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Degree sum condition for Z_3-connectivity in graphs

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ABSTRACT

Let G be a 2-edge-connected simple graph on n vertices, let A denote an abelian group with the identity element 0, and let D be an orientation of G. The *boundary* of a function $f : E(G) \to A$ is the function $\partial f : V(G) \to A$ given by $\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e)$, where $E^+(v)$ is the set of edges with tail v and $E^-(v)$ is the set of edges with head v. A graph G is *A-connected* if for every $b : V(G) \to A$ with $\sum_{v \in V(G)} b(v) = 0$, there is a function $f : E(G) \to A - \{0\}$ such that $\partial f = b$. In this paper, we prove that if $d(x) + d(y) \ge n$ for each $xy \in E(G)$, then G is not Z_3-connected if and only if G is either one of 15 specific graphs or one of $K_{2,n-2}$, $K_{3,n-3}$, $K^+_{2,n-2}$ or $K^+_{3,n-3}$ for $n \ge 6$, where $K^+_{r,s}$ denotes the graph obtained from $K_{r,s}$ by adding an edge joining two vertices of maximum degree. This result generalizes the result in [G. Fan, C. Zhou, Degree sum and Nowhere-zero 3-flows, Discrete Math. 308 (2008) 6233–6240] by Fan and Zhou.

1. Introduction

Graphs in this paper are finite, loopless, and may have multiple edges. Terminology and notations not defined here are from [1]. Let G be a graph, H a subgraph of G, and $v \in V(G)$. Let $d_H(v)$ denote the number of edges joining v to vertices of $V(H) - v$. In particular, when $H = G$, $d_G(v)$ is the degree of v and we simply write $d(v)$ for it. For two subsets $A, B \subseteq V(G)$, $e_G(A, B)$ (or simply $e(A, B)$) denotes the number of edges with one endpoint in A and the other endpoint in B. For simplicity, if H_1 and H_2 are two subgraphs of G, we write $e(H_1, H_2)$ to mean $e(V(H_1), V(H_2))$.

A cycle is a connected 2-regular graph. An n-cycle is a cycle on n vertices. For simplicity, a 3-cycle with vertex set $\{x, y, z\}$ is denoted by xyz. The complete graph on n vertices is denoted by K_n. Let K_n^- denote the graph obtained from K_n by deleting an edge, and let $K^+_{r,s}$ denote the simple graph obtained from the complete bipartite graph $K_{r,s}$ by adding an edge joining two vertices of maximum degree. Throughout this paper, when $K_{2,n-2}$ and $K^+_{2,n-2}$ are mentioned, we mean $n \ge 4$; when $K_{3,n-3}$ and $K^+_{3,n-3}$ are mentioned, $n \ge 6$.

Let G be a graph, and let D be an orientation of G. If an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then let *tail* $(e) = u$ and *head* $(e) = v$. For a vertex $v \in V(G)$, let $E^+(v)$ denote the set of edges with tail v and $E^-(v)$ the set of edges with head v. Let A denote an (additive) abelian group with the identity element 0. Let A^* denote the set of nonzero elements of A. We define $F(G, A)$ to be the set of labelings of $E(G)$ using elements of A and define $F^*(G, A)$ to be the set of labelings of $E(G)$ using nonzero elements of A.

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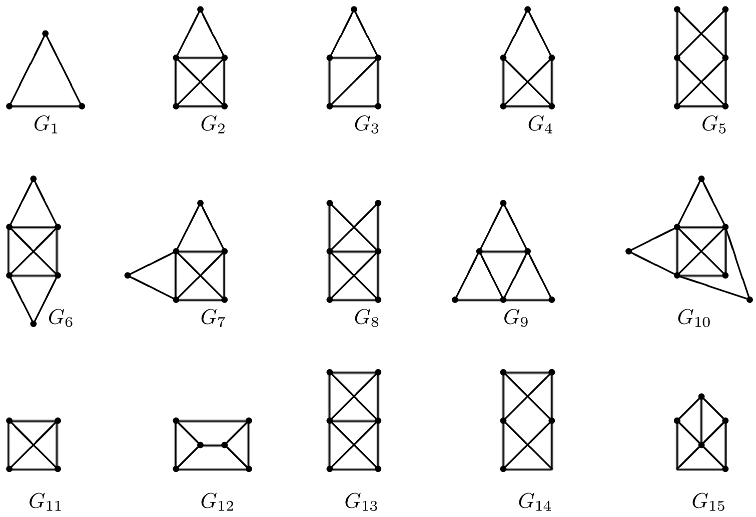

Fig. 1. Exceptional graphs for the main theorem.

Given a function $f \in F(G, A)$, define $\partial f : V(G) \to A$ by

$$\partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e),$$

where "$\sum$" refers to the addition in A. The value $\partial f(v)$ is known as the *net flow out of* v *under* f.

For a graph G, a function $b : V(G) \to A$ is an *A-valued zero-sum function* on G if $\sum_{v \in V(G)} b(v) = 0$. The set of all A-valued zero-sum functions on G is denoted by $\mathcal{Z}(G, A)$. Given $b \in \mathcal{Z}(G, A)$, a function $f \in F^*(G, A)$ is an (A, b)*-nowhere-zero flow* if G has an orientation D such that $\partial f = b$. A graph G is *A-connected* if for every $b \in \mathcal{Z}(G, A)$, G admits an (A, b)-nowhere-zero flow. A *nowhere-zero A-flow* is an $(A, 0)$-nowhere-zero flow, where here 0 denotes the function on $V(G)$ that is identically zero. More specifically, a *nowhere-zero k-flow* is a nowhere-zero Z_k-flow, where Z_k is the cyclic group of order k. Tutte [12] proved that G admits a nowhere-zero A-flow with $|A| = k$ if and only if G admits a nowhere-zero k-flow. We use *group connectivity* to refer to the general properties of a graph being A-connected for some particular A. Let $\langle A \rangle$ denote the family of graphs which are A-connected.

Integer flow problems were introduced by Tutte [11,13]. Group connectivity was introduced by Jaeger et al. [7] as a generalization of nowhere-zero flows. This paper is mainly motivated by the following two conjectures.

Conjecture 1.1 (*[11]*). *Every 4-edge-connected graph admits a nowhere-zero Z_3-flow.*

Conjecture 1.2 (*[7]*). *Every 5-edge-connected graph is Z_3-connected.*

Conjecture 1.2 implies Conjecture 1.1 by a result of Kochol [8] that reduces Conjecture 1.1 to a consideration of 5-edge-connected graphs. So far, both conjectures are still open. Recently, degree conditions have been used to guarantee the existence of nowhere-zero Z_3-flows and Z_3-connectivity. Let G be a graph on n vertices. If $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices u and v, then G is said to satisfy *Ore's condition*. Throughout this paper, we say G satisfies the *given degree-sum condition* if $d(u) + d(v) \geq n$ for every edge $uv \in E(G)$. Fan and Zhou [5] investigated the relationship between Ore's condition and nowhere-zero Z_3-flows; Lou et al. [10] studied Z_3-connectivity in graphs satisfying Ore's condition. Fan and Zhou [5] also studied the relationship between the given degree-sum condition and nowhere-zero Z_3-flows. We investigate Z_3-connectivity in graphs satisfying the given degree-sum condition and prove the following theorem in this paper.

Theorem 1.3. *Let G be a 2-edge-connected simple graph on n vertices. If $d(x) + d(y) \geq n$ for each $xy \in E(G)$, then $G \notin \langle Z_3 \rangle$ if and only if G is one of $K_{2,n-2}$, $K_{3,n-3}$, $K_{2,n-2}^+$, $K_{3,n-3}^+$ or one of the 15 exceptional graphs in Fig. 1.*

2. Lemmas

For a subset $X \subseteq E(G)$, the contraction G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting all loops generated by this process. Note that even if G is simple, G/X may have multiple edges. For convenience, we write G/e for $G/\{e\}$, where $e \in E(G)$. If H is a subgraph of G, then G/H denotes $G/E(H)$.

The *wheel* W_k $(k \geq 2)$ is the graph obtained from a k-cycle by adding a new vertex, called the *center* of the wheel, which is adjacent to every vertex of the k-cycle. We define W_k to be *odd* (even) if k is odd (or even, respectively). For technical reasons, we define the wheel W_1 to be a 3-cycle.

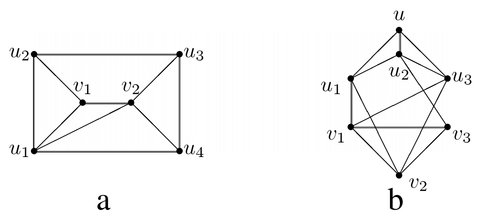

Fig. 2. Two Z_3-connected graphs.

In this section, we establish several lemmas. Some results in [2–4,7,9] on group connectivity are summarized as follows.

Lemma 2.1. *Let A be an abelian group with $|A| \geq 3$. The following results are known:*

(1) *K_n and K_n^- are A-connected if $n \geq 5$.*
(2) *C_n is A-connected if and only if $|A| \geq n + 1$.*
(3) *$K_{m,n}$ is A-connected if $m \geq n \geq 4$; neither $K_{2,t}$ $(t \geq 2)$ nor $K_{3,s}$ $(s \geq 3)$ is Z_3-connected.*
(4) *$W_{2k} \in \langle Z_3 \rangle$ and $W_{2k+1} \notin \langle Z_3 \rangle$, where k is a positive integer.*
(5) *If $G \notin \langle A \rangle$, then also $H \notin \langle A \rangle$ when H is a spanning subgraph of G.*
(6) *If $H \subseteq G$, $H \in \langle A \rangle$, and $G/H \in \langle A \rangle$, then $G \in \langle A \rangle$.*

When H_1 and H_2 are two subgraphs of a graph G, we say that G is the 2-*sum* of H_1 and H_2, denoted by $H_1 \oplus H_2$, if $E(H_1) \cup E(H_2) = E(G)$, $|V(H_1) \cap V(H_2)| = 2$ and $|E(H_1) \cap E(H_2)| = 1$. Note that the definition of 2-sum of two graphs here is not that of 2-sum used in graph minor theory, which allows the edge joining the two common vertices to be dropped when forming the 2-sum.

A graph G is *triangularly connected* if whenever $e_1, e_2 \in E(G)$, there exists a list $C_1, \ldots, C_k$ of cycles such that $e_1 \in E(C_1)$, $e_2 \in E(C_k)$, $|E(C_i)| \leq 3$ for $1 \leq i \leq k$, and such that $E(C_j) \cap E(C_{j+1}) \neq \emptyset$ for $1 \leq j \leq k - 1$. For triangularly connected graphs, the following characterization of group connectivity is known.

Lemma 2.2 (*[4]*). *If G is a triangularly connected graph on $n \geq 3$ vertices, then G is not Z_3-connected if and only if there is an odd wheel W and a subgraph G_1 such that $G = W \oplus G_1$, where G_1 is triangularly connected and not Z_3-connected.*

Lemma 2.3 (*[3]*). *If G is a triangularly connected graph with $\delta(G) \geq 4$, then G is Z_3-connected.*

For a graph G with $u, v, w \in V(G)$ such that $v, w \in N(u)$, let $G_{[uv,uw]}$ be the graph obtained from G by deleting two edges uv and uw and then adding edge vw, that is, $G_{[uv,uw]} = G \cup \{wv\} - \{uv, uw\}$.

Lemma 2.4 (*[2]*). *Let A be an abelian group, let G be a graph, and let u, v, w be three vertices of G such that $d(u) \geq 4$ and $v, w \in N(u)$. If $G_{[uv,uw]}$ is A-connected, then so is G.*

The edge v_1v_2 in Fig. 2(a) is called a *distinguished edge*. By the result in [10, Lemma 2.2], Fig. 2(a) is Z_3-connected. The same is true for Fig. 2(b).

Lemma 2.5. *Both graphs in Fig. 2 are Z_3-connected.*

Proof. Let G be the graph (b) shown in Fig. 2. The graph $G_{[u_1v_1,u_1v_2]}$ has two copies of the edges v_1v_2. Iteratively contracting 2-cycles leads eventually to K_1, which is Z_3-connected. By Lemma 2.1(2) and (6), $G_{[u_1v_1,u_1v_2]} \in \langle Z_3 \rangle$. By Lemma 2.4, $G \in \langle Z_3 \rangle$. $\square$

By the results in [6, Proposition 1.3] and [5, Theorem 1.7], no graph in $\{G_4, G_{11}, G_{12}, G_{13}, G_{15}, K_{3,n-3}^+\}$ admits a nowhere-zero Z_3-flow. By definition, every graph not admitting a nowhere-zero Z_3-flow is not Z_3-connected. We summarize this result in the following lemma (also see [10, Theorem 1.7]).

Lemma 2.6. *No graph in $\{G_4, G_{11}, G_{12}, G_{13}, G_{15}, K_{3,n-3}^+\}$ is Z_3-connected.*

Lemma 2.7. *No graph in Fig. 1 or in $\{K_{2,n-2}, K_{3,n-3}, K_{2,n-2}^+, K_{3,n-3}^+\}$ is Z_3-connected.*

Proof. By Lemmas 2.1 and 2.6, we only need to check the graphs G_1, G_2, G_3, G_6, G_7, G_9, G_{10}, and $K_{2,n-2}^+$, since each of the others (except $\{G_4, G_{11}, G_{12}, G_{13}, G_{15}\}$) is a spanning subgraph of G_{13} or $K_{3,n-3}^+$. It follows from Lemma 2.2 that no graph in $\{G_1, G_2, G_3, G_6, G_7, G_9, G_{10}, K_{2,n-2}^+\}$ is Z_3-connected. $\square$

Since an even wheel W_4 and the graph in Fig. 2(a) play an important role in the proof of our main theorem, we establish the following two technical lemmas.

Lemma 2.8. *Suppose that G is a 2-edge-connected simple graph on $n \geq 6$ vertices such that $d(x)+d(y) \geq n$ for each $xy \in E(G)$. If G_1 is a Z_3-connected subgraph of G, then*

(1) *if $n \in \{6, 7\}$ and $|V(G_1)| \geq 4$, then $G \in \langle Z_3 \rangle$.*
(2) *if $n = 8$ and $|V(G_1)| \geq 5$, then $G \in \langle Z_3 \rangle$.*
(3) *if $n \geq 9$ and $|V(G_1)| \geq n-4$, then $G \in \langle Z_3 \rangle$.*

Proof. Let G^* be a maximum Z_3-connected subgraph of G such that G^* contains G_1. Note that $|V(G^*)| \geq |V(G_1)|$. If $G^* = G$, we are done. Thus, assume that $G^* \neq G$. Let $G_2 = G - V(G^*)$. By Lemma 2.1(2) and (6), $e(v, G^*) \leq 1$ for each vertex $v \in V(G_2)$. Since G is 2-edge-connected, G_2 has no isolated vertex. Thus, G_2 has an edge v_1v_2. When $n = 6$ and $|V(G_1)| \geq 4$, $|V(G_2)| \leq 2$. Since $G^* \neq G$, $|V(G_2)| = 2$ and G_2 contains only one edge v_1v_2. Since for each vertex $v \in V(G_2)$, $e(v, G^*) \leq 1$. Thus, $d(v_1) + d(v_2) = 4 < 6$. Similarly, when $n = 7$ and $|V(G_1)| \geq 4$, $|V(G_2)| \leq 3$ and $d(v_1) + d(v_2) \leq 6 < 7$; when $n = 8$ and $|V(G_1)| \geq 5$, $|V(G_2)| \leq 3$ and thus $d(v_1) + d(v_2) \leq 6 < 8$; when $n \geq 9$ and $|V(G_1)| \geq n-4$, $|V(G_2)| \leq 4$ and thus $d(v_1) + d(v_2) \leq 8 < n$. This contradicts the given degree-sum condition. $\square$

Corollary 2.9. *Suppose that G is a 2-edge-connected simple graph on $n \geq 6$ vertices such that $d(x)+d(y) \geq n$ for all $xy \in E(G)$. If u is a vertex of G such that $d(u) = \delta(G) = 3$ and $N(u) = \{u_1, u_2, u_3\}$, then the following hold:*

(1) *if G contains an even wheel W_4 with $\{u, u_1, u_2, u_3\} \subseteq V(W_4)$, then $G \in \langle Z_3 \rangle$.*
(2) *if G contains the subgraph H in Fig. 2(a) with $\{u, u_1, u_2, u_3\} \subseteq V(H)$, then $G \in \langle Z_3 \rangle$.*

Proof. (1) Let the vertex set of the non-central 4-cycle in the wheel be $\{v_1, v_2, v_3, v_4\}$, with $u = v_1$, and let v_5 be the central vertex. Let $M = V(G) - V(W_4)$. When $n \in \{6, 7, 8\}$, the wheel is a Z_3-connected subgraph of G with 5 vertices. By Lemma 2.8, G is Z_3-connected. Thus, let $n \geq 9$. Applying the given degree-sum condition to v_1v_2 and v_1v_4, respectively, $e(v_2, M) \geq n-6$ and $e(v_4, M) \geq n-7$. On the other hand, $|M| = n-5$. Thus $|N(v_2) \cap N(v_4) \cap M| \geq n-8$. By Lemma 2.1(2) and (6), G has a Z_3-connected subgraph G_1 containing an even wheel W_4 and all the vertices of $N(v_1) \cap N(v_4) \cap M$. This means $|V(G_1)| \geq 5 + n - 8 - 1 = n - 4$. Lemma 2.8 shows that G is Z_3-connected.
(2) The proof is similar. $\square$

Lemma 2.10. *Let G be a 2-edge-connected simple graph on n vertices with $\delta(G) = 2$. If $d(x) + d(y) \geq n$ for each $xy \in E(G)$, then $G \notin \langle Z_3 \rangle$ if and only if G is $K_{2,n-2}$ or $K^+_{2,n-2}$ or G_i in Fig. 1 with $1 \leq i \leq 10$.*

Proof. The sufficiency follows immediately from Lemma 2.7. Conversely, suppose that $G \notin \langle Z_3 \rangle$. We shall prove that G must be $K_{2,n-2}$ or $K^+_{2,n-2}$ or G_i, where $1 \leq i \leq 10$ in Fig. 1. Since G is a 2-edge-connected simple graph, we have that $n \geq 3$. If $n = 3$, then $G = G_1$. If $n = 4$, then G must be $K_{2,2}$ or $K^+_{2,2}$, since $\delta(G) = 2$. Suppose therefore that $n \geq 5$. Let $d(u) = \delta(G) = 2$, $N(u) = \{u_1, u_2\}$, and $N = N(u_1) \cap N(u_2)$. By applying the given degree-sum condition to uu_1 and uu_2, respectively, $d(u_1) \geq n-2$ and $d(u_2) \geq n-2$. It follows that $n-4 \leq |N| \leq n-2$. In the remainder of the proof we shall use two claims.

Claim 1. *$G[N]$ does not contain a pair of incident edges.*

Proof of Claim 1. Suppose, to the contrary, that $v_1v_2 \in E(G)$ and $v_2v_3 \in E(G)$, where $v_1, v_2, v_3 \in N$. The subgraph induced by u_1, u_2, v_1, v_2 and v_3 contains an even wheel W_4 with the center at v_2. Since $|N| \geq n-4$, G has a Z_3-connected subgraph G_1 containing an even wheel W_4 and all the vertices in N. Obviously, $|V(G_1)| \geq n-2$. By Lemma 2.8, $G \in \langle Z_3 \rangle$, a contradiction.

Claim 2. *If $v_0 \in N(u_1) - (N(u_2) \cup \{u_2\})$, then $e(v_0, N) \leq 1$.*

Proof of Claim 2. Suppose otherwise that v_0 has two neighbors v_1 and v_2 in N. In this case, applying the given degree sum condition to uu_2, we get $u_1u_2 \in E(G)$. It follows that G contains an even wheel W_4 induced by v_0, v_1, v_2, u_1, and u_2 with the center at u_1. As in Claim 1, G has a Z_3-connected subgraph G_1 containing an even wheel W_4 and all the vertices in N, and $|V(G_1)| \geq n-2$. Lemma 2.8 proves that $G \in \langle Z_3 \rangle$, a contradiction.

Now we are ready to complete the proof of our lemma. We assume first that $|N| = n-2$. If there is no edge in $G[N]$, then G is $K_{2,n-2}$ if $u_1u_2 \notin E(G)$ and G is $K^+_{2,n-2}$ otherwise. Suppose now that v_1 and v_2 in N are adjacent. When $n = 5$, $G = G_4$ if $u_1u_2 \notin E(G)$ and $G = G_2$ otherwise. Let $n = 6$. When $G[N]$ has only one edge, $G = G_5$ if $u_1u_2 \notin E(G)$ and $G = G_8$ if $u_1u_2 \in E(G)$; when $G[N]$ has two edges, these two edges are incident, contrary to Claim 1. When $n \geq 7$, by applying the given degree-sum condition to v_1v_2, $G[N]$ contains a pair of incident edges in $G[N]$, contrary to Claim 1.

We next assume that $|N| = n-3$. Now there is a vertex $v_0 \notin N(u_1) \cap N(u_2)$. We assume, without loss of generality, that $v_0 \notin N(u_2)$. By applying the given degree-sum condition to uu_2, we get $u_1u_2 \in E(G)$. When $n = 5$, $G = G_3$. Thus, let $n \geq 6$.

Suppose first that $v_0 \notin N(u_1) \cup N(u_2)$. When $n = 6$, $G = G_6$. Assume $n = 7$ or $n \geq 9$. Since $\delta(G) = 2$, let $v_1, \ldots, v_k$ be the neighbors of v_0 in $N(u_1) \cap N(u_2) - \{u\}$, where $k = d(v_0)$. For each $1 \leq j \leq k$, by applying the given degree-sum condition to v_0v_j, $G[N]$ has a pair of incident edges, contrary to Claim 1. When $n = 8$, $G[N]$ has a pair of incident edges or a pair of independent edges. In the former case, it is contrary to Claim 1. In the latter case, $G - u$ is a triangularly connected graph with $\delta(G) \geq 4$. By Lemma 2.3, $G - u \in \langle Z_3 \rangle$; also $G \in \langle Z_3 \rangle$ since $d(u) = 2$, a contradiction.

Next suppose that $v_0 \in N(u_1) \cup N(u_2)$. Without loss of generality, let $v_0 \in N(u_1) - (N(u_2) \cup \{u_2\})$. Since $\delta(G) = 2$, v_0 has at least one neighbor in N. By Claim 2, v_0 has only one neighbor v_1 in N. It follows that $v_0v \notin E(G)$ for $v \in N - v_1$. When $n = 6$, $G = G_7$. When $n \geq 7$, $G[N]$ has a pair of incident edges by applying the given degree-sum condition to v_0v_1, contrary to Claim 1.

Finally, we assume that $|N| = n - 4$. By applying the given degree-sum condition to uu_1 and uu_2, respectively, $N(u_1) - (N(u_2) \cup \{u_2\}) \neq \emptyset$, $N(u_1) - (N(u_1) \cup \{u_1\}) \neq \emptyset$ and $u_1u_2 \in E(G)$. Let $v_1 \in N(u_1) - (N(u_2) \cup \{u_2\})$, $v_2 \in N(u_2) - (N(u_1) \cup \{u_1\})$. If $v_1v_2 \notin E(G)$, by $\delta(G) = 2$ and by Claim 2, v_1 (v_2) has only one neighbor v_1' (v_2') in N. By symmetry, when $n = 6$, $G = G_9$; when $n = 7$, $G = G_{10}$. When $n \geq 8$, $G[N]$ has a pair of incident edges in $G[N]$ by applying the given degree-sum condition to v_1v_1' and v_2v_2', respectively, contrary to Claim 1. Thus, we assume that $v_1v_2 \in E(G)$. When $n = 6$, let $v_3 \in N - \{u\}$. Now $v_1v_3, v_2v_3 \in E(G)$. G contains an even wheel W_4 induced by u_1, u_2, v_1, v_2 and v_3 with the center at v_3. We contract this W_4 and get a 2-cycle. We contract this 2-cycle and get a K_1 which is Z_3-connected. By Lemma 2.1, $G \in \langle Z_3\rangle$, a contradiction. When $n \geq 7$, applying the given degree-sum condition to v_1v_2, one of v_1 and v_2 has at least two neighbors in N, contrary to Claim 2. □

In order to prove Lemma 2.13, we establish the following two lemmas.

Lemma 2.11. *Suppose that G is a 2-edge-connected simple graph on n vertices and that $d(x) + d(y) \geq n$ for each $xy \in E(G)$ and that $\delta(G) = 3$ and $N(u) = \{u_1, u_2, u_3\}$. Let $M = V(G) - \{u, u_1, u_2, u_3\}$. Assume that $G \notin \langle Z_3\rangle$.*

(1) *If $u_1u_2, u_2u_3 \in E(G)$, then there is no vertex $v \in N(u_1) \cap N(u_2) \cap N(u_3) - \{u\}$.*
(2) *If $u_iu_j, u_iu_k \in E(G)$ and if $v_1 \in N(u_i) \cap N(u_j) \cap M$ and $v_2 \in N(u_j) \cap N(u_k) \cap M$, then $v_1v_2 \notin E(G)$, where $\{i, j, k\} = \{1, 2, 3\}$.*
(3) *If $u_1u_2, u_2u_3, u_3u_1 \in E(G)$, then $|N(u_i) \cap N(u_j) - \{u\}| \leq 2$, where $1 \leq i < j \leq 3$.*

Proof. (1) Suppose otherwise that u_1, u_2 and u_3 have a common vertex v except for u. It follows that G contains an even wheel W_4 induced by u, u_1, u_2, u_3 and v, contrary to Corollary 2.9.
(2) Suppose otherwise that $v_1v_2 \in E(G[M])$. This means that G contains the graph in Fig. 2(a) induced by u_i, u_j, u_k, u, v_1 and v_2 with the distinguished edge u_iv_1, contrary to Corollary 2.9.
(3) Suppose, to the contrary, that there are $i_0, j_0 \in \{1, 2, 3\}$ such that $|N(u_{i_0}) \cap N(u_{j_0}) - \{u\}| \geq 3$. Let $v_1, v_2, v_3 \in N(u_{i_0}) \cap N(u_{j_0}) - \{u\}$. Let $k \in \{1, 2, 3\} - \{i_0, j_0\}$. On the other hand, by applying the given degree-sum condition to uu_k, $e(u_k, M) \geq n - 6$. It follows that there are at most two vertices in M which are not adjacent to u_{i_0} since $|M| = n - 4$. Thus, there is $v \in \{v_1, v_2, v_3\}$ such that $vu_k \in E(G)$. This means that $e(v, \{u_1, u_2, u_3\}) = 3$, contrary to (1). □

Lemma 2.12. *Suppose that G is a 2-edge-connected simple graph on $6 \leq n \leq 10$ vertices such that $d(x) + d(y) \geq n$ for each $xy \in E(G)$. Assume further that $d(u) = 3$, $N(u) = \{u_1, u_2, u_3\}$ and $u_1u_2u_3$ is a 3-cycle. Let $M = V(G) - \{u, u_1, u_2, u_3\}$, $N = \{v \in M : e(v, \{u_1, u_2, u_3\}) \leq 1\}$. If $G \notin \langle Z_3\rangle$, then each of the following holds.*

(1) *$|N| \leq 2$.*
(2) *If $N \neq \emptyset$, then $7 \leq n \leq 9$.*
(3) *If there are two vertices $v_1, v_2 \in M$ such that $e(v_i, \{u_1, u_2, u_3\}) = 1$ or there is one vertex $v \in M$ such that $e(v, \{u_1, u_2, u_3\}) = 0$, then $n = 8$.*

Proof. By Lemma 2.11, for each vertex $v \in M$, $e(v, \{u_1, u_2, u_3\}) \leq 2$. Let $N = \{x_1, \ldots, x_s, y_1, \ldots, y_t\}$ such that $e(x_i, \{u_1, u_2, u_3\}) = 1$ for $1 \leq i \leq s$ and $e(y_j, \{u_1, u_2, u_3\}) = 0$ for $1 \leq j \leq t$. If N has no vertex x with $e(x, \{u_1, u_2, u_3\}) = 1$, s is defined to be 0; if N has no vertex y with $e(y, \{u_1, u_2, u_3\}) = 0$, t is defined to be 0. On the other hand, by applying the given degree-sum condition to each edge uu_i for $i = 1, 2, 3$, $e(u_i, M) \geq n - 6$. It follows that $3(n - 6) \leq e(\{u_1, u_2, u_3\}, M) \leq 2(n - 4 - s - t) + s$, which implies that $n \leq 10 - s - 2t$. When $n = 6$, $e(M, \{u_1, u_2, u_3\}) \geq 4$ since $\delta(G) = 3$. It follows that $4 \leq e(\{u_1, u_2, u_3\}, M) \leq 4 - s - 2t$, which implies that $s = t = 0$ and $N = \emptyset$. Thus, if $N \neq \emptyset$, then $7 \leq n \leq 10 - s - 2t$.

Suppose that $|N| \geq 3$. Since $7 \leq n \leq 10 - s - 2t$, we have that $t = 0$, $s = 3$ and $n = 7$. In this case, $G[M]$ is a 3-cycle, say $x_1x_2x_3$. $d(x_1) + d(x_2) = 6 < 7$. This contradiction proves (1).

If $N \neq \emptyset$, then $s \geq 1$ or $t \geq 1$. It follows immediately from $7 \leq n \leq 10 - s - 2t$ that $7 \leq n \leq 9$ and (2) holds.

If there are two vertices $v_1, v_2 \in M$ such that $e(v_i, \{u_1, u_2, u_3\}) = 1$, then $s \geq 2$ and $t \geq 0$. Thus, $7 \leq n \leq 8$. If $n = 7$, then $|M| = 3$. Since $\delta(G) = 3$, $v_1v_2 \in E(G)$. In this case, $d(v_1) + d(v_2) = 6 < 7$, contrary to the given degree-sum condition. Thus, $n = 8$. Suppose that there is a vertex $v \in M$ such that $e(v, \{u_1, u_2, u_3\}) = 0$. By $\delta(G) = 3$, v is adjacent to three vertices in M. Thus, $n \geq 8$. In this case, $t \geq 1$ and $n \leq 8$ and (3) holds. □

Lemma 2.13. *Let G be a 2-edge-connected simple graph on n vertices with $\delta(G) = 3$. If $d(x) + d(y) \geq n$ for each $xy \in E(G)$, then $G \notin \langle Z_3\rangle$ if and only if G is $K_{3,n-3}$ or $K^+_{3,n-3}$ or G_i, where $11 \leq i \leq 15$ in Fig. 1.*

Proof. If G is $K_{3,n-3}$ or $K^+_{3,n-3}$ or G_i, where $11 \leq i \leq 15$ in Fig. 1, then by Lemma 2.7, $G \notin \langle Z_3\rangle$. Conversely, suppose that $G \notin \langle Z_3\rangle$. We shall prove that it must be $K_{3,n-3}$ or $K^+_{3,n-3}$ or G_i, where $11 \leq i \leq 15$ in Fig. 1. Since $\delta(G) = 3$, for $n = 4$, $G = G_{11}$. For $n = 5$, since n is odd, there must be a vertex v such that $d(v) = 4$. For any $w \in V(G) - v$, $d_{G-v}(w) \geq 2$, so $G - v$ contains a 4-cycle. This means that G contains an even wheel W_4 with the center at v as a spanning subgraph. By Lemma 2.1, $G \in \langle Z_3\rangle$, a contradiction. Therefore we assume that $n \geq 6$. Let $d(u) = 3$, $N(u) = \{u_1, u_2, u_3\}$ and $M = V(G) - \{u, u_1, u_2, u_3\}$.

Case 1. There is no edge in $G[N(u)]$.

If there is no edge in $G[M]$, then G is $K_{3,n-3}$. Thus, assume that $G[M]$ contains an edge xy. For $n = 6$, G is $K^{+}_{3,3}$. When $n \geq 7$, applying the given degree-sum condition to uu_i for $i = 1, 2, 3$, $d(u_i) \geq n - 4$. This means that each vertex in M is adjacent to each vertex in $\{u_1, u_2, u_3\}$. G must contain K_4^-, the union of u_1xy and u_2xy, and $d(u_1) \geq 4$. The graph $G_{[u_1x,u_1y]}$ contains a 2-cycle; by iterative contracting 2-cycles, we obtain the graph K_1 which is Z_3-connected. By Lemma 2.1, $G_{[u_1x,u_1y]} \in \langle Z_3\rangle$, so by Lemma 2.4, $G \in \langle Z_3\rangle$, a contradiction.

Case 2. There is exactly one edge in $G[N(u)]$.

We assume, without loss of generality, that $u_1u_2 \in E(G)$. Applying the given degree-sum condition to uu_i for $i = 1, 2, 3$, u_3 is adjacent to each vertex of M and u_i is adjacent to at least $n-5$ vertices of M for $i = 1, 2$. Thus, $n-5 \leq |N(u_1)\cap N(u_2)| \leq n - 3$ since $u \in N(u_1) \cap N(u_2)$ and $u \notin M$.

Assume first that $|N(u_1) \cap N(u_2)| = n - 3$. In this case, $N(u_1) = N(u_2) = N(u_3) = M$. If there is no edge in $G[M]$, then $G = K^{+}_{3,n-3}$. Thus, we assume that there is an edge v_1v_2 in $G[M]$. It follows that G contains the subgraph H in Fig. 2(a) induced by u, u_1, u_2, u_3, v_1 and v_2 with the distinguished edge u_1v_1. This contradicts Corollary 2.9.

We next assume that $|N(u_1) \cap N(u_2)| = n - 4$. In this case, there is only one vertex in M which is not in $N(u_1) \cap N(u_2)$. Let $v_0 \in M - N(u_1) \cap N(u_2)$. If $v_0 \in N(u_1) \cup N(u_2)$, without loss of generality, let $v_0 \in N(u_1) - N(u_2)$. Since $\delta(G) \geq 3$ and $v_0u, v_0u_2 \notin E(G)$, there is a vertex $v_3 \in N(u_1) \cap N(u_2)$ such that $v_0v_3 \in E(G)$. Applying the given degree-sum condition to uu_3, u_3 is adjacent to all the vertices in M. Thus, $v_0u_3, v_3u_3 \in E(G)$. Then G contains the subgraph H in Fig. 2(a) induced by u, u_1, u_2, u_3, v_0 and v_3 with distinguished edge u_1v_0, contrary to Corollary 2.9. Next, suppose that $v_0 \notin N(u_1) \cup N(u_2)$. Since $\delta(G) = 3$, v_0 has three neighbors in $V(G) - \{u, u_1, u_2\}$ and hence $n \geq 7$. If there is an edge v_1v_2 in the subgraph induced by $N(u_1) \cap N(u_2)$, then G contains the subgraph H in Fig. 2(a) induced by u, u_1, u_2, u_3, v_1 and v_2 with distinguished edge u_1v_1, contrary to Corollary 2.9. Assume that there is no edge in the subgraph induced by $N(u_1) \cap N(u_2)$. In this case, applying the given degree-sum condition to u_3v_0 and $n \geq 6$, there are $v_1, v_2 \in N(u_1) \cap N(u_2)$ such that $v_0v_1, v_0v_2 \in E(G)$. Note that $d(v_2) \geq 4$, $u_3v_1, u_3v_2 \in E(G)$. Let $G' = G_{[v_2u_2,v_2u_3]}$. This implies that G' contains an even wheel W_4 induced by u, u_1, u_2, u_3 and v_1 with the center at u_2. We contract this W_4 and contract every 2-cycle obtained in the process. Since $|N(u_1) \cap N(u_2)| \geq n - 4$, $\kappa'(G') \geq 2$ and $v_0u_3, v_0v_1 \in E(G')$, the resulting graph is K_1 which is Z_3-connected. By Lemma 2.1, $G' \in \langle Z_3\rangle$, and so by Lemma 2.4, $G \in \langle Z_3\rangle$, a contradiction.

Next, assume that $|N(u_1) \cap N(u_2)| = n - 5$. Recall that $n \geq 6$. When $n = 6$, $G = G_{12}$. Thus, $n \geq 7$. Recall that u_i is adjacent to at least $n-5$ vertices of M for $i = 1, 2$. Let $v_1 \in N(u_1) - (N(u_2) \cup \{u_2\})$ and $v_2 \in N(u_2) - (N(u_1) \cup \{u_1\})$. If there is a vertex v_3 in $N(u_1) \cap N(u_2)$ such that $v_1v_3 \in E(G)$ or $v_2v_3 \in E(G)$, by symmetry, let $v_1v_3 \in E(G)$. In this case, G contains the subgraph H induced by u, u_1, u_2, u_3, v_1 and v_3 with distinguished edge u_1v_1, contrary to Corollary 2.9. Thus, neither v_1 nor v_2 is adjacent to any vertex in $N(u_1) \cap N(u_2)$. By applying the given degree-sum condition to $u_1v_1, v_1v_2 \in E(G)$, $d(v_1) + d(v_2) = 6 < n$, a contradiction.

Case 3. There are exactly two edges in $G[N(u)]$.

In this case, we assume, without loss of generality, that $u_1u_2, u_2u_3 \in E(G)$.

Assume first that $n \geq 9$. In this case, we claim that u_1, u_2 and u_3 have a common neighbor v except for u. Suppose, to the contrary, that for each vertex $v \in M$, $e(v, \{u_1, u_2, u_3\}) \leq 2$. By applying the given degree-sum condition to uu_i, $d(u_i) \geq n - 3$ for $i = 1, 2, 3$. On the other hand, each vertex in M is adjacent to at most two of u_1, u_2 and u_3. It follows that $2(n - 4) \geq d(u_1) + d(u_2) + d(u_3) - 7 \geq 3(n - 3) - 7$, which implies that $n \leq 8$. Thus, when $n \geq 9$, u_1, u_2 and u_3 have a common neighbor v except for u, contrary to Lemma 2.11.

Assume then that $n = 8$. By applying the given degree-sum condition to uu_i for $i = 1, 2, 3$, we have $e(u_1, M) \geq 3$, $e(u_2, M) \geq 2$ and $e(u_3, M) \geq 3$. Since $|M| = 4$, $|N(u_1) \cap N(u_2) \cap M| \geq 1$, $|N(u_2) \cap N(u_3) \cap M| \geq 1$ and $|N(u_1) \cap N(u_3) \cap M| \geq 2$. Let $v_1 \in N(u_1) \cap N(u_2) \cap M$, $v_2, v_3 \in N(u_1) \cap N(u_3) \cap M$ and $v_4 \in N(u_2) \cap N(u_3) \cap M$. By Lemma 2.11(1), $v_1 \notin N(u_3)$, $v_2, v_3 \notin N(u_2)$ and $v_4 \notin N(u_1)$. Thus, $M = \{v_1, v_2, v_3, v_4\}$. Since $\delta(G) = 3$, by Lemma 2.11(2), $v_1v_4 \in E(G)$ and $v_1v_2, v_1v_3, v_4v_2, v_4v_3 \notin E(G)$. Thus, $d(v_1) + d(v_4) = 6 < 8$. This contradicts the given degree-sum condition.

Next, let $n = 7$. By applying the given degree-sum condition to uu_i for $i = 1, 2, 3$, we obtain $e(u_1, M) \geq 2$, $e(u_2, M) \geq 1$ and $e(u_3, M) \geq 2$. It follows that $|N(u_1) \cap N(u_3) \cap M| \geq 1$. Let $v_2 \in N(u_1) \cap N(u_3) \cap M$, $v_1 \in N(u_1) \cap M - \{v_2\}$ and $v_3 \in N(u_3) \cap M - \{v_2\}$. Assume first that $v_1 \neq v_3$. By Lemma 2.11(1), $u_2v_2 \notin E(G)$. Since $e(u_2, M) \geq 1$, either $v_1u_2 \in E(G)$ or $u_2v_3 \in E(G)$. In the former case, by Lemma 2.11(2), $v_1v_2 \notin E(G)$. Applying $\delta(G) = 3$ and Lemma 2.11(1) to v_1 and v_2, respectively, we have $v_3v_1, v_3v_2 \in E(G)$, $d(v_3) = 3$ and $d(v_2) = 3$. By Lemma 2.11(2), $v_3u_2, v_3u_1 \notin E(G)$. Thus, $d(v_2) + d(v_3) = 6 < 7$, contrary to the given degree-sum condition. In the latter case, by applying $\delta(G) = 3$ and Lemma 2.11(1) to v_3 and v_2, we have $v_2v_1, v_3v_1 \in E(G)$, $d(v_3) = 3$ and $d(v_2) = 3$. By Lemma 2.11(2), $v_1u_3, v_1u_2 \notin E(G)$. Thus, $d(v_1) + d(v_2) = 6 < 7$, contrary to the given degree-sum condition.

Now we suppose that $v_1 = v_3$. Let $v \in M - \{v_1, v_2\}$. It follows that $vu_2 \in E(G)$. By Lemma 2.11(1), $v_2u_2, v_1u_2 \notin E(G)$. Since $\delta(G) = 3$, $vv_1, vv_2 \in E(G)$. By applying the given degree-sum condition to vv_1 and vv_2, respectively, $v_1v_2 \in E(G)$. In this case, G is the graph in Fig. 2(b) which is Z_3-connected by Lemma 2.5, a contradiction.

Finally, let $n = 6$. Let $v_1, v_2 \in V(G) - \{u, u_1, u_2, u_3\}$. By Lemma 2.11(1) and by $\delta(G) = 3$, $e(v_i, \{u_1, u_2, u_3\}) = 2$ and $v_1v_2 \in E(G)$. If $v_1u_1, v_1u_3 \in E(G)$, by Lemma 2.11(2), $v_2u_1, v_2u_3 \in E(G)$. In this case, $G = G_{14}$. If $v_1u_1, v_1u_2 \in E(G)$, by Lemma 2.11, $v_2u_2, v_2u_3 \in E(G)$. In this case, G is G_{15}.

Case 4. There are three edges in $G[N(u)]$.

When $n \geq 11$, as in the proof in Case 3, $|N(u_1) \cap N(u_2) \cap N(u_3)| \geq 2$. By Lemma 2.11(1), $G \in \langle Z_3 \rangle$, a contradiction. Thus, we assume that $6 \leq n \leq 10$. Let $N = \{v \in M : e(v, \{u_1, u_2, u_3\}) \leq 1\}$.

First, we assume that $N = \emptyset$. In this case, $e(v, \{u_1, u_2, u_3\}) = 2$ for each vertex $v \in M$. Let $v_1 \in M \cap N(u_1) \cap N(u_2)$. Since $\delta(G) = 3$, there must be a vertex $v_2 \in M$ such that $v_1v_2 \in E(G[M])$. By Lemma 2.11(2), $v_2 \in N(u_1) \cap N(u_2)$. When $n = 6$, G is G_{13}. When $n \geq 7$, by the given degree-sum condition to v_1v_2, there is a vertex $v_3 \in M$ such that $v_1v_3 \in E(G[M])$ or $v_2v_3 \in E(G[M])$. By symmetry, let $v_1v_3 \in E(G)$. By Lemma 2.11(3), $v_3 \notin N(u_1) \cap N(u_2) \cap M$, that is, $v_3 \in N(u_1) \cap N(u_3) \cap M$ or $v_3 \in N(u_2) \cap N(u_3) \cap M$. Both cases contradict Lemma 2.11(2). Thus, $N \neq \emptyset$.

We next assume that there exists a vertex $v_0 \in N$ such that $e(v_0, \{u_1, u_2, u_3\}) = 0$. By Lemma 2.12, $n = 8$. As in the proof of Lemma 2.12, for each vertex v in $M - \{v_0\}$, $e(v, \{u_1, u_2, u_3\}) = 2$. Let $M = \{v_0, v_1, v_2, v_3\}$. By $d(v_0) \geq 3$, $v_0v_1, v_0v_2, v_0v_3 \in E(G)$. By applying the given degree-sum condition to each edge v_0v_1, v_0v_2, v_0v_3 and by Lemma 2.11, the subgraph induced by M is a complete graph. By Lemma 2.11(3), $(N(v_1) \cup N(v_2) \cup N(v_3)) \cap \{u_1, u_2, u_3\} = \{u_1, u_2, u_3\}$. Thus, there are $s, t \in \{1, 2, 3\}$ such that $v_s \in N(u_i) \cap N(u_j) \cap M$ and $v_t \in N(u_i) \cap N(u_k) \cap M$, which contradicts Lemma 2.11(2).

So far we have proved that $N \neq \emptyset$ and N does not have a vertex v such that $e(v, \{u_1, u_2, u_3\}) = 0$. Thus, assume that there is one vertex $v_0 \in M$ such that $e(v_0, \{u_1, u_2, u_3\}) = 1$. We assume, without loss of generality, that $v_0u_3 \in E(G)$. By Lemma 2.12, $7 \leq n \leq 9$. Since $d(v_0) \geq 3$, there exists a vertex $v_1 \in M$ such that $v_1v_0 \in E(G)$. By applying the given degree-sum condition to v_0v_1, $M - \{v_0, v_1\}$ contains at least one vertex, say v_2, adjacent to both v_0 and v_1, for otherwise, $e(v_1, M - \{v_0, v_1\}) + e(v_0, M - \{v_0, v_1\}) \leq |M| - 2$, which implies that $d(v_0) + d(v_1) \leq |M| - 2 + 2 + 3 = n - 1 < n$, a contradiction.

Suppose that $e(v_i, \{u_1, u_2, u_3\}) = 2$ for each $i = 1, 2$. If $v_1 \in N(u_i) \cap N(u_j) \cap M$, then by Lemma 2.11(2), $v_2 \in N(u_i) \cap N(u_j) \cap M$ for $i \neq j$. If $\{i, j\} = \{1, 3\}$, then G contains an even wheel W_4 induced by u_1, u_3, v_0, v_1 and v_2 with the center at v_1; if $\{i, j\} = \{2, 3\}$, then G contains an even wheel W_4 induced by u_2, u_3, v_0, v_1 and v_2 with the center at v_1. We contract this W_4 and iteratively contracting 2-cycles leads eventually to a K_1 which is Z_3-connected. By Lemma 2.1, $G \in \langle Z_3 \rangle$, a contradiction. Thus, $N(u_i) \cap N(u_j) \cap M = \{v_1, v_2\}$ and $\{u_i, u_j\} = \{u_1, u_2\}$. In this case, G contains the graph H in Fig. 2(a) with a 4-cycle $v_1v_0u_3u_1$ and a distinguished edge u_2v_2. We contract this H and iteratively contracting 2-cycles leads eventually to a K_1 which is Z_3-connected. By Lemma 2.1, $G \in \langle Z_3 \rangle$, a contradiction.

Thus, there is one of v_1 and v_2, say v_1, such that $e(v_1, \{u_1, u_2, u_3\}) = 1$, by Lemma 2.12, $n = 8$. Pick $v_3 \in M - \{v_0, v_1, v_2\}$. This implies that $e(v_3, \{u_1, u_2, u_3\}) = 2$. Since $d(v_0) + d(v_1) \geq 8$, $v_0v_3 \in E(G)$ and $v_1v_3 \in E(G)$. Thus, $d(v_0) = 4$. Since $e(\{u_1, u_2\}, v_0) = 0$ and $e(\{u_1, u_2\}, v_1) \leq 1$, $e(\{u_1, u_2\}, \{v_2, v_3\}) \geq 3$. We assume, without loss of generality, that $v_2u_1, v_2u_2 \in E(G)$. If $v_1u_1 \in E(G)$, by assumption that $e(v_1, \{u_1, u_2, u_3\}) = 1$, $e(v_1, \{u_2, u_3\}) = 0$. Applying the given degree-sum condition to uu_2 and uu_3, respectively, then $v_3u_2, v_3u_3 \in E(G)$; if $v_1u_3 \in E(G)$, then $v_3u_1, v_3u_2 \in E(G)$. For both cases, let $G' = G_{[v_2u_1, v_2u_2]}$. It follows that G' contains a 2-cycle $u_1u_2u_1$. Iteratively contracting 2-cycles leads eventually to a K_1, which is Z_3-connected. By Lemmas 2.1 and 2.5, $G \in \langle Z_3 \rangle$, a contradiction. □

Lemma 2.14. *Let G be a 2-edge-connected simple graph on n vertices with $\delta(G) \geq 4$, where $n \geq 7$. If $d(x) + d(y) \geq n$ for each $xy \in E(G)$, then $G \in \langle Z_3 \rangle$ or G contains K_4^-.*

Proof. Let $v \in V(G)$ be a vertex such that $d(v) = \delta(G) \geq 4$. Suppose that $N(v) = \{u_1, u_2, \ldots, u_k\}$. It follows that $k \geq 4$. If there is no edge in $G[N(v)]$, then for each $1 \leq i \leq k$, u_i is adjacent to all the vertices in $V(G) - N_G(v)$ by the given degree-sum condition. Therefore, G contains $K_{k,n-k}$ as a subgraph. Since $\delta(G) \geq 4$, $k \geq 4$ and $n - k = d_G(u_i) \geq 4$. By Lemma 2.1(3), G is Z_3-connected.

So we may assume that $G[N(v)]$ contains some edge, say $u_1u_2 \in E(G)$. This implies that vu_1u_2 is a 3-cycle of G. If there is no K_4^- in G, then each vertex in $V(G) - \{v, u_1, u_2\}$ is adjacent to at most one vertex in $\{v, u_1, u_2\}$. Thus, $d_G(v) + d_G(u_1) + d_G(u_2) \leq n - 3 + 6 = n + 3$. By the given degree-sum condition, $d_G(v) + d_G(u_1) + d_G(u_2) \geq 3n/2$. Thus, $3n/2 \leq n + 3$, and so $n \leq 6$, a contradiction. Therefore, G contains a K_4^-. □

3. Proof of Theorem 1.3

If G is one of $K_{2,n-2}, K_{3,n-3}, K_{2,n-2}^+, K_{3,n-3}^+$ and the 15 exceptional graphs in Fig. 1, by Lemma 2.7, $G \notin \langle Z_3 \rangle$. Conversely, suppose that $G \notin \{K_{2,n-2}, K_{3,n-3}, K_{2,n-2}^+, K_{3,n-3}^+\}$ and no graph in Fig. 1 is G. We shall prove that $G \in \langle Z_3 \rangle$. If $2 \leq \delta(G) \leq 3$, then by Lemmas 2.10 and 2.13, $G \in \langle Z_3 \rangle$. Suppose therefore that $\delta(G) \geq 4$.

We proceed by induction on $n = |V(G)|$. When $n = 5$, G is K_5 and $G \in \langle Z_3 \rangle$ by Lemma 2.1(1). When $n = 6$, if G is K_6, then by Lemma 2.1(1), $G \in \langle Z_3 \rangle$. Thus, assume that G is not a K_6. In this case, $\delta(G) = 4$ and let $d(u) = 4$. Let $G' = G - u$. Then for each vertex $v \in V(G')$, $d_{G'}(v) \geq 3$. Since $|V(G')| = 5$, G' has an even wheel W_4 as a spanning subgraph. By Lemma 2.1, $G' \in \langle Z_3 \rangle$ and hence $G \in \langle Z_3 \rangle$. Suppose thus that $n \geq 7$ and the theorem holds for every graph G with $|V(G)| < n$. By Lemma 2.14, we may assume that G contains a K_4^-, the union of two triangles xyz and xyw with $d(z) \geq 4$. Let G' be the graph obtained from G by deleting zx, zy, and adding xy.

We claim that G' is 2-edge connected. Suppose otherwise that G' is not connected or G' has an cut edge e. Define G'' as follows. $G'' = G'$ if G' is not connected and $G'' = G' - e$ otherwise. It follows that x, y, w are in one component F_1 of G'' and z is in other component F_2 of G''. We further assume $e = z_1z_2$ such that if $G'' = G' - e$, then $z_1 \in V(F_1)$ and $z_2 \in V(F_2)$.

If G' is not connected, w has a neighbor $w' \in V(G) - \{x, y, z\}$ and define $e_0 = ww'$; if G' has an cut edge e and $w \neq z_1$, then $zw \notin E(G)$. Since $\delta(G) \geq 4$, w has a neighbor $w' \in V(G) - \{x, y, z_1\}$ and define $e_0 = ww'$; if G' has an cut edge e and $w = z_1$, then we also have $zw \notin E(G)$. Since $\delta(G) \geq 4$, w have a neighbor $w_1 \in V(G) - \{x, y, z_1\}$. By $\delta(G) \geq 4$ again, w_1 has a neighbor $w_2 \in V(G) - \{x, y, z_1\}$ and define $e_0 = w_1w_2$. Thus, F_1 contains an edge $e_0 = a_1a_2$ such that $e(a_1a_2, z) = 0, z_1 \notin \{a_1, a_2\}$. Similarly, F_2 contains an edge b_1b_2 such that $\{b_1, b_2\} \cap \{z, z_2\} = \emptyset$. By the given degree sum condition, $n \leq d(a_1) + d(a_2) \leq 2|V(F_1)| - 2$ and $n \leq d(b_1) + d(b_2) \leq 2|V(F_2)| - 2$. It follows that $|V(F_i)| \geq (n + 2)/2$ for $i = 1, 2$. Thus, $n \geq |V(F_1)| + |V(F_2)| \geq n + 2$, a contradiction.

Let H be the maximal Z_3-connected subgraph containing 2-cycle xyx of G' and $G^* = G'/H$. Since G' is 2-edge connected, G^* is 2-edge connected. Denote by u^* the new vertex into which H is contracted. Note that G^* is a simple graph, in which all vertices except for u^* and z, have the same degree as in G and $e(t, H) \leq 1$ for any $t \in V(G) - V(H)$. If $G^* \in \langle Z_3 \rangle$, by Lemma 2.1 $G' \in \langle Z_3 \rangle$ and so is G. Let $|V(G^*)| = n^*$. Note that each vertex in G^* other than u^* and z has degree at least 4, then it is a routine work to verify that if $n^* \leq 5$, then $G^* \in \langle Z_3 \rangle$, which implies that $G' \in \langle Z_3 \rangle$ and so is G. Therefore, assume that $n^* \geq 6$, that is $n \geq 8$.

Note that $|V(G^*)| = n^* < n$. To prove that $G^* \in \langle Z_3 \rangle$, we need to prove that $d_{G^*}(v_1) + d_{G^*}(v_2) \geq n^*$ for any two distinct $v_1, v_2 \in V(G^*)$ and $v_1v_2 \in E(G^*)$. There are four cases to discuss, as follows.

If $v_1, v_2 \in V(G^*) \setminus \{z, u^*\}$, then $d_{G^*}(v_1) + d_{G^*}(v_2) = d_G(v_1) + d_G(v_2) \geq n > n^*$.

If $v_1 \neq u^*$ and $v_2 = z$, then using $d_{G^*}(z) = d_G(z) - 2$, $d_{G^*}(v_1) + d_{G^*}(v_2) = d_G(v_1) + d_G(v_2) - 2 \geq n - 2 \geq n^*$.

If $v_1 = u^*$ and $v_2 \neq z$, then there is $\lambda \in V(H)$ such that $v_2\lambda \in E(G)$. Since $d_{G^*}(u^*) \geq d_G(\lambda) - (|V(H)| - 1)$, we have that $d_{G^*}(v_1) + d_{G^*}(v_2) \geq d_G(\lambda) - (|V(H)| - 1) + d_G(v_2) \geq n - (|V(H)| - 1) = n^*$.

It remains to us that $v_1 = u^*$ and $v_2 = z$. Let $T = G - V(H)$. It follows that there is $\mu \in V(H) - \{x, y\}$ such that $\mu z \in E(G)$.

If $|V(H)| = 3$, then $V(H) = \{x, y, w\}$. In this case, $\mu \in \{x, y, w\}$. We have $d_{G^*}(z) + d_{G^*}(u^*) = d(z) - 2 + e(xy, T - z) + e(w, T) = d(z) - 2 + d(x) + d(y) - 6 + d(w) - 2 \geq 2n - 10$. Since $n \geq 8$, $d_{G^*}(v_1) + d_{G^*}(v_2) \geq n - 2 = n^*$.

If $|V(H)| = 4$, then $V(H) = \{x, y, w, s\}$. Because $n^* \geq 6$, $|V(H)| = 4$ implies that $n \geq 9$. Since $|V(H)| = 4$, $e(xy, T - z) \geq d(x) + d(y) - 8$, $e(w, T) \geq d(w) - 3$ and $e(s, T) \geq d(s) - 3$. Since $\delta(G) \geq 4$, $d_{G^*}(z) + d_{G^*}(u^*) = d(z) - 2 + d(xy, T - z) + e(w, T) + e(s, T) \geq d(z) - 2 + d(x) + d(y) - 8 + d(w) - 3 + d(s) - 3 = (d(z) + d(x) - 5) + (d(y) + d(w) - 8) + (d(s) - 3) \geq n - 5 + n - 8 + 1 \geq n - 3 = n^*$.

Therefore we suppose that $|V(H)| \geq 5$. Let $H_0 = H - \{x, y, \mu\}$. If H_0 contains an edge ss' and $e(ss', T) \geq 2$, then $d_{G^*}(u^*) + d_{G^*}(z) \geq d(z) - 2 + d(\mu) - (|V(H)| - 1) + 2 \geq n - (|V(H)| - 1) = n^*$. Thus, assume $e(ss', T) \leq 1$. In this case, $n \leq d(s) + d(s') \leq 2|V(H)| - 1$, that is, $|V(H)| \geq \frac{n+1}{2}$, so $|V(T)| \leq \frac{n-1}{2}$. For any $t \in V(T - z)$, there exist $t' \in V(T - z)$ such that $tt' \in E(G)$ since $d(t) \geq 4$ and $e(t, H) \leq 1$ for any $t \in V(T - z)$. By the given degree sum condition, $d(t) + d(t') \geq n$. Then $|V(T)| \geq \frac{n}{2}$, a contradiction. So assume that there is no edge in H_0. It follows that $e(H - \mu, T - z) \geq 2$ since $|V(H)| \geq 5$ and $\delta(G) \geq 4$. We obtain that $d_{G^*}(u^*) + d_{G^*}(z) \geq d(z) - 2 + d(\mu) - (|V(H)| - 1) + e(H - \mu, T - z) \geq n - (|V(H)| - 1) = n^*$.

By the induction hypothesis, either $G^* \in \langle Z_3 \rangle$ or G^* is one of $K_{2,n-2}, K_{3,n-3}, K^+_{2,n-2}, K^+_{3,n-3}$ and the 15 exceptional graphs in Fig. 1. Note that there are $(|V(G^*)| - 2)$ vertices of degree at least 4, since each such vertex has the same degree in G^* as that in G. This shows that $G^* \notin \{K_{2,n-2}, K_{3,n-3}, K^+_{2,n-2}, K^+_{3,n-3}\}$ and no graph in Fig. 1 except G_6 is G^*. Suppose that $G^* = G_6$. Let v_1, v_2 be two vertices of degree 2 in G_6 and other vertices in G_6 has degree 4 which implies that $n = 8$, $n^* = 6$ and $|V(H)| = 3$. Thus, $\{v_1, v_2\} = \{u^*, z\}$. Since $\delta(G) \geq 4$ and $d(x) + d(y) \geq 8$, $d_{G^*}(u^*) \geq 4$, contrary to $d_{G^*}(u^*) = 2$. Therefore we complete our proof.

Corollary 3.1. *Let G be a 2-edge-connected simple graph on n vertices. If $d(x) + d(y) \geq n + 1$ for each $xy \in E(G)$, then $G \notin \langle Z_3 \rangle$ if and only if G is either $K^+_{2,n-2}$ or G_1 or G_2 or G_{11}.*

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