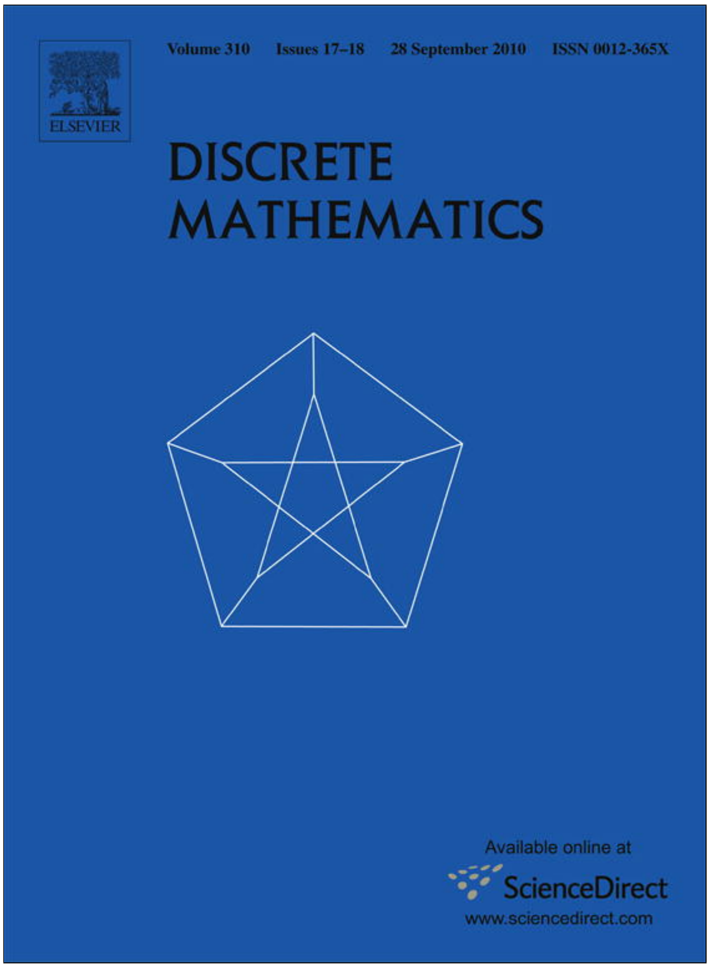
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Note

On 3-edge-connected supereulerian graphs in graph family $C(l, k)$[☆]

Xiaomin Li[a,*], Dengxin Li[a], Hong-Jian Lai[b]

[a] Department of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, PR China
[b] Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA

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ABSTRACT

Let $l > 0$ and $k \geq 0$ be two integers. Denote by $C(l, k)$ the family of 2-edge-connected graphs such that a graph $G \in C(l, k)$ if and only if for every bond $S \subset E(G)$ with $|S| \leq 3$, each component of $G - S$ has order at least $(|V(G)| - k)/l$. In this paper we prove that if a 3-edge-connected graph $G \in C(12, 1)$, then G is supereulerian if and only if G cannot be contracted to the Petersen graph. Our result extends some results by Chen and by Niu and Xiong. Some applications are also discussed.

1. Introduction

We consider finite, undirected and loopless graphs, and follow Bondy and Murty [5] for undefined notations and terminology. For a graph G with a connected subgraph H, the *contraction* G/H is the graph obtained from G by contracting all edges of H, and by deleting any resulting loops. Let v_H denote the vertex in G/H to which H is contracted. H is called the *preimage* of v_H. Let $O(G)$ denote the set of all odd degree vertices of a graph G. A graph G is an *Eulerian graph* if it is a connected graph with $O(G) = \emptyset$. A graph is *supereulerian* if it has a spanning Eulerian subgraph. In particular, K_1 is both Eulerian and supereulerian. Following the notation of Catlin [11], we denoted by SL the family of all supereulerian graphs. Boesch, Suffel, and Tindell first proposed the characterization of supereulerian graphs in [4], and they suggested that this would be a difficult problem. Pulleyblank [22] indicated that determining if a planar graph with maximum degree 3 is supereulerian is NP-complete, confirming the suggestion of Boesch, et al. in [4].

A minimum edge-cut of a graph G is called a *bond* of G. For two integers $l > 0$ and $k \geq 0$, let $C(l, k)$ denote the family of 2-edge-connected graphs such that a graph $G \in C(l, k)$ if and only if for every bond $S \subset E(G)$ with $|S| \leq 3$, each component of $G - S$ has order at least $(|V(G)| - k)/l$. For a graph G, $\kappa(G)$ and $\kappa'(G)$ denote the connectivity and the edge-connectivity of graph G, respectively.

Catlin and Li [12] first investigated the characterization problem of supereulerian graphs in the family of $C(l, k)$. Later, Broersma and Xiong [6], Li et al. [18] continued the research and proved the following results concerning the characterization of supereulerian graphs in a certain family $C(l, k)$.

Theorem 1 (*Theorem 6 of [12]*)**.** *If $G \in C(5, 0)$, then $G \in SL$ if and only if G cannot be contracted to $K_{2,3}$.*

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* Corresponding author.
E-mail addresses: lxm@ctbu.edu.cn, lxm19701222@gmail.com, lxm_19701222@sohu.com (X. Li).

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Theorem 2 (*Theorem 7 of [6]*). *If $G \in C(5,2)$ and $|V(G)| \geq 13$, then $G \in SL$ if and only if G cannot be contracted to $K_{2,3}$ or $K_{2,5}$.*

Theorem 3 (*Theorem 1.3 of [18]*). *If $G \in C(6,0)$, then $G \in SL$ if and only if G cannot be contracted to $K_{2,3}$, $K_{2,5}$ or $K_{2,3}(e)$, where $K_{2,3}(e)$ is the graph obtained from $K_{2,3}$ by replacing an edge $e \in E(K_{2,3})$ with a path of length 2.*

Theorem 4 (*[19]*). *Let $G \in C(6,5)$ be a graph with $|V(G)| > 35$. Then $G \in SL$ if and only if G cannot be contracted to a member in $\{S(1,2), S(3,2), S(4,1), J(2,2), K_{2,3}, K_{2,5}\}$, where $S(i,j)$ is a graph obtained by identifying one vertex of degree i in $K_{2,i}$ with one vertex of degree j in $K_{2,j}$, and adding a new edge joining the other two vertices of degree i and j in $K_{2,i}$ and $K_{2,j}$ respectively. Picking one edge with two ends u_1 and u_2 in $K_{2,i+1}$ and w_1, w_2 being the two vertices of degree j in $K_{2,j}$, then the graph $J(i,j)$ is obtained by identifying u_1 with w_1 and adding a new edge joining u_2 and w_2.*

Let P_{10} denote the Petersen graph. When restricted to 3-edge-connected graphs, the four theorems above can be improved by the following stronger conclusions.

Theorem 5 (*Theorem 2 of [13]*). *Let G be a graph. If $G \in C(10,0)$ with $\kappa'(G) \geq 3$, then $G \in SL$ if and only if G cannot be contracted to P_{10}.*

Theorem 6 (*[21]*). *Let G be a graph. If $G \in C(10,k)$ with $\kappa'(G) \geq 3$, then when $|V(G)| > 11k$, $G \in SL$ if and only if G cannot be contracted to P_{10}.*

The main purpose of this note is to prove the following result which extends Theorems 5 and 6.

Theorem 7. *Let G be a graph. If $G \in C(12,1)$ with $\kappa'(G) \geq 3$, then $G \in SL$ if and only if G cannot be contracted to P_{10}.*

In Section 2 we will present Catlin's reduction method using collapsible graphs, and some associated results, which will be employed. The proof of the main theorem will be given in Section 3, and some applications of the main result can be found in Section 4.

2. Reduction method and some associated results

A graph G is *collapsible* if for every even subset $R \subset V(G)$, G has a spanning connected subgraph H_R such that $O(H_R) = R$. In particular, K_1 is both supereulerian and collapsible. Following the notation of Catlin [7], we denote the family of collapsible graphs by CL. Note that any collapsible graph G is supereulerian since $\emptyset$ is an even subset of $V(G)$. Therefore, we have $CL \subset SL$.

In [8], Catlin showed that every graph G has a unique collection of pairwise disjoint maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$. The contraction of G obtained from G by contracting each nontrivial maximal collapsible subgraphs of G into a single vertex, is called the *reduction* of G. A graph is *reduced* if it is the reduction of itself.

Throughout this paper, we say a graph G can be contracted to a graph G' means that G has G' as its reduction.

The following theorems are useful in the proof of our main result.

Theorem 8 (*Theorems 5 and 8 of [8]*). *Let G be a connected graph.*

(i) *Let G' be the reduction of G. Then $G \in SL$ if and only if $G' \in SL$, and $G \in CL$ if and only if $G' = K_1$.*
(ii) *If G is reduced, then every subgraph of G is reduced also.*

Let $F(G)$ denote the minimum number of extra edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees.

Theorem 9 (*Theorem 1.5 of [10]*). *Let G be a connected reduced graph. If $F(G) \leq 2$, then $G \in \{K_1, K_2, K_{2,t}(t \geq 1)\}$.*

Theorem 10 (*Theorem 7 of [9]*). *If G is a connected reduced graph, then $F(G) = 2|V(G)| - |E(G)| - 2$.*

Theorem 11 (*[14]*). *Let G be a connected graph with $|V(G)| \leq 13$ and $\delta(G) \geq 3$. Then either G is a supereulerian graph with 12 vertices and with an odd cycle, or the reduction of G is in $\{K_1, K_2, K_{1,2}, K_{1,3}, P_{10}\}$.*

For a graph G, an edge-cut $X \subset E(G)$ is called an *essential edge-cut*, if each component of $G - X$ has at least one edge.

Theorem 12 (*Lemma 9 of [20]*). *Let G be a 3-edge-connected nonsupereulerian and reduced graph with $F(G) = 3$. Then every edge-cut of size 3 is not an essential edge-cut (i.e. the number of edge-cut of size 3 is equal to the number of vertex of degree 3 in G).*

Theorem 13 (*Theorem 3 of [20]*). *Let G be a 3-edge-connected graph such that G has at most 11 edge-cuts of size 3. Then $G \in SL$ if and only if G cannot be contracted to P_{10}.*

3. Proof of the main result

Proof of Theorem 7. Let G be a 3-edge-connected graph that is in $C(12, 1)$. Note that if G is supereulerian, then any contraction of G is also supereulerian. Therefore, as P_{10} is not supereulerian, if G is supereulerian, then G cannot be contracted to P_{10}. Thus it suffices to show that if G cannot be contracted to P_{10}, then G must be supereulerian.

Let G' be the reduction of G. By Theorem 8 (i), it suffices to show that if $G' \neq P_{10}$, then $G' \in SL$. Arguing by contradiction, we assume that

$$G' \neq P_{10} \quad \text{and} \quad G' \notin SL. \tag{1}$$

Let $D_i(G') = \{v | d_{G'}(v) = i, v \in V(G')\}$ and $d_i = |D_i(G')|$ for $i = 1, 2, \ldots$.

As $|V(G')| = \sum_{j\geq 1} d_j$ and $2|E(G')| = \sum_{j\geq 1} jd_j$, by Theorem 10, we have

$$2F(G') = 3d_1 + 2d_2 + d_3 - \sum_{j\geq 5}(j-4)d_j - 4. \tag{2}$$

As $\kappa'(G) \geq 3$, we have $\kappa'(G') \geq 3$ and $d_1 = d_2 = 0$. Then by (2) we have

$$2F(G') = d_3 - 4 - \sum_{j\geq 5}(j-4)d_j. \tag{3}$$

Claim 1. $F(G') \geq 3$.

If $F(G') \leq 2$, then by Theorem 9, $G' \in \{K_1, K_2, K_{2,t}(t \geq 1)\}$. As $G' \notin SL$, $G' \neq K_1$. So $G' \in \{K_2, K_{2,t}(t \geq 1)\}$, contrary to $\kappa'(G') \geq 3$. Hence Claim 1 holds.

Claim 2. $12 \geq d_3 \geq 10$. *Moreover, if* $d_3 = 12$, *then* $|V(G')| \leq 13$.

If $d_3 \leq 9$, then $F(G') < 3$ by (3), contrary to Claim 1. Hence $d_3 \geq 10$.

Let $c = d_3$, and $T_1, T_2, \ldots, T_c$ be the preimage of vertices of degree 3 in G'. Let $|V(G)| = n$. Hence by $G \in C(12, 1)$,

$$n \geq \sum_{i=1}^{c} |V(T_i)| \geq c\frac{n-1}{12}. \tag{4}$$

If $n > 13$, then by (4),

$$c \leq \frac{12n}{n-1} = 12 + \frac{12}{n-1} < 13.$$

Since c is an integer, we conclude that $c \leq 12$ if $n > 13$.

If $n \leq 13$, then $c \leq |V(G')| \leq n \leq 13$. If $c = 13$, then $d_3 = c = |V(G')| = 13$ and $V(G') = D_3(G')$. This contradicts to the fact that $|O(G')|$ must be even. Hence we always have $c \leq 12$, and so

$$12 \geq d_3 \geq 10.$$

Let $V_4 = \{v | d_{G'}(v) \geq 4, v \in V(G')\}$. It follows by $G \in C(12, 1)$ that

$$n \geq \sum_{i=1}^{c} |V(T_i)| + |V_4| \geq c\frac{n-1}{12} + |V_4|. \tag{5}$$

If $c = 12$, then by (5), $|V_4| \leq 1$, and so $|V(G')| \leq c + |V_4| \leq 13$ follows. This verifies Claim 2.

By Claim 2, we shall distinguish two cases to finish the proof.

Case 1. $d_3 = 12$.

If $d_3 = 12$, then $|V(G')| \leq 13$ by Claim 2. Then by $\kappa'(G') \geq 3$ and by Theorem 11, $G' \in SL$, contrary to (1).

Case 2. $10 \leq d_3 \leq 11$.

By Claim 1, $F(G') \geq 3$. If $F(G') = 3$, by Theorem 12, G' has at most 11 edge-cuts of size 3, then by Theorem 13, either $G' \in SL$ or $G' = P_{10}$, contrary to (1).

Therefore, we may assume that $F(G') \geq 4$. It follows by (3) that $d_3 = 2F(G') + 4 + \sum_{j\geq 5}(j-4)d_j$, and so

$$12 \leq 12 + \sum_{j\geq 5}(j-4)d_j \leq 2F(G') + 4 + \sum_{j\geq 5}(j-4)d_j = d_3 \leq 11,$$

a contradiction as $\sum_{j\geq 5}(j-4)d_j \geq 0$.

As either case leads to a contradiction, the proof for Theorem 7 is now completed. □

4. Application

Strengthening a conjectured result by Benhocine, et al. in [3] within the 3-edge-connected graphs, Chen and Lai [15] present the following result:

Theorem 14 (*[15]*)**.** *Let G be a 3-edge-connected simple graph with n vertices. If n is large and if for every edge $uv \in E(G)$, $d(u) + d(v) \geq \frac{n}{6} - 2$, then either $G \in SL$ or G has P_{10} as its reduction.*

As a corollary of Theorem 7, we present the following result which is on the same line of Theorem 14.

Theorem 15. *Let G be a 3-edge-connected simple graph with n vertices. If $\delta(G) \geq 4$ and if*

$$\min\{\max\{d(x), d(y)\} | xy \in E(G)\} \geq \frac{n-1}{12} - 1,$$

then either $G \in SL$ or G has P_{10} as its reduction.

For $v \in V(G)$, we define the *neighborhood* $N(v)$ of v in G to be the set of vertices adjacent to v in G. First we present a lemma:

Lemma 16. *Let G be a k-edge-connected simple graph with $\delta(G) \geq k+1$. Let S be an edge-cut of G such that $|S| = k$. Let G_1 be a component of $G - S$. Then G_1 has an edge not adjacent to any edge of S.*

Proof. We pick a vertex $x \in V(G_1)$. Since $\delta(G) \geq k+1$, we have $d_G(x) \geq k+1$. By $|S| = k$ and G is simple, in G_1 there is a vertex $y \in N(x)$ such that y is not incident with any edge of S, i.e. $N(y) \subseteq V(G_1)$. Similarly, since $d_G(y) \geq k+1$, there is a vertex $z \in N(y)$ such that $N(z) \subseteq V(G_1)$(note that z may be x). Thus edge yz is not adjacent to any edge of S. □

Proof of Theorem 15. Let S be an edge-cut of G with $|S| = 3$, and let G_1 and G_2 be the two components of $G - S$ with $|V(G_1)| \leq |V(G_2)|$. By Theorem 7, it suffices to prove that $|V(G_1)| \geq \frac{n-1}{12}$.

Since $\delta(G) \geq 4$, by Lemma 16, G_1 has at least one edge, say uv, such that both of u, v are not incident with any edge of S. Thus $N(u) \subseteq V(G_1)$ and $N(v) \subseteq V(G_1)$. By the hypothesis of Theorem 15, we have

$$|V(G_1)| \geq \max\{d(u), d(v)\} + 1 \geq \frac{n-1}{12}.$$

Thus Theorem 15 follows from Theorem 7. □

Solving the open problems proposed by Bauer in [1,2], Catlin and Lai proved the following results:

Theorem 17 (*Theorem 9 of [8]*)**.** *Let G be a 2-edge-connected simple graph on n vertices. If $\delta(G) > \frac{n}{5} - 1$ and $n > 20$, then $G \in SL$.*

Theorem 18 (*Theorem 5 of [17]*)**.** *Let G be a 2-edge-connected triangle-free simple graph on $n > 30$ vertices. If $\delta(G) > \frac{n}{10}$, then $G \in SL$.*

If G is 3-edge-connected, then Theorems 17 and 18 can be improved by the following strong conclusions.

Theorem 19. *Let G be a 3-edge-connected simple graph with n vertices. Then each of the following holds.*

(i) *If $\delta(G) \geq \frac{n-1}{12} - 1$ and $n \geq 61$, then $G \in SL$ if and only if G cannot be contracted to P_{10}.*
(ii) *If G is triangle-free such that $\delta(G) \geq \frac{n-1}{24}$ and $n \geq 97$, then $G \in SL$ if and only if G cannot be contracted to P_{10}.*

Proof. (i) By $\delta(G) \geq \frac{n-1}{12} - 1$ and $n \geq 61$, we have $\delta(G) \geq 4$. $\delta(G) \geq \frac{n-1}{12} - 1$ implies that $\min\{\max\{d(x), d(y)\} | xy \in E(G)\} \geq \frac{n-1}{12} - 1$. Hence Theorem 19(i) follows by Theorem 15.

(ii) Let S be an edge-cut of G with $|S| = 3$, and let G_1 and G_2 be the two components of $G - S$ with $|V(G_1)| \leq |V(G_2)|$. By Theorem 7, it suffices to prove that $|V(G_1)| \geq \frac{n-1}{12}$.

By $\delta(G) \geq \frac{n-1}{24}$ and $n \geq 97$, we have $\delta(G) \geq 4$. Then by Lemma 16, G_1 has at least one edge, say uv, such that both of u, v are not incident with any edge of S. Thus $N(u) \subseteq V(G_1)$ and $N(v) \subseteq V(G_1)$. Since G is triangle-free, $N(u) \cap N(v) = \emptyset$. Then

$$|V(G_1)| \geq |N(u)| + |N(v)| \geq 2\delta(G) \geq \frac{n-1}{12}.$$

Thus Theorem 19(ii) follows by Theorem 7. □

5. Remark

Theorems 5–7 suggest the following question: if G is 3-edge-connected, and if l is an integer, what is the maximum value of l such that the statement $G \in C(l, 0) \cap SL$ if and only if G cannot be contracted to P_{10} remains valid? Theorem 7 in this note suggests that $l \geq 12$. However, the graph obtained from any one of the two Blanuša snarks B [16] by replacing each vertex of B by a 3-edge-connected spanning subgraph of $K_{n/18}$ seems to suggest that $l \leq 17$. We therefore conclude this note with the following conjecture that if $\kappa'(G) \geq 3$ and if $G \in C(17, 0)$, then $G \in SL$ if and only if G cannot be contracted to P_{10}.

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