a connected graph.*

(i) *If* G *is a* 4*-edge-connected* 2*-triangular graph, then* $\Lambda_g(G) \leq 3$*.*

(ii) *If* G *is* 3*-triangular, then* $\Lambda_g(G) \leq 3$*.*

Another subclass that has been well investigated is the class of triangularly connected graphs.

Definition 7.43 *For an integer* $m \geq 3$*, a graph* G *is* m-circuit-connected *if for every* $e_1, e_2 \in E(G)$*, there exists a sequence of circuits* $C_1, C_2, \ldots, C_k$ *such that* $e_1 \in E(C_1)$*,* $e_2 \in E(C_k)$*,* $|E(C_i)| \leq m$ *for* $1 \leq i \leq k$*, and* $E(C_j) \cap E(C_{j+1}) \neq \emptyset$ *for* $1 \leq j \leq k-1$*.*

Definition 7.44 *Let* G_1, G_2 *be two disjoint graphs. Then* $G_1 \oplus_2 G_2$ *is defined to be the graph obtained from* $G_1 \cup G_2$ *by identifying one edge. Let* $\mathcal{WF}$ *be the family of graphs that satisfy the following conditions:*

(WF1) $K_3, W_{2n+1} \in \mathcal{WF}$;

(WF2) *If* $G_1, G_2 \in \mathcal{WF}$*, then* $G_1 \oplus_2 G_2 \in \mathcal{WF}$*.*

Theorem 7.45 (Fan et al. [13]) *Let* G *be a* 3*-circuit-connected graph. Then* $\Lambda_g(G) \leq 3$ *if and only if* $G \notin \langle WF \rangle$*.*

Note that every connected and locally connected graph is 3-circuit-connected, and that 2-connected chordal graphs are locally connected. The following results follow immediately.

Theorem 7.46 ([20, Theorem 3.1]) *Let A be an abelian group with $|A| \geq 3$. Then every 2-edge-connected, locally 3-edge-connected graph is in $\langle A \rangle$.*

Theorem 7.47 ([10, (4.7)]) *Let G be a 3-edge-connected chordal graph. Then one of the following statements holds:*

(i) *G is A-connected, for any abelian group A with $|A| \geq 3$, or*

(ii) *G has a block isomorphic to a K_4, or*

(iii) *G has a subgraph G_1 such that $G_1 \notin \mathbf{Z}_3$ and such that $G = G_1 \oplus K_4$.*

The following conjectures are still open.

Conjecture 7.48 ([72]) *Let G be a 4-edge-connected graph. If each edge of G is contained in a circuit with length at most 3, then G admits a $\mathbf{Z}_3$-NZF.*

Conjecture 7.49 ([73]) *Let G be a 5-edge-connected graph with each edge contained in a circuit with length at most 3. Then G is $\mathbf{Z}_3$-connected.*

Theorem 7.50 ([73]) *If G is 6-edge-connected and if $E(G)$ is an edge disjoint union of circuits of length at most 3, then G is $\mathbf{Z}_3$-connected.*

More generally, the following problems and conjectures with circuit structures are waiting for solutions.

Problem 7.51 Let $m \geq 3$ be an integer. Then every 4-edge-connected, m-circuit-connected graph has a 3-NZF.

Problem 7.52 Let $m \geq 3$ be an integer. Then every 5-edge-connected, m-circuit-connected graph G has its group connectivity number $\Lambda_g(G) \leq 3$.

Conjecture 7.53 *Let $m \geq 3$ be an integer. Let G be a 4-edge-connected graph. If each edge of G is contained in a circuit with length at most m, then G admits a $\mathbf{Z}_3$-NZF.*

Conjecture 7.54 *Let $m \geq 3$ be an integer. Let G be a 5-edge-connected graph with each edge contained in a circuit with length at most m. Then G is $\mathbf{Z}_3$-connected.*

7.8 Claw-decompositions and All Tutte-orientations

In [47], Barat and Thomassen proposed another way to tackle the 3-flow problem and the $\mathbf{Z}_3$-connectivity problem using claw-decompositions. The definition given below, presented in [75], is a slight generalization of what is given in [47].

Definition 7.55 *A connected loopless graph with 3 edges and at least one vertex of degree 3 is called a* generalized claw. *When restricted to simple graphs, a generalized claw must be isomorphic to a $K_{1,3}$. A graph G with $|E(G)| \equiv 0 \pmod 3$ has a* claw-decomposition *or* $K_{1,3}$-decomposition *if $E(G)$ can be partitioned into disjoint unions $E(G) = X_1 \cup X_2 \cup \cdots \cup X_k$ such that for each i with $1 \leq i \leq k$, $G[X_i]$ is a generalized claw.*

Several conjectures concerning claw-decompositions were proposed by Barat and Thomassen.

Conjecture 7.56 (Barat, Thomassen and Jaeger) (i) ([47, Conjecture 2.1]) *There exists a* *smallest natural number k_c such that every simple k_c-edge-connected graph G with $|E(G)| \equiv 0 \pmod 3$ has a claw-decomposition.*

(ii) (Jaeger's weak 3-flow conjecture, which is Conjecture 1.1 (iv) restated as Conjecture 2.2 in [47]) *There exists a smallest natural number k_t such that every simple k_t-edge-connected graph G has a* 3-NZF.

Definition 7.57 (Barat and Thomassen [47]) *Let $w : V(G) \mapsto \{0, 1, 2\}$ be a map satisfying $\sum_{v\in V(G)} w(v) \equiv |E(G)| \pmod 3$. If G has an orientation D such that $d^+_D(v) \equiv w(v) \pmod 3$, $\forall v \in V(G)$, then G admits a* generalized Tutte-orientation prescribed by w. *If for every map $w : V(G) \mapsto \{0, 1, 2\}$ satisfying $\sum_{v\in V(G)} w(v) \equiv 0 \pmod 3$, G always admits a generalized Tutte-orientation prescribed by w, then G admits* all generalized Tutte-orientations.

It has been observed in [48] that the two statements "a graph G admits all generalized Tutte-orientations" and "G is $\mathbf{Z}_3$-connected" are equivalent. We present a simple proof of this fact below.

Proposition 7.58 *Let G be a connected graph. The following statements are equivalent:*

(i) *G is $\mathbf{Z}_3$-connected.*

(ii) *G admits all generalized Tutte-orientations.*

Proof Let $d_G(v)$ denote the degree of v in G. Suppose that (i) holds. Let $w : V(G) \mapsto \{0, 1, 2\}$ be a map satisfying $\sum_{v\in V(G)} w(v) \equiv |E(G)| \pmod 3$. Define $b : V(G) \mapsto \mathbf{Z}_3$ by $b(v) \equiv 2w(v) - d(v) \pmod 3$. As $\sum_{v\in V(G)} w(v) \equiv |E(G)| \pmod 3$, we have

$$\sum_{v\in V(G)} b(v) \equiv \sum_{v\in V(G)} [2w(v) - d(v)] \equiv 0 \pmod 3,$$

and so $b \in Z(G, \mathbf{Z}_3)$. Since G is $\mathbf{Z}_3$-connected, $\exists\, f \in F^*(G, \mathbf{Z}_3)$ such that $\partial f = b$. By reversing the orientation of each edge if necessary, we may assume that $f(e) = 1$, $\forall\, e \in E(G)$, and let D denote the corresponding orientation of G. Then as $\partial f(v) = d^+_D(v) - d^-_D(v) \equiv b(v) \equiv 2w(v) - d_G(v) \pmod 3$, we have $d^+_D(v) \equiv w(v) \pmod 3$, and so G admits a generalized Tutte-orientation prescribed by w. Since w is arbitrary, (ii) holds.

Conversely, we assume that (ii) holds. Let $b \in Z(G, \mathbf{Z}_3)$ be given, and let $w(v) \equiv 2(b(v) + d_G(v)) \pmod 3$. Then by $b \in Z(G, \mathbf{Z}_3)$, $\sum_{v\in V(G)} w(v) \equiv |E(G)| \pmod 3$. By (ii), G has an orientation D such that $d^+_D(v) \equiv w(v) \equiv 2(b(v) + d(v)) \pmod 3$, which is equivalent to $d^+_D(v) - d^-_D(v) \equiv b(v) \pmod 3$, and so (i) holds. □

Conjecture 7.59 ([47, Conjecture 2.5], Conjecture 1.4 (iii) restated) *There exists a smallest natural number k_g such that every k_g-edge-connected graph G is $\mathbf{Z}_3$-connected.*

Barat and Thomassen showed that all these numbers k_c, k_t and k_g are related.

Theorem 7.60 (Barat and Thomassen [47]) *Each of the following statements holds:*

(i) ([47, Theorem 2.4]) *If $k_c \le 8$, then $k_t \le 8$.*

(ii) ([47, Theorem 2.7]) *If one of k_c, k_t and k_g is finite, then all three are finite. Moreover, $\max\{k_t, k_c\} \le k_g \le 2k_t + 2$ and $k_t \le k_c + 5$.*

Barat and Thomassen also proposed the following conjecture.

Conjecture 7.61 ([47, Conjecture 2.3]) *Every 4-edge-connected simple planar graph G with $|E(G)| \equiv 0 \pmod 3$ has a claw-decomposition.*

This conjecture is disproved in [75].

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