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# Group Connectivity in Products of Graphs

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**Abstract**

Let $G$ be a 2-edge-connected undirected graph, $A$ be an (additive) abelian group and $A^* = A - \{0\}$. A graph $G$ is $A$-connected if $G$ has an orientation $D(G)$ such that for every function $b : V(G) \mapsto A$ satisfying $\sum_{v \in V(G)} b(v) = 0$, there is a function $f : E(G) \mapsto A^*$ such that for each vertex $v \in V(G)$, the total amount of $f$ values on the edges directed out from $v$ minus the total amount of $f$ values on the edges directed into $v$ equals $b(v)$. For a 2-edge-connected graph $G$, define $\Lambda_g(G) = \min\{k :$ for any abelian group $A$ with $|A| \geq k$, $G$ is $A$-connected$\}$.

Let $G_1 \otimes G_2$ and $G_1 \times G_2$ denote the strong and Cartesian product of two connected nontrivial graphs $G_1$ and $G_2$. In this paper, we prove that $\Lambda_g(G_1 \otimes G_2) \leq 4$, where equality holds if and only if both $G_1$ and $G_2$ are trees and $\min\{|V(G_1)|, |V(G_2)|\}$=2; $\Lambda_g(G_1 \times G_2) \leq 5$, where equality holds if and only if both $G_1$ and $G_2$ are trees and either $G_1 \cong K_{1,m}$ and

$G_2 \cong K_{1,n}$, for $n, m \geq 2$ or $\min\{|V(G_1)|, |V(G_2)|\}$=2. A similar result for the lexicographical product graphs is also obtained.

**Keywords:** nowhere-zero flows, group connectivity, products of graphs

# 1 Introduction

We consider finite graphs which may have multiple edges but no loops. Undefined terms and notations will follow Bondy and Murty [1]. For $m, n \geq 1$, $P_m$ is a path with $m$ edges, $K_m$ is a complete graph with $m$ vertices and $K_{m,n}$ is a complete bipartite graph with bipartition $(X, Y)$ such that $|X| = m$ and $|Y| = n$. And $\kappa(G), \kappa'(G)$ and $\delta(G)$ denote the connectivity, edge-connectivity and minimum degree of a graph $G$, respectively. Different from [1], a 2-regular nontrivial connected graph is called a **circuit**, and a circuit with $k$ edges is referred as a $k$-circuit. For an edge subset $J \subseteq E(G)$ of a graph $G$, we take the convention of using $J$ to denote both the edge subset as well as the subgraph $G[J]$ induced by the edge set $J$. Throughout this paper, $G_1$ and $G_2$ denote two nontrivial connected simple graphs and $A$ denotes an (additive) abelian group with identity 0. and $A^* = A - \{0\}$.

Let $D = D(G)$ be an orientation of a graph $G$. If an edge $e \in E(G)$ is directed from a vertex $u$ to a vertex $v$, then let $tail(e) = u$ and $head(e) = v$. For a vertex $v \in V(G)$, define

$$E_D^+(v) = \{e \in E(G) : v = \text{ tail}(e)\}, \text{ and } E_D^-(v) = \{e \in E(G) : v = \text{ head}(e)\}.$$

Following Jaeger et al [9], we define $F(G, A) = \{f | f : E(G) \mapsto A\}$ and $F^*(G, A) = \{f | f : E(G) \mapsto A^*\}$. For a function $f : E(G) \to A$, define $\partial f : V(G) \mapsto A$ by

$$\partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e),$$

where "$\sum$" refers to the addition in $A$.

Assume that $G$ has an orientation $D(G)$. A function $b : V(G) \mapsto A$ is called an $A$**-valued zero sum function** on $G$ if $\sum_{v \in V(G)} b(v) = 0$. The set of all $A$-valued zero sum functions on $G$ is denoted by $Z(G, A)$. A function $f \in F(G, A)$ is an $A$**-flow** of $G$ if $\partial f(v) = 0$ for every vertex $v \in V(G)$. An $A$-flow $f$ is a **nowhere-zero** $A$**-flow** (abbreviated as $A$-NZF) if $f \in F^*(G, A)$. For a $b \in Z(G, A)$, a function $f \in F^*(G, A)$ is **a nowhere-zero** $(A, b)$**-flow** (abbreviated as $(A, b)$-NZF) if $\partial f = b$. A graph $G$ is $A$**-connected** if $\forall b \in Z(G, A)$, $G$ has an $(A, b)$-NZF. Let $\langle A \rangle$ be the family of graphs that are $A$-connected. The **group connectivity number** of a graph $G$ is defined as

$$\Lambda_g(G) = \min\{k : G \in \langle A \rangle \text{ for every abelian group } A \text{ with } |A| \geq k\}.$$

The concept of group connectivity was first introduced by Jaeger, Linial, Payan and Tarsi in [9] as a nonhomogeneous form of the nowhere-zero flow problem. The nowhere-zero flow problem was first introduced by Tutte [17] in his way to attach the 4-color-conjecture. Tutte left with several fascinating conjectures in this area, which have remained open as of today.

**Conjecture 1.1** *(Tutte [17], [8])*
*(i) Every graph $G$ with $\kappa'(G) \geq 2$ has a nowhere-zero $Z_5$-flow.*
*(ii) Every graph $G$ with $\kappa'(G) \geq 2$ without a subgraph contractible to the Peterson graph admits a nowhere-zero $Z_4$-flow.*
*(iii) Every graph $G$ with $\kappa'(G) \geq 4$ admits a nowhere-zero $Z_3$-flow.*

Jaeger et al made the following conjectures about group connectivity. The truth of these conjectures will imply the truth of Tutte's $Z_5$-flow conjecture and $Z_3$-flow conjecture, as indicated by Kochol [10].

**Conjecture 1.2** *(Jaeger, Linial, Payan and Tarsi [9])*
*(i) If $G$ is a 3-edge-connected graph, then $\Lambda_g(G) \leq 5$.*
*(ii) If $G$ is a 5-edge-connected graph, then $\Lambda_g(G) \leq 3$.*

While many have contributed to the literature of nowhere-zero flows, all these conjectures remain open. Most recently, X. Yao et al [18] and X. Zhang et al [21] have found best possible degree conditions for graphs with group connectivity 4 and 3, respectively. X. Yao and D. Gong also investigated the group connectivity of Kneser graphs. A survey on group connectivity can be found in [15]. Several researchers have investigated the problem that what kind of products graphs will have nowhere-zero $A$-flows when $|A|$ is small, as seen in [7], [16] and [20], among others.

The purpose of this paper is to determine the group connectivity number for all strong product and lexicographical product graphs, and to investigate the group connectivity number of the Cartesian product graphs.

For graph products, we adopt the notation in [6]. Let $G_1, G_2$ be two graphs. The **Cartesian product** graph $G = G_1 \times G_2$ is a graph with vertex set $V(G) = V(G_1) \times V(G_2)$ and edge set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)\}$. The **strong product** graph $G = G_1 \otimes G_2$ is a graph with vertex set $V(G) = V(G_1) \times V(G_2)$ and edge set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1$ and $u_2v_2 \in E(G_2)$, or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$, or both $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)\}$. And the **lexicographic product** (sometimes called composition, tensor or wreath product) $G = G_1[G_2]$ is a graph with vertex set $V(G) = V(G_1) \times V(G_2)$ and edge set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G_1)$, or $u_1 = v_1$ and $u_2v_2 \in E(G_2)\}$.

The following are immediate from the above definitions.

**Proposition 1.3** *Each of the following holds.*
*(i)* $G_1 \times G_2$ *is a spanning subgraph of* $G_1 \otimes G_2$*, and* $G_1 \otimes G_2$ *is a spanning subgraph of* $G_1[G_2]$.
*(ii) If* $G_2 \cong K_m$ *is a complete graph, then* $G_1[K_m] = G_1 \otimes K_m$.

In this paper we will determine the group connectivity number of certain products of connected graphs by proving the following main results.

**Theorem 1.4** $\Lambda_g(G_1 \otimes G_2) \leq 4$*, where equality holds if and only if both* $G_1$ *and* $G_2$ *are trees and* $min\{|V(G_1)|, |V(G_2)|\} = 2$.

**Corollary 1.5** $G_1 \otimes G_2$ *has a nowhere-zero 3-flow if and only if either one of* $G_1$ *and* $G_2$ *is not a tree, or both* $G_1$ *and* $G_2$ *are trees with* $min\{|V(G_1)|, |V(G_2)|\} \geq 3$.

**Theorem 1.6** $\Lambda_g(G_1[G_2]) \leq 4$*, where equality holds if and only if both* $G_1$ *and* $G_2$ *are trees and* $min\{|V(G_1)|, |V(G_2)|\} = 2$.

**Corollary 1.7** $G_1[G_2]$ *has a nowhere-zero 3-flow if and only if either one of* $G_1$ *and* $G_2$ *is not a tree, or both* $G_1$ *and* $G_2$ *are trees with* $min\{|V(G_1)|, |V(G_2)|\} \geq 3$.

**Theorem 1.8** $\Lambda_g(G_1 \times G_2) \leq 5$*, where equality holds if and only if either* $G_1 \cong K_{1,m}$ *and* $G_2 \cong K_{1,n}$*, for* $n, m \geq 2$ *or* $G_1$ *is a tree and* $G_2 \cong K_2$.

This paper is organized as follows: In Section 2, we present the preliminaries as a preparation for the proofs. Sections 3 and 4 are devoted to the investigation of the group connectivity of strong products and lexicographical products, and Cartesian products of graphs, respectively.

# 2 Preliminaries

The purpose of this section is to lay down the preparation for the proofs of the main results in the next two sections.

Let $G$ be a graph and let $X \subseteq E(G)$ be an edge subset. The **contraction** $G/X$ is the graph obtained from $G$ by identifying the two ends of each edge in $X$ and deleting the resulting loops. For convenience, we use $G/e$ for $G/\{e\}$; and if $H$ is a subgraph of $G$, we write $G/H$ for $G/E(H)$.

**Proposition 2.1** *(Proposition 2.2, [9]) Let* $G$ *be a connected graph and* $A$ *be an abelian group. Then following are equivalent.*
*(i)* $G \in \langle A \rangle$.
*(ii)* $\forall \overline{f} \in F(G, A), \exists f \in F_0(G, A)$ *such that* $\forall e \in E(G), f(e) \neq \overline{f}(e)$.
*(iii)* $\forall b \in Z(G, A)$*, and* $\forall \overline{f} \in F(G, A), \exists f \in F(G, A)$ *such that* $\partial f = b$ *and* $\forall e \in E(G), f(e) \neq \overline{f}(e)$.

**Proposition 2.2** *(Lai, Proposition 3.2 of [13] and Proposition 2.2 of Chen et al, [4]) Let $A$ be an abelian group with $|A| \geq 3$. Then $\langle A\rangle$ satisfies each of the following:*
*(C1) $K_1 \in \langle A\rangle$,*
*(C2) if $G \in \langle A\rangle$ and $e \in E(G)$, then $G/e \in \langle A\rangle$,*
*(C3) if $H$ is a subgraph of $G$ and if both $H \in \langle A\rangle$ and $G/H \in \langle A\rangle$, then $G \in \langle A\rangle$.*

**Lemma 2.3** *(Lemma 2.1 of [14]) Let $G$ be a graph and $A$ be an abelian group. If for every edge $e$ in a spanning tree of $G$, $G$ has a subgraph $H_e \in \langle A\rangle$ with $e \in E(H_e)$, then $G \in \langle A\rangle$.*

**Proposition 2.4** *([9], Lemma 3.3 of [13]) For any abelian group $A$, $C_n \in \langle A\rangle$ if and only if $|A| \geq n+1$.*

Let $H_1$ and $H_2$ be two subgraphs of a graph $G$. We say that $G$ is a **parallel connection** of $H_1$ and $H_2$, denoted by $H_1 \oplus_2 H_2$, if $E(H_1) \cup E(H_2) = E(G)$, $|V(H_1) \cap V(H_2)| = 2$ and $|E(H_1) \cap E(H_2)| = 1$. The edge $e \in E(H_1) \cap E(H_2)$ is usually referred as the **base edge**.

A **wheel** $W_k$ is the graph obtained from a $k$-circuit by adding a new vertex **the center of the wheel**, and then by joining the center to every vertex of the $k$-circuit. A fan $F_k$ is the graph obtained from $W_k$ by deleting an edge not incident with the center. Note that $F_2$ is the 3-circuit, and $W_3$ is the complete graph $K_4$. The family $WF$ can now be recursively defined as follows:
(WF1) For all $k \geq 1$, and $n \geq 2$, $W_{2k+1}, F_n \in \langle WF\rangle$.
(WF2) If $G, H \in \langle WF\rangle$, then any parallel connection of $G$ and $H$ is also in $\langle WF\rangle$.

Graphs in $\langle WF\rangle$ are usually referred as $WF$-graphs. For an integer $k \geq 3$, graph $G$ is **$k$-circuit connected** if for any pair of edges $e, e' \in E(G)$, $G$ has a sequence of circuits $C_1, C_2, \cdots, C_m$ such that $|E(C_i)| \leq k$, $(1 \leq i \leq m)$, $e \in E(C_1)$, $e' \in E(C_m)$ and $E(C_i) \cap E(C_{i+1}) \neq \emptyset$, $(1 \leq i \leq m-1)$. The sequence $C_1, C_2, \cdots, C_m$ is often referred as an **$(e, e')$-$k$-circuit-path**. A 3-circuit connected graph is also referred as a **triangularly connected** graph. By definition, every $WF$-graph is triangularly connected.

**Theorem 2.5** *(Fan et al, [5]) Let $G$ be a triangularly connected graph with $|V(G)| \geq 3$. Then*
*(i) $G$ is $Z_3$-connected if and only if $G$ contains a nontrivial $Z_3$-connected subgraph.*
*(ii) $G$ is $Z_3$-connected if and only if $G \notin \langle WF\rangle$.*

A graph $G$ is **collapsible** if for every even subset $R \subseteq V(G)$, $G$ has a subgraph $\Gamma_R$ (called the **$R$-subgraph** of $G$) such that $G - E(\Gamma_R)$ is connected and $R$ is the set of odd-degree vertices of $\Gamma_R$. The collection of all collapsible graphs is denoted by $\mathcal{CL}$. The following summarizes some useful result related to collapsible graphs.

**Theorem 2.6** *Let $G$ be a graph and $H$ be a collapsible subgraph of $G$. Each of the following holds.*
*(i) (Catlin, Theorem 3 and its Corollary of [3]) $G$ is collapsible if and only if $G/H$ is collapsible.*
*(ii) (Catlin, Lemma 3 of [3]) If $G$ is collapsible, then for any $e \in E(G)$, $G/e$ is collapsible.*
*(iii) (Catlin, [3] and Lemma 1 of [2]) Let $e \in E(K_{3,3})$. Then $C_2$, $C_3$, $K_{3,3}$ and $K_{3,3} - e$ are collapsible.*
*(iv) (Theorem 1.5, [12]) Let $A$ be an abelian group with $|A| = 4$. Then $H \in \langle A \rangle$.*

We follow the notations in [11]. Let $G$ be a graph with $C_4$ as a subgraph, and $\pi = \{X, Y\}$ the bipartition of $V(C_4)$ so that both $X$ and $Y$ are independent sets of $C_4$. Let $G/\pi$ denote the graph obtained from $G$ by identifying all vertices of $X$ to form a single vertex $x$, identifying all vertices of $Y$ to form a single vertex $y$, and then joining $x, y$ with a new edge $e_\pi = xy$, so that

$$E(G) - E(C_4) = E(G/\pi) - \{e_\pi\}.$$

Catlin had the following result.

**Theorem 2.7** *(Catlin, [2]) Let $G/\pi$ be defined as above. If $G/\pi \in \mathcal{CL}$, then $G \in \mathcal{CL}$.*

# 3 Group Connectivity of Strong Products and Lexicographical Products

The following observation follows from the definition of strong product immediately.

$$G_1 \otimes G_2 \text{ is triangularly connected.} \tag{1}$$

Thus every edge of $G_1 \otimes G_2$ is contained in a 3-circuit. It follows by Lemma 2.3 and Proposition 2.4, that

$$\Lambda_g(G_1 \otimes G_2) \leq 4. \tag{2}$$

We shall prove Theorem 1.4 by proving each of the following lemmas.

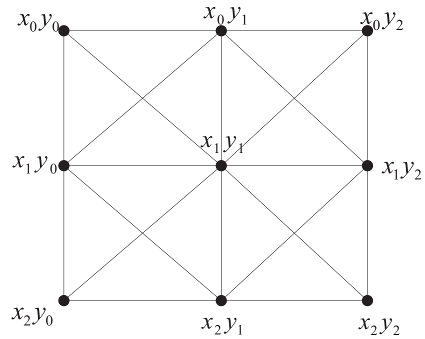



**Figure 1** $G = P_2 \otimes P_2$

**Lemma 3.1** *Each of the following holds.*
*(i)* $\Lambda_g(C_n \otimes K_2) = 3$ *and* $\Lambda_g(P_2 \otimes P_2) = 3$.
*(ii) If* $G_1$ *or* $G_2$ *contains a circuit, then* $\Lambda_g(G_1 \otimes G_2) = 3$.
*(iii) If both* $G_1$ *and* $G_2$ *contains a path of length at least 2, then* $\Lambda_g(G_1 \otimes G_2) = 3$.

**Proof:** (i) By (2), $\Lambda_g(C_n \otimes K_2) \leq 4$. By the definition of strong product, $C_n \otimes K_2$ is not a $WF$-graph. It follows by (1) and by Theorem 2.5 that $C_n \otimes K_2$ is $Z_3$-connected. Thus $\Lambda_g(C_n \otimes K_2) = 3$.

Let $G = P_2 \otimes P_2$ (see Figure 1). Then the subgraph $G'$ induced by the vertex subset $\{x_0y_1, x_1y_0, x_1y_1, x_1y_2, x_2y_1\}$ is an even wheel. By Theorem 2.5(ii), $\Lambda_g(G') = 3$. And by Theorem 2.5(i), $\Lambda_g(G) = 3$.

(ii) and (iii) Both conclusions follow from (i) and from Theorem 2.5(i). □

**Lemma 3.2** *If* $G_1$ *is a tree, and* $G_2 \cong K_2$*, then each of the following holds.*
*(i)* $G_1 \otimes G_2 \in \langle WF \rangle$.
*(ii)* $\Lambda_g(G_1 \otimes G_2) = 4$.
*(iii) If* $H$ *is a nontrivial connected graph, then* $H$ *has a nowhere-zero* $Z_3$*-flow if and only if* $H \oplus_2 K_4$ *has a nowhere-zero* $Z_3$*-flow.*
*(iv)* $G_1 \otimes G_2$ *does not have a nowhere-zero* $Z_3$*-flow.*

**Proof:** (i) We first argue by induction on $|V(G_1)|$ to show that, under the assumption of this lemma, $G_1 \otimes G_2 \in \langle WF \rangle$. If $|V(G_1)| = 2$, then $G_1 \cong K_2$ as well, and so $G_1 \otimes G_2 \cong K_4 \in \langle WF \rangle$. Assume that for smaller values of $|V(G_1)|$, if $G_1$ is a tree, then $G_1 \otimes G_2 \in \langle WF \rangle$. Assume now $|V(G_1)| \geq 3$. Since $G_1$ is a tree, $G_1$ has an edge $uv$ such that $u$ has degree 1 in $G_1$. It follows by assumption that $(G_1 - u) \otimes G_2 \in \langle WF \rangle$. By the definition of strong product, $G_1 \otimes G_2$ is a parallel connection of $(G_1 - u) \otimes G_2$ and $G_1[\{u, v\}] \otimes G_2 \cong K_4$. It follows by (WF2) in the definition of $\langle WF \rangle$ that $G_1 \otimes G_2 \in \langle WF \rangle$. Hence (i) holds by induction.

(ii) By (2), (i) and Theorem 2.5, (ii) follows as well.

(iii) Suppose that the $K_4$, as a subgraph of $H \oplus_2 K_4$, has four vertices $u_1, v_1, u_2, v_2$ such that $u_1, v_1$ are the two vertices not in $V(H)$. If $H$ has a nowhere-zero 3-flow, then extend the orientation of $H$ to $H \oplus_2 K_4$ by orienting all edges incident with $u_1$ away from $u_1$ and all edges incident with $v_1$ into $v_1$, and by extending the flow ion $H$ to $E(K_4) - E(H)$ taking a constant value 1. Then we obtain a nowhere-zero $Z_3$-flow of $H \oplus_2 K_4$. Conversely, if $H \oplus_2 K_4$ has a nowhere-zero $Z_3$-flow, then since both $u_1$ and $v_1$ are adjacent degree 3 vertices, the restriction of this $Z_3$-flow to $E(H)$ is also a nowhere-zero $Z_3$-flow of $L$.

(iv) This follows from (iii) and by induction on $|V(G_1)|$. $\square$

**Proof of Theorem 1.4**: By (2) and by Lemma 3.2, we may assume that $\Lambda_g(G_1 \otimes G_2) = 4$ to prove that both $G_1$ and $G_2$ are trees, and $\min\{|V(G_1)|, |V(G_2)|\} = 2$.

If $G_1$ or $G_2$ has a circuit, then by Lemma 3.1(ii), $G_1 \otimes G_2 \in \langle Z_3 \rangle$. Hence we may assume that both $G_1$ and $G_2$ are trees. If $\min\{|V(G_1)|, |V(G_2)|\} \geq 3$, then since $G_1$ and $G_2$ are connected, each of $G_1$ and $G_2$ contains a path of length 2. It follows by Lemma 3.1(i) that $G_1 \otimes G_2$ has a nontrivial subgraph in $\langle Z_3 \rangle$, and so by (1) and by Theorem 2.5(i), $G_1 \otimes G_2 \in \langle Z_3 \rangle$. Therefore, we must have $\min\{|V(G_1)|, |V(G_2)|\} = 2$. $\square$

**Proof of Corollary 1.5**: Since $G \in \langle Z_3 \rangle$ implies that $G$ has a nowhere-zero 3-flow, the sufficiency follows from Theorem 1.4. Conversely, if both $G_1$ and $G_2$ are trees, and $\min\{|V(G_1)|, |V(G_2)|\} = 2$, then by Lemma 3.2(iv), $G_1 \otimes G_2$ does not have a nowhere-zero $Z_3$-flow. $\square$

**Proof of Theorem 1.6**: By Proposition 1.3(i), $G_1 \otimes G_2$ is a spanning subgraph of $G_1[G_2]$. By Theorem 1.4 and Lemma 2.3, $\Lambda_g(G_1[G_2]) \leq 4$. If $G_1 \otimes G_2 \in \langle Z_3 \rangle$, then by Lemma 2.3, $G_1[G_2] \in \langle Z_3 \rangle$ as well. If both $G_1$ and $G_2$ are trees, and $\min\{|V(G_1)|, |V(G_2)|\} = 2$, then by Proposition 1.3(ii), $G_1[G_2] = G_1 \otimes G_2$, and so by Theorem 1.4, $G_1[G_2] \notin \langle Z_3 \rangle$. $\square$

**Proof of Corollary 1.7**: The proof is similar to that for Corollary 1.5, and so it is omitted. $\square$

# 4 Group Connectivity of Cartesian Products

Then the following observation follows from the definition of Cartesian product immediately.

$$G_1 \times G_2 \text{ is 4-circuit connected.} \tag{3}$$

Thus every edge of $G_1 \times G_2$ is contained in a 4-circuit. It follows by Lemma 2.3 and Proposition 2.4 that

$$\Lambda_g(G_1 \times G_2) \le 5. \tag{4}$$

**Lemma 4.1** *Let $G$ be 4-circuit connected and $A$ be an abelian group with $|A| = 4$. Each of the following holds.*
*(i) $G \in \langle A \rangle$ if and only if $G$ has a nontrivial $A$-connected subgraph.*
*(ii) $G \in \mathcal{CL}$ if and only if $G$ has a nontrivial collapsible subgraph.*
*(iii)If $G$ has a nontrivial $A$-connected subgraph, then $\Lambda_g(G) \le 4$; If $G$ has a nontrivial collapsible subgraph, then $G$ is collapsible and $\Lambda_g(G) \le 4$.*

**Proof:** (i) If $G \in \langle A \rangle$, then $G$ is a nontrivial $A$-connected subgraph of $G$.

Conversely, let $H$ be a nontrivial maximal $A$-connected subgraph of $G$. If $G = H$, then done. Assume that $H \ne G$. Since $|E(H)| \ge 1$, there is an edge $e_1 \in E(H)$. Since $E(G) - E(H) \ne \emptyset$, there is an edge $e_2 \in E(G) - E(H)$. By the definition of 4-circuit-connectedness, $G$ has an $(e_1, e_2)$-4-circuit-path. By the choice of $e_1$ and $e_2$, this 4-circuit-path has a circuit $T$ with $|E(T)| \le 4$ such that $T_1 = E(T) \cap E(H) \ne \emptyset$ and $T_2 = E(T) - T_1 \ne \emptyset$. By Proposition 2.4, $T/T_1$ is $A$-connected. Let $H' = H \cup T$. Since $H'/H = T/T_1$ is $A$-connected, and since $H$ is $A$-connected, it follows by Proposition 2.2(C3) that $H'$ is $A$-connected, contrary to the maximality of $H$. Thus we must have $H = G$, and so $G$ is $A$-connected. This, together with (4), implies that $\Lambda_g(G) \le 4$.

(ii) If $G \in \langle A \rangle$, then $G$ is a nontrivial collapsible subgraph of $G$.

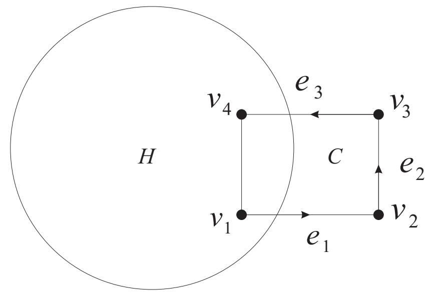
Conversely, let $H$ be a nontrivial maximal collapsible subgraph of $G$. If $G = H$, then done. Assume that $H \ne G$. Since $|E(H)| \ge 1$, there is an edge $e_1 \in E(H)$. Since $E(G) - E(H) \ne \emptyset$, there is an edge $e_2 \in E(G) - E(H)$. By the definition of 4-circuit-connectedness, $G$ has an $(e_1, e_2)$-4-circuit-path. By the choice of $e_1$ and $e_2$, this 4-circuit-path has a circuit $T$ with $|E(T)| \le 4$ such that $T_1 = E(T) \cap E(H) \ne \emptyset$ and $T_2 = E(T) - T_1 \ne \emptyset$. By Theorem 2.6(ii), $T/T_1$ is collapsible. Let $H' = H \cup T$. Since $H'/H = T/T_1$ is collapsible, and since $H$ is collapsible, it follows by Theorem 2.6(i) that $H'$ is collapsible, contrary to the maximality of $H$. Thus we must have $H = G$, and so $G$ is collapsible.

(iii) This follows from (ii), (4) and Theorem 2.6(iv). □

**Lemma 4.2** *Let $C$ be a 4-circuit and $A$ be an abelian group with $|A| = 4$. Let $G = H \oplus_2 C$. Then $H$ is $A$-connected if and only if $G$ is $A$-connected.*

**Proof:** Let $V(C) = \{v_1, v_2, v_3, v_4\}$, $E(C) = \{e_1, e_2, e_3, e_4\}$ and assume $V(H) \cap V(C) = \{v_1, v_4\}$ (see Figure 2). Let $D$ be an orientation of $G$ such that the edge $v_iv_{i+1}$ is directed from $v_i$ from $v_{i+1}$, for $i = 1, 2, 3$.

If $H \in \langle A \rangle$, since $G/H$ is a 3-circuit, by Proposition 2.4, $G/H \in \langle A \rangle$, then, by Proposition 2.2 (C3), $G \in \langle A \rangle$.




$G$

**Figure** 2 $G = H \bigoplus_2 C$

Conversely, if $G \in A$, since for any abelian group $A$ of order 4, either $A \cong Z_4$ or $A \cong Z_2 \times Z_2$, then we have the following two cases.

**Case 1:** $A \cong Z_4$. Let $b \in Z(H, Z_4)$. Define $b' : V(G) \mapsto Z_4$ to be

$$b'(v) = \begin{cases} 1, & \text{if } v = v_2, v_3, \\ b(v) + 1, & \text{if } v = v_1, v_4, \\ b(v), & \text{otherwise.} \end{cases}$$

Then $\sum_{v \in V(G)} b'(v) = \sum_{v \in V(H)} b(v) + 4 \equiv 0 \pmod 4$. So $b' \in Z(G, Z_4)$. By the definition of $Z_4$-connectedness, there is a $Z_4$-NZF $f'$ of $G$ such that $\partial f'(v) = b'(v)$, for any $v \in V(G)$. So $f'(e_3) = f'(e_2) + 1 = f'(e_1) + 2$. Therefor $\{f'(e_3), f'(e_2), f'(e_1)\} = \{1, 2, 3\}$. This concludes that $f'(e_1) = 1, f'(e_2) = 2$ and $f'(e_3) = 3$. Define $f : E(H) \mapsto Z_4^*$ to be $f(e) = f'(e)$, for any $e \in E(H)$. Then

$$\partial f(v) = \begin{cases} \partial f'(v) - 1 = b'(v) - 1 = b(v), & \text{if } v = v_1, \\ \partial f'(v) + 3 = b'(v) + 3 = b(v) + 4 \equiv b(v), & \text{if } v = v_4, \\ b(v), & \text{otherwise.} \end{cases}$$

That is, for any $v \in V(H)$, $\partial f(v) = b(v)$. Therefore by the definition of $Z_4$-connectedness, $H \in \langle Z_4 \rangle$.

**Case 2:** $A \cong Z_2 \times Z_2$.

Let $b \in Z(H, Z_2 \times Z_2)$. Define $b' : V(G) \mapsto Z_2 \times Z_2$ to be

$$b'(v) = \begin{cases} b(v) + (0, 1), & \text{if } v = v_1, \\ (1, 0), & \text{if } v = v_2, \\ (0, 1), & \text{if } v = v_3, \\ b(v) - (1, 0), & \text{if } v = v_4, \\ b(v), & \text{otherwise.} \end{cases}$$

Then $\sum_{v\in V(G)} b'(v) = \sum_{v\in V(H)} b(v) + 2(0,1) = 0$. So $b' \in Z(G, Z_4)$. By the definition of $Z_2 \times Z_2$-connectedness, there is a $Z_2 \times Z_2$-NZF $f'$ of $G$ such that $\partial f'(v) = b'(v)$, for any $v \in V(G)$. So $f'(e_3) = f'(e_2) + (0,1) = f'(e_1) + (1,1)$. Therefor $\{f'(e_3), f'(e_2), f'(e_1)\} = \{(0,1), (1,0), (1,1)\}$. This concludes that $f'(e_1) = (0,1), f'(e_2) = (1,1)$ and $f'(e_3) = (1,0)$. Let $f : E(H) \mapsto Z_2 \times Z_2^*$ be $f(e) = f'(e)$, for any $e \in E(H)$. Then

$$\partial f(v) = \begin{cases} \partial f'(v) - (0,1) = b'(v) - (0,1) = b(v), & \text{if } v = v_1, \\ \partial f'(v) + (1,0) = b'(v) + (1,0) = b(v), & \text{if } v = v_4, \\ b(v), & \text{otherwise.} \end{cases}$$

That is, for any $v \in V(H)$, $\partial f(v) = b(v)$. Therefore by the definition of $Z_2 \times Z_2$-connectedness, $H \in \langle A \rangle$.

By Case 1 and Case 2, we prove that if $G \in \langle A \rangle$, then $H \in \langle A \rangle$. □

**Lemma 4.3** *Each of the following holds.*
*(i) Let $G$ be a tree. Then $\Lambda_g(G \times K_2) = 5$.*
*(ii) Let $m \geq 2, n \geq 2$. Then $\Lambda_g(K_{1,m} \times K_{1,n}) = 5$.*

**Proof:** (i) If we can prove that $G \times K_2$ is not $Z_4$-connected, then, by (4), $\Lambda_g(G \times K_2) = 5$. We will prove by induction on $|V(G)|$ that $G \times K_2$ is not $Z_4$-connected.

If $|V(G)| = 2$, then $G \cong K_2$, and so $G \times K_2 \cong C_4$. By Proposition 2.4, $G \times K_2$ is not $Z_4$-connected. Assume that for smaller values of $|V(G)|$, if $G$ is a tree, then $G \times K_2$ is not $Z_4$-connected. Assume now $|V(G)| \geq 3$. Since $G$ is a tree, $G$ has an edge $uv$ such that $u$ has degree 1 in $G$. It follows by assumption that $(G-u) \times K_2$ is not $Z_4$-connected. By the definition of Cartesian product, $G \times K_2$ is a parallel connection of $(G-u) \times K_2$ and $G[\{u,v\}] \times K_2 \cong C_4$. It follows by Lemma 4.2 that $G \times K_2$ is not $Z_4$-connected. This completes the proof of (i).

(ii) By (4), if we can prove that $K_{1,m} \times K_{1,n}$ is not $Z_4$-connected, then $\Lambda_g(K_{1,m} \times K_{1,n}) = 5$. By contradiction, we assume that $K_{1,m} \times K_{1,n}$ is $Z_4$-connected.

Suppose $V(K_{1,m}) = \{x_0, x_1, ..., x_m\}$ with $d_{K_{1,m}}(x_0) = m$ and $V(K_{1,n}) = \{y_0, y_1, ..., y_n\}$ with $d_{K_{1,n}}(y_0) = n$. Let $I = \{1, 2, ..., m\}$, $I_0 = \{0, 1, 2, ..., m\}$, $J = \{1, 2, ..., n\}$ and $J_0 = \{0, 1, 2, ..., n\}$. By the definition of Cartesian product, $V(K_{1,m} \times K_{1,n}) = \{v_{ij} = x_i y_j : \text{for } i \in I_0 \text{ and } j \in J_0\}$. Let $E_1 = \{v_{0j}v_{ij} : \text{for } i \in I,\ j \in J_0\}$ and $E_2 = \{v_{i0}v_{ij} : \text{for } i \in I_0,\ j \in J\}$. Then $E(K_{1,m} \times K_{1,n}) = E_1 \bigcup E_2$ and $v_{ij}$ has degree 2, for $i \in I$ and $j \in J$.

Let $D$ be an orientation of $K_{1,m} \times K_{1,n}$ such that $v_{0j}v_{ij} \in E_1$ is directed from $v_{0j}$ to $v_{ij}$; $v_{i0}v_{ij} \in E_2$ is directed from $v_{ij}$ to $v_{i0}$(see $K_{1,3} \times K_{1,3}$ in Figure 3).

Let $b : V(K_{1,m} \times K_{1,n}) \mapsto Z_4$ such that

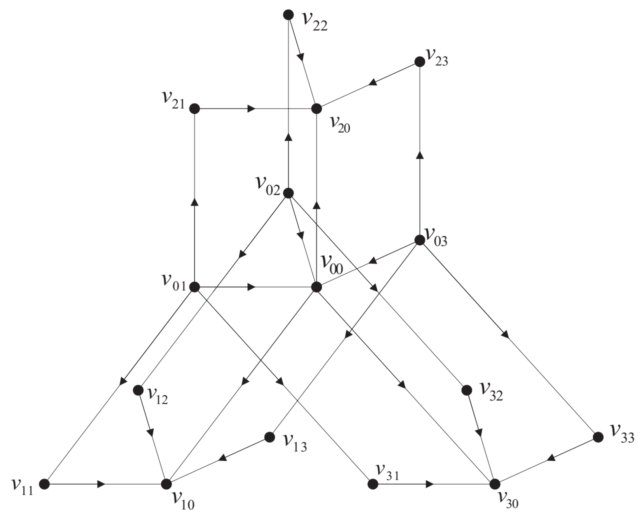



**Figure** 3 $K_{1,3} \times K_{1,3}$

$$b(v) = \begin{cases} 1, & \text{if } v = v_{01}, \\ 3, & \text{if } v = v_{10}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $b \in Z(K_{1,m} \times K_{1,n}, Z_4)$.
Let $\overline{f} : K_{1,m} \times K_{1,n} \mapsto Z_4$ such that

$$\overline{f}(e) = \begin{cases} 1, & \text{if } e = v_{01}v_{00}, v_{ij}v_{i0}, \text{ for } i \in I, j \in J, \\ 3, & \text{if } e = v_{10}v_{00}, v_{ij}v_{0j}, \text{ for } i \in I, j \in J, \\ 2, & \text{otherwise.} \end{cases}$$

By Proposition 2.1, there is an $f \in F(K_{1,m} \times K_{1,n}, Z_4)$ such that $\partial f = b$, and $\overline{f}(e) \neq f(e)$, for any $e \in E(K_{1,m} \times K_{1,n})$. For $v_{ij}$, where $i \in I$ and $j \in J$, since $b(v_{ij}) = f(v_{ij}v_{i0}) - f(v_{ij}v_{0j}) = 0$, $f(v_{ij}v_{i0}) = f(v_{ij}v_{0j})$. Together with $f(v_{ij}v_{i0}) \neq \overline{f}(v_{ij}v_{i0}) = 1$ and $f(v_{ij}v_{0j}) \neq \overline{f}(v_{ij}v_{i0}) = 3$, we have

$$f(v_{ij}v_{i0}), f(v_{ij}v_{0j}) \in \{0, 2\}, \text{ for } i \in I \text{ and } j \in J. \tag{5}$$

For vertex $v_{i0}$, $i = 2, ..., m$, since $b(v_{i0}) = 0 = -\sum_{j=1}^{n} f(v_{ij}v_{i0}) - f(v_{i0}v_{00})$, $f(v_{i0}v_{00}) = -\sum_{j=1}^{n} f(v_{ij}v_{i0})$. By (5), $f(v_{i0}v_{00}) \in \{0, 2\}$, and since $f(v_{i0}v_{00}) \neq \overline{f}(v_{i0}v_{00}) = 2$, $f(v_{i0}v_{i0}) = 0$, for $i = 2, ..., m$. By the similar argument, $f(v_{00}v_{0j}) = 0$, for $j = 2, ...., n$. That is

$$f(v_{i0}v_{00}) = 0, \text{ for } i = 2, ..., m; \; f(v_{00}v_{0j}) = 0, \text{ for} j = 2, ...., n. \tag{6}$$

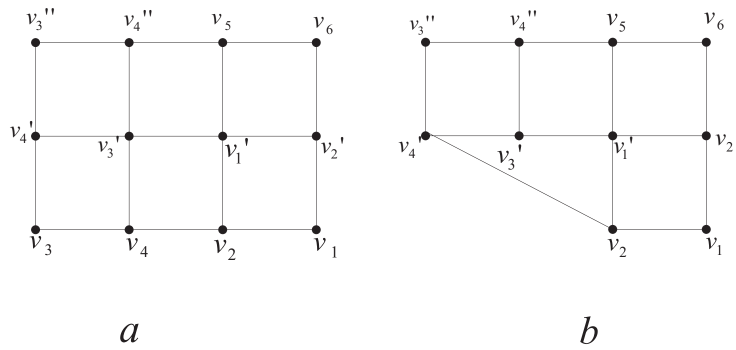



**Figure 4**

Since $b(v_{00}) = 0$, by (6), $f(v_{10}v_{00}) = f(v_{01}v_{00})$. And $f(v_{10}v_{00}) \neq \overline{f}(v_{10}v_{00}) = 3$, $f(v_{01}v_{00}) \neq \overline{f}(v_{01}v_{00}) = 1$, so $f(v_{10}v_{00}), f(v_{01}v_{00}) \in \{0, 2\}$. For vertex $v_{10}$, $b(v_{10}) = 3 = -\sum_{j=1}^{n} f(v_{1j}v_{10}) - f(v_{10}v_{00})$. But by (5) and (6), $-\sum_{j=1}^{3} f(v_{1j}v_{10}) - f(v_{10}v_{00}) \in \{0, 2\}$. This is a contradiction. Therefore $K_{1,m} \times K_{1,n}$ is not $Z_4$-connected.

Thus by (4), $\Lambda_g(K_{1,m} \times K_{1,n}) = 5$. □

**Lemma 4.4** *Each of the following holds.*
*(i)*$\Lambda_g(P_2 \times P_3) \leq 4$.
*(ii)Let* $n \geq 3$. *Then* $\Lambda_g(C_n \times K_2) \leq 4$.
*(iii)If one of* $G_1$ *and* $G_2$ *is not a tree, then* $\Lambda_g(G_1 \times G_2) \leq 4$.

**Proof:** (i) We label most of the vertices of $P_2 \times P_3$ as in Figure 4 *a*. Let $\pi_1 = \langle \{v_3, v_3'\}, \{v_4, v_4'\} \rangle$ and $H_1 = (P_2 \times P_3)/\pi_1$ (see Figure 4 *b*). Let $\pi_2 = \langle \{v_1, v_1'\}, \{v_2, v_2'\} \rangle$ and $H_2 = H_1/\pi_2$ (see Figure 5 *a*). Let $\pi_3 = \langle \{v_3', v_3''\}, \{v_4', v_4''\} \rangle$ and $H_3 = H_2/\pi_3$ (see Figure 5 *b*). If we redraw $H_3$(see Figure 5 c), then $H_3 \cong K_{3,3} - e$. By Theorem 2.6(iii), $H_3 \in \mathcal{CL}$. It follows by Theorem 2.7 that $H_2 \in \mathcal{CL}$. Similarly by Theorem 2.7, $H_1 \in \mathcal{CL}$ and $P_2 \times P_3 \in \mathcal{CL}$. Then by Theorem 2.6(iv) and (4), $\Lambda_g(P_2 \times P_3) \leq 4$.

(ii) First we will prove by induction on $n$ that $C_n \times K_2 \in \mathcal{CL}$. When $n = 3$, by Proposition 2.6(iii), $C_3 \in \mathcal{CL}$, and by Lemma 4.1(iii), $C_3 \times K_2 \in \mathcal{CL}$. When $n = 4$, let $C = v_1v_2v_3v_4$ be a 4-circuit in $C_4 \times K_2$ (see Figure 6 *a*). Let $\pi = \langle \{v_1, v_3\}, \{v_2, v_4\} \rangle$ and $G' = (C_4 \times K_2)/\pi$ (see Figure 6 *b*). If we redraw $G'$ (see Figure 6 *c*), then $G' \cong K_{3,3}$. It follows by Proposition 2.6(iii) that $G'$ is collapsible. Therefor by Lemma 2.7, $C_4 \times K_2 \in \mathcal{CL}$.

For a fixed $n > 4$, assume that for any $m < n$, $C_m \times K_2 \in \mathcal{CL}$. Let $C = v_1v_2v_1'v_2'$ and $C' = v_1'v_2v_1''v_2''$ be two 4-circuits contained in $C_n \times K_2$ (see Figure 7 *a*). Let $\pi_1 = \langle \{v_1, v_1'\}, \{v_2, v_2'\} \rangle$ and $\pi_2 = \langle \{v_1, v_1''\}, \{v_2, v_2''\} \rangle$. Let $G' = (C_n \times K_2)/\pi_1$(see Figure 7 a) and $G'' = (C_n \times K_2)/\pi_1/\pi_2$ (see Figure 7

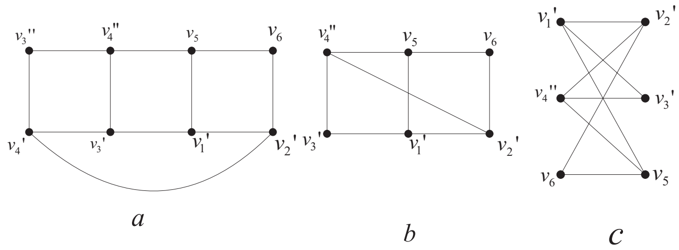


**Figure 5**

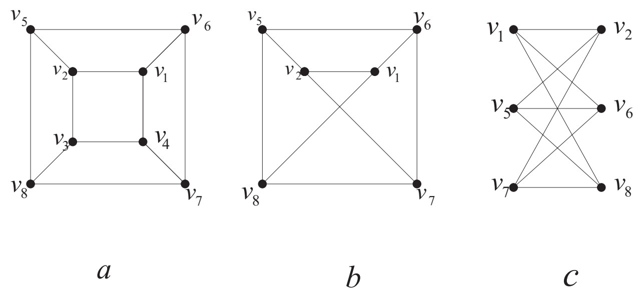


**Figure 6**

b). Then by assumption $(C_n \times K_2)/\pi_1/\pi_2 \cong C_{n-2} \times K_2 \in \mathcal{CL}$ (see Figure 7 $c$). By Lemma 2.7, $(C_n \times K_2)/\pi_1 \in \mathcal{CL}$. And by Lemma 2.7 again, $(C_n \times K_2) \in \mathcal{CL}$. Thus $C_n \times K_2 \in \mathcal{CL}$, for $n \geq 3$. It follows by (4) and Theorem 2.6 (iv) that $\Lambda_g(C_n \times K_2) \leq 4$, for $n \geq 3$.

(iii) Suppose $G_1$ is not a tree, then there is a circuit $C_n \subseteq G_1$, where $n \geq 3$. Therefor $G_1 \times G_2$ contains a nontrivial collapsible subgraph $H \cong C_n \times K_2$. It follows by Theorem 4.1(iii) that $\Lambda_g(G_1 \times G_2) \leq 4$. □

**Proof of Theorem 1.8:** By (4) and by Lemma 4.3, we may assume that $\Lambda_g(G_1 \otimes G_2) = 5$ to prove that either $G_1 \cong K_{1,m}$ and $G_2 \cong K_{1,n}$, where $n, m \geq 2$ or $G_1$ is a tree and $G_2 \cong K_2$.

If $G_1$ or $G_2$ has a circuit, then by Lemma 4.4(iii), $\Lambda_g(G_1 \otimes G_2) \leq 4$. Hence we may assume that both $G_1$ and $G_2$ are trees.

Case 1: If $\min\{|V(G_1)|, |V(G_2)|\} = 2$, assume $V(G_2) = 2$. Since $G_2$ is connected, $G_2 \cong K_2$, by Lemma 4.3(ii), $\Lambda_g(G_1 \times G_2) = 5$.

Case 2: If $\min\{|V(G_1)|, |V(G_2)|\} \geq 3$, then since $G_1$ and $G_2$ are connected, both $G_1$ and $G_2$ contain a path of length 2. If one of $G_1$ and $G_2$ contains a path of length 3, then it follows by Lemma 4.4(i) that $G_1 \otimes G_2$ has a nontrivial subgraph $H \cong P_2 \times P_3$ with $\Lambda_g(H) \leq 4$, and so by (1) and by Theorem 4.1(iii), $\Lambda_g(G_1 \otimes G_2) \leq 4$. Therefor $G_1$ and $G_2$ contains only paths with length 2. So

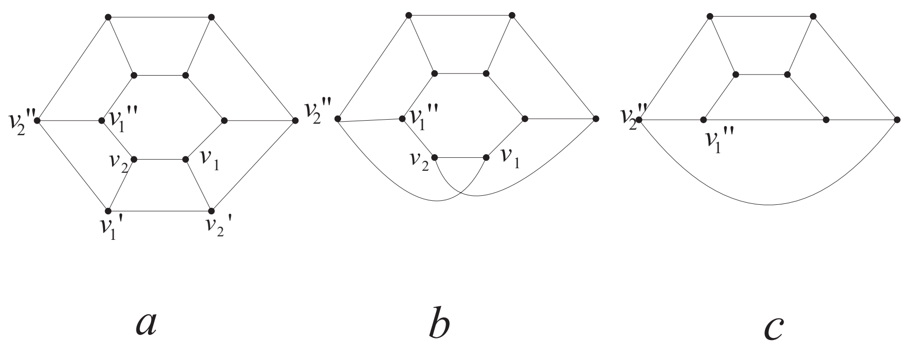



**Figure** 7

$G_1 \cong K_{1,m}$ and $G_2 \cong K_{1,n}$, for $m, n \geq 2$. □

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