# Every 3-Connected Claw-Free $Z_8$-Free Graph Is Hamiltonian

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**Abstract:** In this article, we first show that every 3-edge-connected graph with circumference at most 8 is supereulerian, which is then applied to show that a 3-connected claw-free graph without $Z_8$ as an induced subgraph is Hamiltonian, where $Z_8$ denotes the graph derived from identifying one end vertex of $P_9$ (a path with 9 vertices) with one vertex of a triangle. The above two results are both best possible in a sense that the number 8 cannot be replaced by 9 and they also extend former results by Brousek *et al.* in (Discrete Math 196 (1999), 29–50) and by Łuczak and Pfender in (J Graph Theory 47 (2004), 111–121). © 2009 Wiley Periodicals, Inc. J Graph Theory 64: 1–11, 2010

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## 1. INTRODUCTION

We use $P_k$ to denote a path of order $k$ and $Z_k$ to denote a graph obtained from the disjoint union of a $P_{k+1}$ and a 3-cycle $K_3$ by identifying one end vertex of $P_{k+1}$ with one vertex of $K_3$. A graph $G$ is $\{H_1, H_2, \ldots, H_s\}$-free if it contains no induced subgraphs that are isomorphic to a copy of $H_i$ for any $i$, where $H_i$ is connected for any $i$. A graph $G$ is called *claw-free* if it is $K_{1,3}$-free.

In 1999, Brousek, Ryjáček, and Favaron proved the following theorem.

**Theorem 1** (Brousek *et al.* [2]). *Let $G$ be a 3-connected claw-free graph. If $G$ is $Z_4$-free, then $G$ is Hamiltonian.*

The purpose of this paper is to extend Theorem 1. We have the following theorem as a main result in this paper.

**Theorem 2.** *Let $G$ be a 3-connected simple claw-free graph. If $G$ is $Z_8$-free, then $G$ is Hamiltonian.*

At the end of Section 4, we give an example of a 3-connected claw-free non-Hamiltonian $Z_9$-free graph. In this sense, Theorem 2 is best possible.

Theorem 2 also has an immediate consequence: every 3-connected claw-free $P_{10}$-free graph is Hamiltonian since a $P_{10}$-free graph must be $Z_8$-free. In fact, it is a slightly weaker case of Theorem 3. In Section 5, we shall reprove Theorem 3 with the help of our results.

**Theorem 3** (Łuczak and Pfender [7]). *Every 3-connected $\{K_{1,3}, P_{11}\}$-free graph is Hamiltonian.*

A graph is called *Eulerian* if it is connected and every vertex has an even degree. Note that the graph $K_1$ is also Eulerian. An Eulerian subgraph $C$ of $G$ is called a *spanning Eulerian subgraph* of $G$ if $V(C)=V(G)$ and is called a *dominating Eulerian subgraph* of $G$ if $E(G-V(C))=\emptyset$. A graph is called *supereulerian* if it contains a spanning Eulerian subgraph. For a graph $G$ which contains at least one cycle, the *circumference* of $G$, denoted by $c(G)$, is the length of a longest cycle contained in $G$; and the *girth* of $G$, denoted by $g(G)$, is the length of a shortest cycle contained in $G$. In order to prove Theorem 2, we need the following associate result.

**Theorem 4.** *Let $G$ be a 3-edge-connected graph. If $c(G) \leq 8$, then $G$ is supereulerian.*

The above result is also best possible since the Petersen graph has circumference 9, but it is not supereulerian. More generally, we let $G$ be a 3-edge-connected graph in which every block is a Petersen graph. Since every cycle of $G$ must be inside a block of $G$, and since the longest cycle in the Petersen graph has length 9, we conclude that $c(G)=9$. On the other hand, a graph is supereulerian if and only if every block of it is supereulerian. Hence such a graph $G$ is not supereulerian. Since the example permits any finite number of blocks, it shows that the bound 8 in Theorem 4 is best possible even if we allow for a finite number of exceptions.

## 2. PRELIMINARIES

We follow [1] for terminology and notation not defined here. In particular, we use $\kappa(G)$ and $\kappa'(G)$ to denote connectivity and edge connectivity of $G$, respectively. The *line graph* of a graph $G$, denoted by $L(G)$, has $E(G)$ as its vertex set, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in $G$ are adjacent. By the length of a path we mean the number of its edges. A *subpath* of a path $P$ is a connected subgraph of $P$. If not otherwise stated subscripts of vertices in a cycle $C$ are taken modulo $|V(C)|$. For a cycle $C=u_1u_2\dots u_mu_1$ we use $C(u_i,u_j)$ to denote the path $u_iu_{i+1}\dots u_j$. We denote by $P(u,v)$ the subpath of $P$ with the first vertex $u$ and the last vertex $v$. For two sets $S$ and $T$, by $S\triangle T$ we denote the symmetric difference of $S$ and $T$.

### A. Reduction Methods for Supereulerian Graphs

Let $G$ be a graph and $X\subseteq E(G)$ be an edge subset. The *contraction* $G/X$ is the graph obtained from $G$ by identifying the two ends of each edge in $X$ and then deleting the resulting loops. We define $G/\emptyset=G$. If $K$ is a subgraph of $G$, then we write $G/K$ for $G/E(K)$. If $K$ is a connected subgraph of $G$, and if $v_K$ is the vertex in $G/K$ onto which $K$ is contracted, then $K$ is called the *preimage* of $v_K$, and is denoted by $PI(v_K)$. A vertex $v$ in a contraction of $G$ is *nontrivial* if $PI(v)$ has at least one edge.

For a graph $G$, let $O(G)$ denote the set of odd degree vertices in $G$. In [3] Catlin defined collapsible graphs. Given a subset $R\subseteq V(G)$ with $|R|$ even, a subgraph $\Gamma$ of $G$ is an *$R$-subgraph* if both $O(\Gamma)=R$ and $G-E(\Gamma)$ is connected. A graph $G$ is *collapsible* if for any even subset $R$ of $V(G)$, $G$ has an $R$-subgraph. Catlin showed in [3] that every vertex of $G$ lies in a unique maximal collapsible subgraph of $G$. The *reduction* of $G$, denoted by $G'$, is obtained from $G$ by contracting all maximal collapsible subgraphs of $G$. A graph is *reduced* if it is the reduction of some graph.

**Theorem 5** (Catlin [3]). *Let $G$ be a connected graph and let $H$ be a collapsible subgraph of $G$. Then each of the following holds*:

(a) *$G$ is collapsible if and only if $G/H$ is collapsible. In particular, $G$ is collapsible if and only if the reduction $G'$ is $K_1$*;
(b) *$G$ is reduced if and only if $G$ has no nontrivial collapsible subgraphs*;
(c) $g(G') \geq 4$ *and* $\delta(G') \leq 3$;
(d) *$G$ is supereulerian if and only if $G'$ is supereulerian.*

**Theorem 6** (Chen [5]). *If $G$ is a 3-edge-connected simple graph with at most 13 vertices, then either $G$ is supereulerian or $G$ is contractible to the Petersen graph.*

It is known that all complete graphs of order at least 3 are collapsible and any cycle of length at least 4 is not collapsible. If $G$ contains a 4-cycle $C = uvzwu$ with a partition $\pi = \langle \{u,z\}, \{v,w\} \rangle$, then we can follow Catlin [4] and define $G/\pi(C)$ to be the graph obtained from $G - E(C)$ by identifying $u$ and $z$ to form a new vertex $x$, identifying $v$ and $w$ to form a new vertex $y$, and adding an edge $e_\pi = xy$.

**Theorem 7** (Catlin [4]). *Let $G$ be a graph containing a 4-cycle $C$ and let $G/\pi(C)$ be defined as above. Each of the following holds*:

(a) *If $G/\pi(C)$ is collapsible, then $G$ is collapsible*;
(b) *If $G/\pi(C)$ has a spanning Eulerian subgraph, then $G$ has a spanning Eulerian subgraph, i.e., if $G/\pi(C)$ is supereulerian, then $G$ is supereulerian.*

Let $G$ be a graph such that $\kappa(L(G)) \geq 3$ and $L(G)$ is not complete. The *core* of the graph $G$, denoted by $G_0$, is obtained by contracting all pendant edges and contracting exactly one edge $xy$ or $yz$ for each path $P = xyz$ in $G$ with $d_G(y) = 2$, where $d(x), d(z) > 2$ since $\kappa(L(G)) \geq 3$. The remaining edge of $P$ will be referred to as a nontrivial edge in the contraction. Shao [9] proved Theorem 8(a)–(c). In a similar way as Theorem 8(c), one can prove Theorem 8(d).

**Theorem 8** (Łuczak and Pfender [7], Shao [9]). *Let $G_0$ be the core of graph $G$, then each of the following holds*:

(a) *$G_0$ is nontrivial and* $\delta(G_0) \geq \kappa'(G_0) \geq 3$;
(b) *$G_0$ is well defined*;
(c) *If $G_0$ has a spanning Eulerian subgraph, then $G$ has a dominating Eulerian subgraph*;
(d) *If $G_0$ has a dominating Eulerian subgraph containing all nontrivial vertices and both end vertices of each nontrivial edges, then $G$ has a dominating Eulerian subgraph.*

## B. Closure of Claw-Free Graphs

Obviously a Hamiltonian graph is supereulerian, but the reverse is not true. We also have that the line graph of a supereulerian graph is Hamiltonian by the following well-known theorem.

**Theorem 9** (Harary and Nash-Williams [6]). *Let G be a connected graph with at least* 3 *edges. The line graph* $L(G)$ *is Hamiltonian if and only if G has a dominating Eulerian subgraph.*

Ryjáček [8] introduced the *closure* of a graph, which works well in the class of claw-free graphs. A vertex $x \in V(G)$ is *locally connected* if the neighborhood of $x$ induces a connected subgraph in $G$. For $x \in V(G)$, the graph $G'_x$ obtained from $G$ by adding the edges $\{yz\colon\ y,z \in N(x)\ \&\ yz \notin E(G)\}$ is called the *local completion of G at x*. The closure of a claw-free graph $G$, denoted by $\mathrm{cl}(G)$, is obtained from $G$ by recursively performing local completions at any locally connected vertex with non-complete neighborhood, as long as it is possible. The following theorem translates claw-free graphs to line graphs when we consider the Hamiltonicity of claw-free graphs.

**Theorem 10** (Ryjáček [8]). *Let G be a claw-free graph. Then*

(a) $\mathrm{cl}(G)$ *is uniquely determined*;
(b) $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph*;
(c) *G is Hamiltonian if and only if* $\mathrm{cl}(G)$ *is Hamiltonian.*

**Theorem 11** (Brousek *et al.* [2]). *Let G be a claw-free graph. Then*

(a) *If G is* $Z_k$*-free*, *then* $\mathrm{cl}(G)$ *is also* $Z_k$*-free for any integer* $k \geq 1$;
(b) *If G is* $P_i$*-free*, *then* $\mathrm{cl}(G)$ *is also* $P_i$*-free for any integer* $i \geq 3$.

## 3. PROOF OF THEOREM 4

We argue by contradiction, and assume that $G$ is a counter-example to Theorem 4 with $|V(G)|$ minimized.

If $G$ is not reduced, then let $G'$ be the reduction of $G$. Since $G$ is not reduced and $G'$ is a contraction of $G$, $|V(G)| > |V(G')|$ and $\kappa'(G') \geq \kappa(G) \geq 3$. Since any cycle of $G'$ can be extended to a cycle of $G$, $c(G') \leq c(G) \leq 8$. By the minimality of $|V(G)|$, $G'$ is supereulerian. By Theorem 5(d), $G$ is supereulerian, contrary to the choice of $G$.

Hence we may assume that $G$ is reduced. If $G$ has a cut-vertex, then by the minimality of $G$, each block of $G$ is supereulerian, and so $G$ is supereulerian, contrary to the choice of $G$. Therefore, we assume that $\kappa(G) \geq 2$ and $G \neq K_1$.

By Theorem 5, $4 \leq g(G) \leq c(G) \leq 8$. If $g(G) = 4$, then we can assume that $H$ is a 4-cycle $x_1x_2y_1y_2x_1$ in $G$ with a partition $\pi = \langle \{x_1, y_1\}, \{x_2, y_2\} \rangle$. We obtain the following three facts to show some good properties of $G$:

**Claim 1.** $\kappa'(G/\pi(H)) \geq 3$.

***Proof of Claim 1.*** By way of contradiction, we assume $\kappa'(G/\pi(H)) \leq 2$. It suffices to distinguish the following two cases to obtain our required contradiction:

**Case 1.1.** $\kappa'(G/\pi(H)) = 1$.

Then $G - E(H)$ has two components $G_1$ and $G_2$ such that $x_i, y_i \in V(G_i)$. Let $P(x_i, y_i)$ be a longest path between $x_i$ and $y_i$ in $G_i$, then $P(x_i, y_i)$ has length at least 3.

Suppose, to the contrary, that $P(x_1,y_1)$ has length 2, say, $P(x_1,y_1)=x_1wy_1$. Since $\kappa'(G)\geq 3$, $G-\{wx_1,wy_1\}$ is still connected. So there exist a pair of paths $P(w,x_1)$ (between $w$ and $x_1$) and $P(w,y_1)$ (between $w$ and $y_1$) in $G-\{wx_1,wy_1\}$ such that either $y_1\notin V(P(w,x_1))$ or $x_1\notin V(P(w,y_1))$, say $y_1\notin V(P(w,x_1))$. Then $y_1wP(w,x_1)$ is a longer path than $P(x_1,y_1)$ in $G_1$, a contradiction. We claim that $P(x_i,y_i)$ has length exactly 3 since otherwise $P(x_1,y_1)y_1x_2P(x_2,y_2)y_2x_1$ is a cycle of length at least 9, a contradiction. Let $P(x_i,y_i)=x_iu_iv_iy_i$. Since $\delta(G)\geq\kappa'(G)\geq 3$, $d(v_1)\geq 3$ and $d(u_1)\geq 3$, and then there are two vertices $s_1$ and $t_1$ such that $s_1u_1$, $t_1v_1\in E(G)$. Since $g(G)=4$, $s_1\neq t_1$. By $\kappa(G)\geq 2$, $u_2v_2$ and each of $\{u_1s_1,v_1t_1\}$ must be in a cycle of length at most 8. Hence the assumption that $g(G)=4$ forces that $s_1y_1$, $t_1x_1\in E(G)$. But then $s_1u_1v_1t_1x_1x_2u_2v_2y_2y_1s_1$ is a 10-cycle, contrary to the assumption that $c(G)\leq 8$.

**Case 1.2.** $\kappa'(G/\pi(H))=2$.

Then $G-E(H)$ has a cut edge $e=z_1z_2$ and $G-(E(H)\cup\{e\})$ has two components $G_1$ and $G_2$ such that $x_i,y_i,z_i\in V(G_i)$. Since $\kappa'(G)\geq 3$ and $g(G)\geq 4$, $G_i$ has at least five vertices.

Let $W_i=\{x_i,y_i,z_i\}$ and suppose $u_i,v_i\in V(G_i)\setminus W_i$ for $i=1,2$.

If every element in the vertex set $V(G_i)\setminus W_i$ has neighbors which are all in $W_i$, then $v_1z_1z_2v_2y_2u_2x_2x_1u_1y_1v_1$ is a 10-cycle of $G$ since $\kappa'(G)\geq 3$, contrary to the assumption that $c(G)\leq 8$.

Suppose only one of $\{G_1,G_2\}$, say $G_1$, has the property that all vertices in $V(G_1)\setminus W_1$ have neighbors only in $W_1$, i.e., there is at least one vertex $u_2$ of $V(G_2)\setminus W_2$ which has a neighbor $v_2\notin W_2$. Since $\kappa(G)\geq 2$, there is a cycle $C_1$ containing two edges $u_2v_2$ and $z_1z_2$. The fact that $z_1z_2$ is a cut edge of $G-E(H)$ implies that $C_1$ contains exactly one element of $W_i\setminus\{z_i\}$. By the supposition of $G_1$, we can take two vertices $u_1,v_1\in V(G_1)\setminus W_1$ such that they are adjacent to every element of $W_1$ and hence there is a cycle of length at least 9, (for the shortest case) say $z_1u_1x_1y_2y_1x_2u_2v_2z_2z_1$, in $G[V(C_1)\cup V(H)\cup\{u_1,v_1,u_2,v_2\}]$, contrary to the assumption that $c(G)\leq 8$.

It remains the case when there are two edges $u_1v_1$ and $u_2v_2$ in $G_1$ and $G_2$, respectively, such that they are not incident with any element of $W_1\cup W_2$. Since $\kappa(G)\geq 2$, there is a cycle $C_1$ containing two edges $u_1v_1$, $u_2v_2$. Hence $|E(C_1)|=8$ by $g(G)\leq 8$. Note that $C_1$ contains exactly two elements of $W_i$. If $z_1z_2\in E(C_1)$, then $G[E(C_1)\triangle E(x_1x_2y_1y_2)]$ is a cycle of length at least 9, a contradiction. Now suppose that $z_1z_2\notin E(C_1)$. Then there is a cycle $C_2$ containing $z_1z_2$ and $u_1v_1$. Hence $G[V(C_1)\cup V(C_2)]$ contains a cycle of length at least 9, contrary to the assumption that $c(G)\leq 8$. This completes the proof of Claim 1. ■

**Claim 2.** $g(G)\geq 5$.

***Proof of Claim 2.*** Assume, in contrast, that $G$ has a 4-cycle $C$, then by Claim 1, $\kappa'(G/\pi(C))\geq 3$. By the definition of $G/\pi(C)$, any cycle of $G/\pi(C)$ can be (possibly trivially) extended to a cycle of $G$, and so $c(G/\pi(C))\leq c(G)\leq 8$. By the minimality of $|V(G)|$, $G/\pi(C)$ is supereulerian. Then by Theorem 7(b), $G$ is also supereulerian, contrary to the choice of $G$. This completes the proof of Claim 2.

By Claims 1 and 2, we can assume that $g(G) \geq 5$, $\kappa'(G) \geq 3$ and $\kappa(G) \geq 2$. Hence $\delta(G) \geq \kappa'(G) \geq 3$.

Now take a longest path $x_1x_2 \dots x_l$ in $G$ and note that the end vertices have neighbors only on the path. As $\delta(G) \geq 3$, $x_1$ has at least three such neighbors. As $g(G) \geq 5$, $N(x_1) = \{x_2, x_5, x_8\}$; otherwise, there is a cycle of length at least 9, a contradiction. Using the alternative longest path $x_4x_3x_2x_1x_5x_6 \dots x_l$, we get $x_4x_8 \in E(G)$ by the same argument, yielding a $C_4 = x_1x_5x_4x_8x_1$, contrary to Claim 2. This completes the proof of Theorem 4. ■

## 4. PROOF OF THEOREM 2

By Theorems 10 and 11, Theorem 2 can be equivalently expressed as: *If G is a 3-connected $Z_8$-free line graph, then G is Hamiltonian.*

A $Y_m$ in $G$ is any subgraph of $G$ isomorphic to the (unique) tree $Y$ on $m+2$ vertices with exactly 3 leaves such that the unique vertex of degree 3 in $Y_m$ is adjacent to two of the three leaves, where *a leaf* of a tree is defined to be a vertex of degree one. Note that the tree is shaped like the letter $Y$. Then $L(G)$ is $Z_8$-free if and only if $G$ is a connected simple graph without subgraphs isomorphic to $Y_{10}$. To describe a $Y_m$, we only list its edges.

Therefore, to show Theorem 2, it suffices to prove the following theorem.

**Theorem 12.** *Let G be a connected simple graph without subgraphs isomorphic to $Y_{10}$. If $\kappa(L(G)) \geq 3$, then $L(G)$ is Hamiltonian.*

Before presenting the proof of Theorem 12, we need the following two lemmas, which will also be used to reprove Theorem 3 in the last section.

**Lemma 13.** *Let G be a reduced graph with $\kappa'(G) \geq 3$, $g(G) \geq 4$ and $c(G) \geq 9$, and let $C = u_1u_2 \dots u_{c(G)}$ be a longest cycle of G. If G contains no subgraphs isomorphic to $Y_{10}$ and $P_{12}$, then any vertex in $V(G) \setminus V(C)$ has at most one neighbor not on C and hence has at least two neighbors on C since $\kappa'(G) \geq 3$.*

***Proof of Lemma 13.*** Suppose there is a vertex $w$ in $V(G) \setminus V(C)$ which has at least two neighbors not on $C$. Let $P(w, u_i)$ be a shortest path between $w$ and $C$ with $u_i \in V(C)$. Then we can take two neighbors, $x, y$ (say), of $w$, such that $x$ and $y$ are not in $V(P(w, u_i)) \cup V(C)$ since $d(w) \geq \kappa'(G_0) \geq 3$. Hence

$$\{xw, yw\} \cup E(P(w, u_i)) \cup E(C(u_i, u_{i+9})) \text{ is a } Y_{9+|E(P(w,u_i))|} \tag{4.1}$$

Let $P(x, y, w)$ be a longest path containing each of $\{x, y, w\}$ such that it contains exactly one vertex of $C$ which is the last vertex of this path. Then $P(x, y, w)$ has length at least 3 by $g(G) \geq 4$. Without loss of generality, we assume $u_1 \in V(P(x, y, w)) \cap V(C)$. Hence

$$P(x, y, w)C(u_1, u_9) \text{ is } P_{9+|E(P(x,y,w))|} \tag{4.2}$$

(4.1) and (4.2) contradict the hypothesis of Lemma 13. This completes the proof of Lemma 13. ■

In the proof of Lemma 14, we use the following additional notation. For the reduction $G'$ of a connected graph $G$, let $\Lambda(G')=\{v\in V(G'): v$ is nontrivial or an end vertex of a nontrivial edge of $G'$".

**Lemma 14.** *Let G be a connected simple graph without subgraphs isomorphic to $Y_{10}$ and $P_{12}$, and $G_0$ be the core of G. If $G_0$ is the reduced graph with $\kappa'(G_0)\geq 3$, $g(G_0)\geq 4$ and $c(G_0)=9$, then $G_0$ has a dominating closed trail H such that $\Lambda(G_0)\subseteq V(H)$.*

***Proof of Lemma 14.*** We argue by contradiction, and assume that $G_0$ is a counterexample to Lemma 14.

Let $C=u_1u_2\cdots u_9u_1$ be a 9-cycle that contains as many vertices of $\Lambda(G_0)$ as possible. We distinguish the following two cases to obtain our desired contradiction:

**Case 1.** $C$ is not dominating.

We can take an edge $xy\in E(G_0)$ such that $x,y\notin V(C)$. By Lemma 13 and $d_{G_0}(x)$, $d_{G_0}(y)\geq 3$, both $x$ and $y$ have at least two neighbors on $C$. Noticing the fact that $4\leq g(G_0)\leq c(G_0)=9$, we claim that $d_C(s,t)\geq 3$ for any pair of vertices $s\in N_{G_0}(x)\cap V(C)$ and $t\in N_{G_0}(y)\cap V(C)$, where $d_C(s,t)$ means the length of the shortest path on $C$ between $s$ and $t$. By this claim and the fact that both $x$ and $y$ have at least two neighbors on $C$, we obtain that there is either a triangle or a cycle of length at least 10, a contradiction.

**Case 2.** $C$ is a dominating closed trail but it does not contain all elements of $\Lambda(G_0)$. There exists at least one vertex in $\Lambda(G_0)\setminus V(C)$. Since $C$ is dominating in $G_0$, we only need to count the number of vertices not in $C$.

Suppose $|V(G_0)|\leq 13$, i.e., $|V(G_0)-V(C)|\leq 4$. Hence by Theorem 6, $G_0$ is supereulerian or the Petersen graph. If $G_0$ is supereulerian, then obviously Lemma 14 holds. So we only need to consider the case when $G_0$ is the Petersen graph, then all the ten vertices in $G_0$ must be in $\Lambda(G_0)$ since otherwise there is a dominating cycle of length 9 containing all such kind of vertices. Let $x_1x_2\dots x_{10}$ be a longest path of $G_0$. For each $i\in\{1,2,\dots,10\}$, if $x_i$ is nontrivial, then $x_i$ has at least a neighbor $v_i$ in the original graph $G$; if $x_i$ is one end vertex of a nontrivial edge of $G_0$, then there is a path of length 2 whose internal vertex $v_i$ has degree two in the original graph $G$. Then $\{x_2v_2,x_{10}v_{10}\}\cup E(x_1x_2\dots x_{10})$ is a $Y_{10}$ and $v_1x_1x_2\dots x_{10}v_{10}$ is a $P_{12}$ in $G$, which is a contradiction.

It remains the case that $|V(G_0)|\geq 14$, i.e., there are at least 5 vertices not on $C$. Let $w\in\Lambda(G_0)\setminus V(C)$. Let $w'$ be the neighbor of $w$ in the original graph $G$. Since $C$ is dominating in $G_0$ and $\kappa'(G_0)\geq 3$, there must be a neighbor $u_1$ (say) of $w$ on $C$ such that there is a vertex $u_8$ (say) of $C$ with distance two from $u_1$ on $C$ and $u_8$ has a neighbor $v$ (say) not on $C$. Let $w'$ be the neighbor of $w$ in the original graph $G$. Then

$$\{vu_8,ww',wu_1\}\cup E(C(u_1,u_9)) \text{ is a } Y_{10} \text{ in } G. \tag{4.3}$$

Similarly, there must be a neighbor $u_1$ (say) of $w$ on $C$ such that there is a vertex $u_9$ (say) of $C$ with distance one from $u_1$ on $C$ and $u_9$ has a neighbor $v_9$ (say) not on $C$. Then

$$w'wu_1C(u_1,u_9)v_9 \text{ is a } P_{12} \tag{4.4}$$

(4.3) and (4.4) contradict the hypothesis of Lemma 14, which completes the proof of Lemma 14. ■

Now we present the proof of Theorem 12.

***Proof of Theorem 12.*** Let $G_0$ be the core of $G$. To show $L(G)$ is Hamiltonian, by Theorems 8 and 9, it suffices to show the following claim:

**Claim 3.** $G_0$ has a dominating closed trail $H$ such that $\Lambda(G_0) \subseteq V(H)$.

***Proof of Claim 3.*** We argue by contradiction, and assume that $G_0$ is a counterexample to Claim 3 with $|V(G_0)|$ minimized.

If $G_0$ is not reduced, then let $G_0'$ be the reduction of $G_0$. Hence $|V(G_0)| > |V(G_0')|$. By the minimality of $|V(G_0)|$, $G_0'$ has a dominating closed trail $H'$ such that $V(H')$ contains all nontrivial vertices and end vertices of all nontrivial edges of $G_0'$. Let $W_1$ be the set of all nontrivial vertices obtained by contracting $G_0$ to $G_0'$ and $W_2$ be the set of all nontrivial vertices and end vertices of all nontrivial edges obtained by contracting $G$ to $G_0$. Then $W_2 = \Lambda(G_0)$. Note that if $w \in W_2$ then there is either a neighbor of $w$ in the original graph $G$ if $w$ is a nontrivial vertex, or a vertex with degree two such that it is in a path containing $w$ if $w$ is an end vertex of a nontrivial edge of $G_0$.

For any vertex $v'$ in $W_1$, we have that $PI(v')$ is collapsible. Let $S = \{z \in V(PI(v')) : z$ is incident with an odd number of edges in $E(H')\}$. Then $|S| \equiv 0 \pmod 2$ since $H'$ is Eulerian. Hence $|S \triangle O(PI(v'))|$ is even. Since $PI(v')$ is collapsible, there exists a spanning connected subgraph $T_{v'} \subseteq PI(v')$ such that $O(T_{v'}) = S \triangle O(PI(v'))$. Let $H = G[E(H') \cup \{\cup E(T_{v'}) : v' \in W_1\}]$. Then $H$ is a dominating closed trail of $G_0$ that contains every element of $W_2$, a contradiction.

Hence we suppose that $G_0$ is reduced. By Theorem 8, we have $\delta(G_0) \geq \kappa'(G_0) \geq 3$. By Lemma 14, it suffices to consider the case that $c(G_0) \geq 10$ since otherwise $G_0$ itself is supereulerian by Theorem 4 which contradicts our assumption.

Let $C = u_1 u_2 \cdots u_{c(G_0)} u_1$ be a longest cycle of $G_0$. If $c(G_0) \geq 11$, then $V(G_0) \setminus V(C) \neq \emptyset$ since otherwise $C$ is a spanning closed trail of $G_0$, a contradiction. Hence we can take a vertex $u \in V(G_0) \setminus V(C)$ such that $u$ is adjacent to one vertex of $C$, say $u_1$, then $\{u_1 u, u_1 u_{c(G_0)}\} \cup E(C(u_1, u_{10}))$ is a $Y_{10}$, which contradicts the assumption of Theorem 12.

The only case left is $c(G_0) = 10$. Theorem 6 implies $|V(G_0)| \geq 14$, i.e., $|V(G_0) \setminus V(C)| \geq 4$. If there are two vertices $x, y$ of $V(G_0) \setminus V(C)$ such that they have the same neighbor $u_1$ (say) on $C$, then $\{xu_1, yu_1\} \cup E(u_1, u_{10})$ is a $Y_{10}$, a contradiction; in the case that every vertex of $V(G_0) \setminus V(C)$ has a different neighbor on $C$ than others, by the fact $|V(G_0) \setminus V(C)| \geq 4$ and Lemma 13, we can find two vertices $x, y$ of $V(G_0) \setminus V(C)$ such that they have neighbors $u_2$ and $u_{10}$ (say) with distance two on $C$, respectively, since $\delta(G_0) \geq \kappa'(G_0) \geq 3$ and $G_0$ is triangle-free. Then $\{xu_2, yu_{10}\} \cup E(C(u_1, u_{10}))$ is a $Y_{10}$, a contradiction.

This completes the proof of Claim 3 and (hence) of Theorem 12. ■

To show the sharpness of Theorem 2, we let $H$ be the graph obtained from the Petersen graph $P$ by adding exactly one pendant edge to every vertex of $P$. Then $L(H)$ is a 3-connected $\{K_{1,3}, Z_9\}$-free graph. However, $L(H)$ is non-Hamiltonian.

## 5. CONCLUDING REMARK

In the last paragraph of Section 4 we presented an extremal graph to show the sharpness of Theorem 12. However it has only 20 vertices. We believe it is the unique non-Hamiltonian 3-connected $\{K_{1,3}, Z_9\}$-free graph. Hence we propose the following conjecture which implies Theorem 3 since a $P_{11}$-free graph must be $Z_9$-free and since the line graph $L(H)$ of a graph $H$ defined in Conjecture 15 is $P_{11}$-free.

**Conjecture 15.** *If $G$ is 3-connected and $\{K_{1,3}, Z_9\}$-free, then $G$ is Hamiltonian unless $G$ is the line graph of $H$ defined in the last paragraph of Section 4.*

To show the sharpness of Conjecture 15, we let $F$ be a graph by adding some pendant edges to every vertex of the Petersen graph where at least one vertex of the Petersen graph is incident to at least 2 pendant edges. Then $\kappa(L(F)) \geq 3$ and $L(F)$ is claw-free and $Z_{10}$-free. Moreover, $L(F)$ has a $Z_9$ and $L(F)$ is non-Hamiltonian.

Although we cannot prove Conjecture 15 at this moment, we will prove Theorem 3 in a similar way to the proof of Theorem 2. Note that $L(G)$ is $P_{11}$-free if and only if $G$ has no subgraph isomorphic to $P_{12}$. Hence in order to prove Theorem 3, it suffices to prove the following result by Theorems 10 and 11.

**Theorem 16.** *Let $G$ be a connected simple graph without subgraphs isomorphic to $P_{12}$. If $\kappa(L(G)) \geq 3$, then $L(G)$ is Hamiltonian.*

***Proof of Theorem 16.*** Let $G_0$ be the core of $G$. To show $L(G)$ is Hamiltonian, by Theorems 8 and 9, it suffices to show the following claim:

**Claim 4.** $G_0$ has a dominating closed trail $H$ such that $V(H)$ contains all nontrivial vertices and end vertices of all nontrivial edges of $G_0$.

We argue by contradiction, and assume that $G_0$ is a counter-example to Claim 4 with $|V(G_0)|$ minimized.

From the proof of Theorem 12, we can suppose that $G_0$ is reduced, $\kappa'(G_0) \geq 3$ and $c(G_0) \geq 9$. It suffices to consider the case that $c(G_0) \geq 10$ by Lemma 14. Note that $G_0$ has no spanning closed trails by its choice.

If $c(G_0) \geq 11$, then $G_0$ has a path $P_{12}$ since $G_0$ has at least one vertex not on the longest cycle.

So we can suppose that $c(G_0) = 10$. Then every cycle of length 10 is dominating since otherwise $G_0$ has a $P_{12}$ obtained by the longest cycle and a path containing an undominated edge. Hence there is a vertex $w$ which is either a nontrivial vertex or one end vertex of a nontrivial edge of $G_0$. If $w$ is a nontrivial vertex, then $w$ has at least a neighbor $w'$ in the original graph $G$; if $w$ is one end vertex of a nontrivial edge of $G_0$, then there is a path of length 2 whose internal vertex $w'$ has degree two in the original graph $G$. In either case we obtain a path $P_{12}$ in the original graph $G$, a contradiction. This completes the proof of Claim 4 and (hence) of Theorem 16, which implies that every 3-connected $\{K_{1,3}, P_{11}\}$-free graph is Hamiltonian. Hence Theorem 3 is proved. ■

The graph $F$ also shows the sharpness of Theorem 16.

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