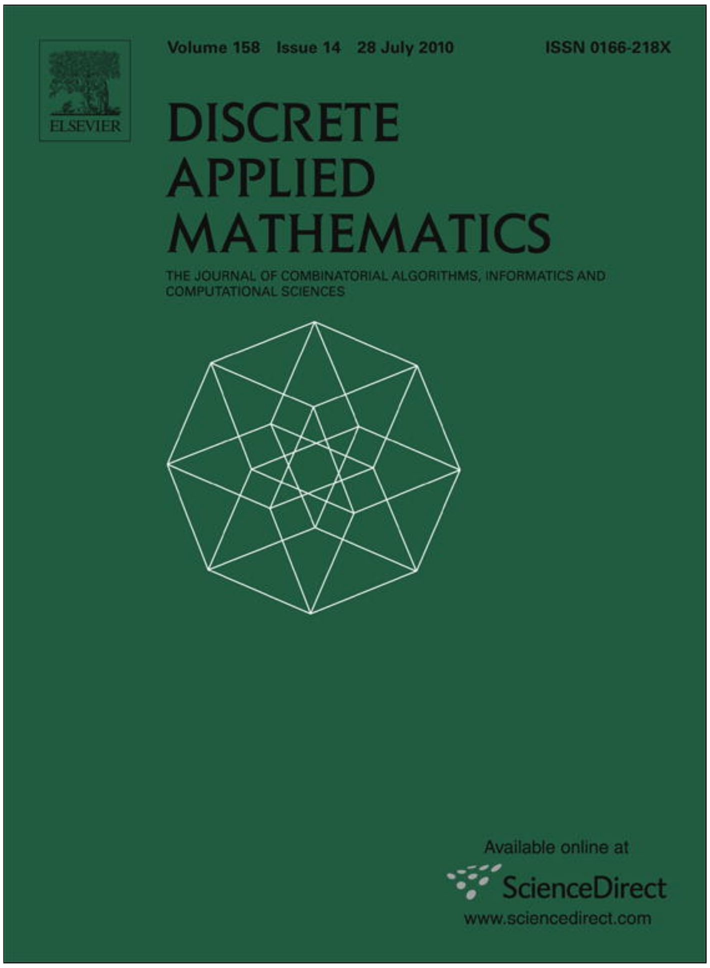
Volume 158 Issue 14 28 July 2010
ISSN 0166-218X
ELSEVIER
DISCRETE
APPLIED
MATHEMATICS
THE JOURNAL OF COMBINATORIAL ALGORITHMS, INFORMATICS AND COMPUTATIONAL SCIENCES
Available online at
ScienceDirect
www.sciencedirect.com

Discrete Applied Mathematics 158 (2010) 1511–1523

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Balanced and 1-balanced graph constructions

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ARTICLE INFO

Article history:
Received 31 July 2009
Received in revised form 9 April 2010
Accepted 9 May 2010
Available online 9 June 2010

Keywords:
Balanced graphs
1-balanced graphs
Cartesian product of graphs
Web graphs

ABSTRACT

There are several density functions for graphs which have found use in various applications. In this paper, we examine two of them, the first being given by $b(G) = |E(G)|/|V(G)|$, and the other being given by $g(G) = |E(G)|/(|V(G)| - \omega(G))$, where $\omega(G)$ denotes the number of components of G. Graphs for which $b(H) \leq b(G)$ for all subgraphs H of G are called *balanced graphs*, and graphs for which $g(H) \leq g(G)$ for all subgraphs H of G are called 1-*balanced* graphs (also sometimes called *strongly balanced* or *uniformly dense* in the literature). Although the functions b and g are very similar, they distinguish classes of graphs sufficiently differently that $b(G)$ is useful in studying random graphs, $g(G)$ has been useful in designing networks with reduced vulnerability to attack and in studying the World Wide Web, and a similar function is useful in the study of rigidity. First we give a new characterization of balanced graphs. Then we introduce a graph construction which generalizes the Cartesian product of graphs to produce what we call a *generalized Cartesian product*. We show that generalized Cartesian product derived from a tree and 1-balanced graphs are 1-balanced, and we use this to prove that the generalized Cartesian products derived from 1-balanced graphs are 1-balanced.

1. Introduction

We follow the notation of Diestel [3] for graphs, with the major exceptions that we use K_n for the complete graph on n vertices and we use $b(G)$ for the function $\epsilon(G) := \frac{|E(G)|}{|V(G)|}$. Graphs considered in this paper are loopless, but multiple edges are allowed. If a graph has an edge, it is called *non-trivial*. In this paper we look at two density functions, both related to the average degree of a graph. The first of these is $b(G) = \frac{|E(G)|}{|V(G)|}$ for a graph G. Graph G is said to be *balanced* if for all non-trivial subgraphs H of G,

$$b(H) \leq b(G)$$

and *strictly balanced* if for all non-trivial proper subgraphs H of G,

$$b(H) < b(G).$$

If G is connected, we also refer to a balanced graph as 0-balanced. Balanced graphs have been widely studied, particularly in the context of random graphs; for example, see [12,4,5,17,21].

The second density function we consider is $g(G) = |E(G)|/\rho(G)$, whose denominator $\rho(G)$ is the *rank* of a graph G given by $|V(G)| - \omega(G)$, where $\omega(G)$ is the number of components of G. (Note that $\rho(G)$ is also the rank of the circuit matroid $M(G)$

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doi:10.1016/j.dam.2010.05.004

derived from the graph G; see Oxley's book [15] for matroid terminology.) If $X \subseteq E(G)$, then the *rank* $\rho(X)$ of X is the rank of the induced graph $G[X]$.

A graph G is 1-*balanced* if, for every non-trivial subgraph $H \subseteq G$, $g(H) \leq g(G)$. The 1-balanced graphs and matroids have been studied by many researchers; see [2,7,13,14,16,19,20], and the references listed in those papers. Other names for a 1-balanced graph include "molecular graph" [13,14], "strongly balanced graph" [12,19], and "uniformly dense" [10,11].

The *prism* on a graph G is the Cartesian product of G with K_2. It can also be seen as being formed by letting G' be a disjoint isomorphic copy of G and joining each vertex a of G to the vertex a' of G' corresponding to a under the isomorphism, thus forming a matching between the two copies of G. In [18], Piazza and Ringeisen generalized the prism on G by taking two disjoint copies G_1 and G_2 of G and a permutation α of the vertices of G_2, and joining each vertex v_i of G_1 to the vertex $\alpha(v_i)$ of G_2. In [11], Hobbs et al. generalized the prism on G further by allowing G_1 and G_2 to be non-isomorphic but on the same numbers of vertices, and by replacing the matching joining them by a k-regular bipartite graph having as its two sides the vertex sets of G and G'. The "generalized prisms" motivate us to consider further generalizations of Cartesian products. In this paper, we present one such generalization, which contains both the Cartesian product and the generalized prisms as special cases. We also present in this paper characterizations for graphs whose generalized products are balanced, and for graphs whose generalized products are 1-balanced.

The construction of bigger 1-balanced graphs from smaller ones would be useful in the context of web-graphs, which have vertices representing web pages and edges corresponding to the links between pages. The structure of the web is often cited [1] as a bow tie, whose knot consists of a strongly connected component, called the *core*; and web-pages on the two sides of the knot consist respectively of those which link towards and away from the core. The core has been observed to be growing in its size over the years [9] and the cause for the growth is attributed to the increasing connectivity between existing web pages. Presences of *hubs* (vertices with high degrees) and *communities* (subgraphs which have more internal links than the external ones) dominate the web [6], which are also involved in the augmentation process of the core. 1-balanced graphs are described to be survivable under attacks on edges [10] and so it is of interest to construct 1-balanced graphs as the cores for the web-graphs. To be able to analyze the properties of the growing core, it would be of interest to design bigger networks from already existing smaller communities that may be modeled as 1-balanced graphs. Constructing bigger 1-balanced graphs from smaller ones would be useful in the context of realizing bigger survivable cores from existing communities.

In this paper, we are interested in constructing bigger 1-balanced graphs from already existing smaller 1-balanced subgraphs of equal density. Our main result is: if all the small 1-balanced graphs have the same number of edges and vertices, then the graphs in a class of generalized product of the 1-balanced graphs is 1-balanced. This generalizes our earlier result [11, Theorem 5] that Cartesian products of 1-balanced graphs are 1-balanced. The rest of the paper is organized as follows. In the next section, we give some preliminary results about 1-balanced graphs that will be used in the paper, and also give the definition of the generalized Cartesian graphs. In Section 3, we prove a new characterization of balanced graphs involving integer-valued functions on the vertices, and use it to prove that the generalized Cartesian product constructed from balanced graphs is balanced. In Section 4, we prove our main result, which makes use of the result on balanced generalized Cartesian products.

2. Preliminaries

2.1. Some results on 1-balanced graphs

We first recall some earlier results that are used in the paper. The following lemma is immediate for 1-balanced graphs.

Lemma 1. *A graph G is* 1*-balanced if and only if for all non-trivial* **connected** *subgraphs H of G, we have $g(H) \leq g(G)$.*

Proof. The necessity is clear. For sufficiency, suppose for all non-trivial, induced, connected subgraphs H of G, $g(H) \leq g(G)$. Let H be a disconnected subgraph of G. Let H_i, $1 \leq i \leq \omega(H)$ be the components of G. Clearly, we may assume that H_i for $1 \leq i \leq \omega(H)$ are non-trivial. By hypothesis, $g(H_i) \leq g(G)$, so $|E(H_i)| \leq g(G)(|V(H_i)| - 1)$ for $1 \leq i \leq \omega(H)$. Hence

$$|E(H)| = \sum_{i=1}^{\omega(H)} |E(H_i)| \leq g(G) \sum_{i=1}^{\omega(H)} (|V(H_i)| - 1) = g(G)(|V(H)| - \omega(H)),$$

and so $g(H) \leq g(G)$. □

As a consequence of the above lemma, we can observe that for a connected graph G, in order to check if G is 1-balanced, it suffices to check if $g(H) \leq g(G)$ for all connected subgraphs H of G. For the purposes of this paper, all graphs considered in the paper are connected. When we refer to Cartesian products, we refer to Cartesian products of connected graphs.

Theorem 2 (*Catlin et al. [2]*)**.** *Let G be a connected graph with $g(G) = \frac{x}{y}$, where x and y are natural numbers. Then, G is* 1*-balanced if and only if there is a family $\mathcal{T}$ of x spanning trees in G such that each edge of G lies in exactly y trees of $\mathcal{T}$.*

The following are consequences of Theorem 2.

Corollary 3. *If a graph G is an edge-disjoint union of spanning 1-balanced subgraphs $G_1, G_2, \ldots, G_p$ for some integer $p \geq 1$, then G is 1-balanced.*

Proof. We show the result for $p = 2$. For $p > 2$, the result follows by induction on p. Since G_1 and G_2 are connected 1-balanced graphs on the same number of vertices, for $x_1 = |E(G_1)|$, $x_2 = |E(G_2)|$ and $y = |V(G)| - 1$, $g(G_1) = \frac{x_1}{y}$ and $g(G_2) = \frac{x_2}{y}$. Thus $g(G) = \frac{x_1+x_2}{y}$. For $i = 1, 2$, since G_i is 1-balanced, G_i has x_i spanning trees such that each edge in G_i is in exactly y of them. Thus G has $x_1 + x_2$ spanning trees such that each edge in G is in exactly y of them. G is 1-balanced by Theorem 2. □

For a positive integer x, let G^x denote the graph obtained by replacing each edge of G by x parallel edges. The next corollary can be derived from Corollary 3.

Corollary 4. *Let x be a positive integer. A graph G is 1-balanced if and only if G^x is 1-balanced.*

2.2. Generalized Cartesian products

Throughout this section, let e, n, ℓ and m be integers with $e \geq n-1 \geq 1$ and $\ell \geq m-1 \geq 1$, let L be a graph with ℓ edges and m vertices, and let the vertices of L be labeled $v_1, v_2, \ldots, v_m$. Label the edges of L as $e_1, e_2, \ldots, e_\ell$. Let $G_1, G_2, \ldots, G_m$ be vertex-disjoint graphs, each having n vertices and e edges. Let k be a positive integer. Let $B_1, B_2, \ldots, B_\ell$ be k-regular bipartite graphs (may be disconnected) such that, if edge e_i of L joins vertices v_r and v_s, then the two sides of B_i are the vertex sets of G_r and G_s. Let $A_k = A_k(G_1, \ldots, G_m; L) = \left(\bigcup_{i=1}^m G_i\right) \cup \left(\bigcup_{i=1}^\ell B_i\right)$. When the value of k is already known, we may use $A = A(G_1, \ldots, G_m; L)$, with the subscript k omitted. Then A_k or A is called a *generalized Cartesian product*. Note that the definition of A_k is ambiguous, since there are many possible k-regular bipartite graphs B_i. We allow this ambiguity because the choices of the B_i make no difference to our results. Also note that if G and L are graphs, then the Cartesian product $G \times L$ is a generalized Cartesian product with $G_i = G$ for $i = 1, 2, \ldots, m$ and $k = 1$.

Let H be a subgraph of A, and suppose H includes one or more vertices of $G_{i_1}, \ldots, G_{i_{\ell'}}$ and no others of the G_i. Let L' be the subgraph of L generated by the vertices $v_{i_1}, \ldots, v_{i_{\ell'}}$. Then we say that *$L'$ is induced by H*.

3. Characterizations of balanced graphs and balanced generalized Cartesian products

In this section, we first provide a new characterization of balanced graphs which is used to construct bigger balanced graphs from smaller ones, which in turn is used in the last section to construct bigger 1-balanced graphs from smaller 1-balanced graphs. The characterization is also used to show that the Cartesian product of balanced graphs is balanced.

The next theorem is our new characterization of balanced graphs. The characterization involves arbitrary non-negative integer vertex weights.

Theorem 5. *Let L be a graph on m vertices $V = \{v_1, \ldots, v_m\}$. Let α be any non-negative integer-valued function on the vertex set V. Let*

$$N_\alpha := \sum_{v_iv_j \in E(L)} \left[\min(\alpha(v_i), \alpha(v_j)) - \frac{1}{m}\sum_{r=1}^m \alpha(v_r)\right].$$

Then L is balanced if and only if $N_\alpha \leq 0$ for all α, and L is strictly balanced if and only if $N_\alpha < 0$ for all non-constant α.

Proof (*Sufficiency of L Balanced*)**.** For a contradiction, suppose L is balanced while there is a non-negative, integer-valued function α on $V(L)$ with $N_\alpha > 0$. Choose α_0 such that $N_{\alpha_0} > 0$ and $s = \max_{1 \leq i \leq m} \alpha_0(v_i)$ is as small as possible. If α_0 were constant on $\{v_1, \ldots, v_m\}$, then $N_{\alpha_0} = 0$. Hence, there is a $j \in \{1, 2, \ldots, m\}$ such that $\alpha_0(v_j) < s$. Then $s \geq 1$ since $\alpha_0(v_j) \geq 0$.

Let $S := \{v_i : \alpha_0(v_i) = s\}$. By the definition of s and j, $S \notin \{\emptyset, V\}$. Consider the function α_0' defined by $\alpha_0'(v_i) = \alpha_0(v_i)$ if $v_i \notin S$ and $\alpha_0(v_i) - 1$ if $v_i \in S$. Thus $\max_{1 \leq i \leq m} \alpha_0(v_i) < s$.

We claim that $N_{\alpha_0'} \geq N_{\alpha_0}$.

Let $L' := L[S]$, and denote $m' := |V(L')| = |S|$ and $\ell' := |E(L')|$. Then

$$\begin{aligned}
\frac{1}{m}\sum_{r=1}^m \alpha_0'(v_r) &= \frac{1}{m}\left[\sum_{r:v_r \notin S} \alpha_0(v_r) + \sum_{r:v_r \in S} (\alpha_0(v_r) - 1)\right] \\
&= \frac{1}{m}\left[\sum_{r:v_r \notin S} \alpha_0(v_r) + \sum_{r:v_r \in S} \alpha_0(v_r) - m'\right] \\
&= \frac{1}{m}\sum_{r=1}^m \alpha_0(v_r) - \frac{m'}{m}.
\end{aligned}$$

Suppose $v_iv_j \in E(L')$. Then $\min(\alpha_0'(v_i), \alpha_0'(v_j)) = \min(\alpha_0(v_i), \alpha_0(v_j)) - 1$. Therefore, in this case we have

$$\min(\alpha_0'(v_i), \alpha_0'(v_j)) - \frac{1}{m}\sum_{r=1}^{m}\alpha_0'(v_r) = \min(\alpha_0(v_i), \alpha_0(v_j)) - 1 - \frac{1}{m}\sum_{r=1}^{m}\alpha_0(v_r) + \frac{m'}{m}.$$

If $v_iv_j \notin E(L')$, then $\min(\alpha_0'(v_i), \alpha_0'(v_j)) = \min(\alpha_0(v_i), \alpha_0(v_j))$. This is true even if, for example, $v_i \in S$ and $v_j \notin S$, for then $\alpha_0'(v_i) = s-1$ and $\alpha_0'(v_j) = \alpha_0(v_j) \le s-1$, and so $\min(\alpha_0'(v_i), \alpha_0'(v_j)) = \min(s-1, \alpha_0(v_j)) = \alpha_0(v_j) = \min(\alpha_0(v_i), \alpha_0(v_j))$. Thus we have

$$\min(\alpha_0'(v_i), \alpha_0'(v_j)) - \frac{1}{m}\sum_{r=1}^{m}\alpha_0'(v_r) = \min(\alpha_0(v_i), \alpha_0(v_j)) - \frac{1}{m}\sum_{r=1}^{m}\alpha_0(v_r) + \frac{m'}{m}.$$

Therefore,

$$N_{\alpha_0'} - N_{\alpha_0} = \ell'\left(-1 + \frac{m'}{m}\right) + (\ell - \ell')\frac{m'}{m} = -\ell' + \frac{\ell m'}{m} = m'\left(-\frac{\ell'}{m'} + \frac{\ell}{m}\right) \ge 0$$

since $\frac{\ell'}{m'} = b(L') \le b(L) = \frac{\ell}{m}$ either because L is balanced or because $\ell' = 0$. Hence the claim.

But $N_{\alpha_0'} \ge N_{\alpha_0} > 0$ is a contradiction to the minimality of s by the definition of $N_{\alpha_0'}$. The contradiction proves sufficiency.

(Necessity of L balanced) Suppose $N_\alpha \le 0$ for all labelings α. Let L' be any non-trivial vertex-induced subgraph of L, and suppose L' has m' vertices and ℓ' edges. Define α on $V(L)$ by letting $\alpha(v) = 1$ if $v \in V(L')$ and 0 if $v \notin V(L')$. Then

$$\frac{1}{m}\sum_{r=1}^{m}\alpha(v_r) = \frac{m'}{m},$$

and

$$\begin{aligned}
0 &\ge N_\alpha \\
&= \sum_{v_iv_j \in E(L')}\left(1 - \frac{m'}{m}\right) + \sum_{v_iv_j \notin E(L')}\left(-\frac{m'}{m}\right) \\
&= \ell' - \frac{m'\ell'}{m} - \frac{m'\ell}{m} + \frac{m'\ell'}{m} \\
&= \ell' - \frac{m'\ell}{m} \\
&= m'\left(\frac{\ell'}{m'} - \frac{\ell}{m}\right).
\end{aligned}$$

Hence we have $\frac{\ell'}{m'} \le \frac{\ell}{m}$ (*i.e.*, $b(L') \le b(L)$), so L is balanced.

The proof for strictly balanced graphs is similar. □

Next, we prove that Cartesian products of balanced graphs are balanced. In fact, we will prove an extension of the result. We present a construction of bigger balanced graphs from smaller ones by joining some additional edges, namely, the following result for generalized Cartesian product defined in Section 2.

Theorem 6. *Let L be a graph on m vertices and ℓ edges. Let k be any positive integer and let $G_1, \ldots, G_m$ be balanced graphs, each on n vertices and e edges. Then $A = A_k(G_1, \ldots, G_m; L)$ is balanced if and only if L is balanced.*

Proof.

$$b(A) = \frac{nkl + me}{mn} = \frac{kl}{m} + \frac{e}{n}. \tag{1}$$

Note that for $i = 1, \ldots, m$,

$$b(G_i) < b(A). \tag{2}$$

(Necessity) Suppose A is balanced. Let $V(L) = \{v_1, v_2, \ldots, v_m\}$. Let L' be any subgraph of L, and suppose L' has ℓ' edges and m' vertices. Form A' on L' as A is formed on L. Then,

$$b(A') = \frac{nk\ell' + m'e}{m'n} = \frac{kl'}{m'} + \frac{e}{n}. \tag{3}$$

Since A is balanced, we have $b(A') \le b(A)$, and by (1) and (3), we have

$$\frac{k\ell'}{m'} \le \frac{k\ell}{m}. \tag{4}$$

Thus

$$\frac{\ell'}{m'} \leq \frac{\ell}{m}. \tag{5}$$

Therefore L is balanced.

(Sufficiency) Suppose L is balanced. Let H be a subgraph of A. If H is a subgraph of a G_j for some $j = 1, \ldots, m$, then since G_j is balanced, we have $b(H) \leq b(G_j)$ and by (2), we have $b(G_j) < b(L)$; thus $b(H) < b(L)$. Otherwise, let $H_i = H \cap G_i$ and $n_i = |V(H_i)|$ for $i = 1, \ldots, m$. Without loss of generality, we may suppose that there is an integer $m' > 0$ such that $1 \leq n_1 \leq n_2 \leq \cdots \leq n_{m'}$ and $n_i = 0$ for $i > m'$. Let L' be the subgraph of L induced by H, and note that $L' = L$ is possible. For each $i \in \{1, 2, \ldots, m'\}$, let $e_i = |E(H_i)|$, and $e' = |E(H) \cap E(\bigcup_{i=1}^{\ell} B_i)|$. Notice that

$$e' \leq k \sum_{v_i v_j \in E(L')} \min(n_i, n_j). \tag{6}$$

Since L is balanced, using $\alpha(v_i) = n_i$ for $i \in \{1, 2, \ldots, m\}$ in Theorem 5 and using $n_i = 0$ for $i > m'$, we have

$$\sum_{v_i v_j \in E(L')} \min(n_i, n_j) = \sum_{v_i v_j \in E(L')} \min(n_i, n_j) \leq \frac{l}{m} \sum_{i=1}^{m} n_i = \frac{l}{m} \sum_{i=1}^{m'} n_i. \tag{7}$$

By (6) and (7), we get

$$e' \leq \frac{kl}{m} \sum_{i=1}^{m'} n_i. \tag{8}$$

Thus,

$$b(H) = \frac{e' + \sum_{i=1}^{m'} e_i}{\sum_{i=1}^{m'} n_i} \tag{9}$$

$$\leq \frac{\frac{kl}{m} \sum_{i=1}^{m'} n_i + \sum_{i=1}^{m'} e_i}{\sum_{i=1}^{m'} n_i} \tag{10}$$

$$= \frac{kl}{m} + \frac{\sum_{i=1}^{m'} e_i}{\sum_{i=1}^{m'} n_i}. \tag{11}$$

By [8, Theorem 1, page 14], we have $\frac{\sum_{i=1}^{m'} e_i}{\sum_{i=1}^{m'} n_i} \leq \max_{1 \leq i \leq m'} \frac{e_i}{n_i} \leq \frac{e}{n}$ since G_i is balanced for every i. Therefore, $b(H) \leq \frac{kl}{m} + \frac{e}{n} = g(A)$ and thus A is balanced. □

Corollary 7. *The Cartesian product of balanced graphs is balanced.*

Proof. Let G and L be two balanced graphs. Then $G \times L = A_1(G, G, \ldots, G; L)$ with suitable choices of the bipartite graphs B_{ij}. By the above theorem, $G \times L$ is balanced. □

4. 1-balanced generalized Cartesian products

The method of generalized Cartesian products defined in Section 1 can be used to construct bigger 1-balanced graphs from smaller ones. In this section, we prove that 1-balanced generalized Cartesian products can be formed from 1-balanced graphs.

In this section we prove that A is 1-balanced if $G_1, \ldots, G_m$ and L are 1-balanced and k is a fixed integer such that

$$\frac{m-1}{\ell} \binom{t}{n} \leq k \leq \frac{m-1}{\ell} (mt). \tag{12}$$

The reason why we need the above bounds for k is explained in the next paragraph.

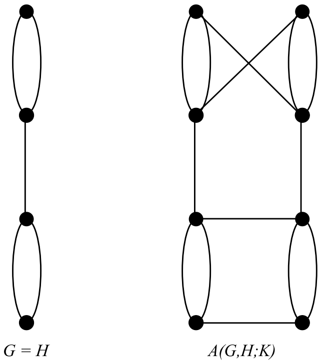

Fig. 1. $A_1(G, H; K_2)$: Example of a generalized Cartesian product that is 1-balanced, but neither G nor H is 1-balanced.

Let $t = g(G_i) = \frac{e}{n-1}$ for all $i \in \{1, 2, \ldots, m\}$. Unlike balanced generalized Cartesian products, the value of the positive integer k in a 1-balanced generalized Cartesian product has a non-trivial lower bound, as the following Lemma shows.

Lemma 8. *If A is 1-balanced, then*

$$k \geq \frac{m-1}{\ell}\left(\frac{t}{n}\right) = \frac{g(G_i)}{g(L)n}.$$

Proof. For each i, we have $|E(B_i)| = 2nk/2 = nk$. Since each G_i is connected and L is connected, A is connected. Hence

$$g(A) = \frac{nk\ell + me}{mn - 1}.$$

Since A is 1-balanced and G_i is a subgraph of A for each i, we have

$$g(A) = \frac{nk\ell + me}{mn - 1} \geq \frac{e}{n-1} = g(G_i).$$

Solving for k, we get

$$k \geq \frac{m-1}{\ell}\left(\frac{t}{n}\right). \quad \square$$

The need for an upper bound for k such as this for k is illustrated by the graph $A = A_3(K_2, K_2; K_2)$. Here, $L = K_2$, $t = 1$ and $m = 2$. We have $\frac{m-1}{\ell}(mt) = 2 < 3$. If H denotes the subgraph on 2 vertices and 3 parallel edges, then $g(H) = 3$. But, $g(A) = \frac{2(3+1)}{3} = \frac{8}{3} < 3 = g(H)$. Therefore A is not 1-balanced. Thus even the usual Cartesian product is not necessarily 1-balanced when $k > \frac{m-1}{\ell}(mt)$, even if $G_1, \ldots, G_m$ and L are 1-balanced.

That a generalized Cartesian product $A = A_k(G_1, \ldots, G_m; L)$ is 1-balanced does not imply that any of $G_1, \ldots, G_m$ or L is 1-balanced. The graph in Fig. 1 is an example of a generalized Cartesian product $A_k(G, H; K_2)$ that is 1-balanced, but neither G nor H is 1-balanced. It is easy to see that $A_1(G, H; K_2)$ is the union of 2 edge-disjoint spanning trees. Thus A is 1-balanced, by Theorem 2. Also, note that K_2 is a subgraph of G and $g(K_2) = 2$, but $g(G) = \frac{5}{3} < 2$. Thus, G is not 1-balanced. Similarly, H is not 1-balanced.

However, we have this result:

Theorem 9. *If A is 1-balanced, then L is strictly balanced.*

Proof. Let $V(L) = \{v_1, v_2, \ldots, v_m\}$. Let L' be any proper connected subgraph of L, and suppose L' has ℓ' edges and m' vertices. Form A' on L' as A is formed on L. Then $g(A') = \frac{nk\ell' + m'e}{m'n-1}$. Since A is 1-balanced, we have

$$\frac{nk\ell' + m'e}{m'n - 1} = g(A') \leq g(A) = \frac{nk\ell + me}{mn - 1}.$$

Cross-multiplying and simplifying,

$$mn^2k\ell' + mm'ne - nk\ell' - m'e \leq m'n^2k\ell + mm'ne - nk\ell - me,$$

or

$$\begin{aligned} mn^2k\ell' - nk\ell' - m'e &\leq m'n^2k\ell - nk\ell - me \\ &< m'n^2k\ell - nk\ell' - m'e \end{aligned}$$

since $\ell > \ell'$ and $m > m'$. Hence,

$$mn^2k\ell' - nk\ell' - m'e < m'n^2k\ell - nk\ell' - m'e,$$

which simplifies to $m\ell' < m'\ell$ since $n^2k > 0$. Thus we have $\frac{\ell'}{m'} < \frac{\ell}{m}$ as required. □

The converse of this theorem is false. Let L be the graph formed by the vertices $\{a, b, c, d\}$ and edges (a, b), (b, c), (a, c) and (c, d). Then $g(L[\{a, b, c\}]) = \frac{3}{2} > 2 = g(L)$. Thus L is not 1-balanced, but it can be easily verified that L is strictly balanced. Now, consider the Cartesian product of K_2 and L, $A = A_1(K_2, K_2, K_2, K_2; L)$. The Cartesian product of K_2 and $H := G[\{a, b, c\}]$ is a subgraph of A, and $g(K_2 \times H) = \frac{9}{5} > \frac{12}{7} = g(A)$. Thus A is not 1-balanced.

Throughout the rest of the paper, for any graph X, we refer to $\gamma(X)$ as

$$\gamma(X) := \max_{X' \subseteq X} g(X'),$$

where the maximum is taken over all non-trivial subgraphs X' of X. We also call a non-trivial subgraph X' of X with $g(X') = \gamma(X)$, as a γ-*achieving* subgraph of X.

From now on, we assume that $G_1, \ldots, G_m$ are connected 1-balanced graphs. We first show that A is 1-balanced if k is as specified in (12) and L is a tree. Our plan of proof is to choose a γ-achieving, connected (in view of Lemma 1) subgraph H of A. We move to the subtree L' of L induced by H and prove $g(H) \le g(A')$ in that case. (It is here that we use the new characterization of balanced graphs, namely Theorem 5.) Using $g(A') \le g(A)$, as shown in the next lemma, we conclude that $g(H) \le g(A)$. Thus $g(A) = \gamma(A)$ and A is 1-balanced.

We start with some lemmas. Let L be any 1-balanced graph, and let L' be a connected induced subgraph of L. Letting A' be constructed from L' as A is constructed from L, we first look at the relationship between $g(A)$ and $g(A')$ (Lemma 10) and between $g(A)$ and $g(G_i)$ (Lemma 11).

Lemma 10. *Let $t = \frac{e}{n-1} = g(G_i)$ for $i \in \{1, 2, \ldots, m\}$, and let $k \ge \frac{t}{g(L)n}$. Let L' be a connected induced subgraph of L. Form A' from L' in the same way A is formed from L. If L is 1-balanced, then $g(A') \le g(A)$.*

Proof.

$$\begin{aligned} g(A) - g(A') &= \frac{nk\ell + me}{mn - 1} - \frac{nk\ell' + m'e}{m'n - 1} \\ &= \frac{nk\ell(m'n - 1) + mm'ne - me - nk\ell'(mn - 1) - mm'ne + m'e}{(mn - 1)(m'n - 1)} \\ &= \frac{nk\ell(m'n - 1) - me - nk\ell'(mn - 1) + m'e}{(mn - 1)(m'n - 1)}. \end{aligned}$$

Since $\ell = g(L)(m - 1)$ and $\ell' \le g(L)(m' - 1)$, we have

$$\begin{aligned} g(A) - g(A') &\ge \frac{g(L)nk(m - 1)(m'n - 1) - me - g(L)nk(m' - 1)(mn - 1) + m'e}{(mn - 1)(m'n - 1)} \\ &= \frac{g(L)nk[mm'n - m'n - m + 1 - mm'n + mn + m' - 1] - (m - m')e}{(mn - 1)(m'n - 1)} \\ &= \frac{g(L)nk[-m'n - m + mn + m'] - (m - m')e}{(mn - 1)(m'n - 1)} \\ &= \frac{g(L)nk[(m - m')(n - 1)] - (m - m')e}{(mn - 1)(m'n - 1)} \\ &= (m - m')\frac{g(L)nk(n - 1) - e}{(mn - 1)(m'n - 1)} \\ &= (m - m')(n - 1)\frac{g(L)nk - g(G_i)}{(mn - 1)(m'n - 1)} \\ &\ge 0 \end{aligned}$$

since $k \ge \frac{g(G_i)}{g(L)n}$. □

Lemma 11. *With $k \ge \frac{m-1}{\ell}\left(\frac{t}{n}\right)$, we have $g(G_i) \le g(A)$.*

Proof. This was noted at the end of the proof of Lemma 8. □

From now on, we assume that k satisfies (12).

Theorem 12. *Let L be a tree. Then A is 1-balanced.*

Proof. If $n = 1, A = L$ and since L is 1-balanced, A is 1-balanced. We may assume that $n > 1$.

Suppose, for a contradiction, that A is not 1-balanced. Then by Lemma 1, there is an induced connected subgraph H of A such that $g(H) = \gamma(A) > g(A)$. Let $H_i = H \cap G_i$ and $n_i = |V(H_i)|$. Without loss of generality, we may suppose there is an integer $m' > 0$ such that $1 \leq n_1 \leq n_2 \leq \cdots \leq n_{m'}$ and $n_i = 0$ for $i > m'$. Let L' be the subgraph of L induced by H, and note that $L' = L$ is possible. L' is the same subgraph of L which is induced by $\{v_1, \ldots, v_{m'}\}$. For each $i \in \{1, 2, \ldots, m'\}$, let $e_i = |E(H_i)|$, $\omega_i = \omega(H_i)$, and $e' = |E(H) \cap E(\bigcup_{i=1}^{\ell} B_i)|$. Notice that

$$e' \leq k \sum_{v_i v_j \in E(L')} \min(n_i, n_j). \tag{13}$$

Since L is a tree and H is connected, L' is a tree. $g(L) = 1$. So,

$$\frac{t}{n} \leq k \leq mt. \tag{14}$$

Recall that $g(A') \leq g(A)$ by Lemma 10. Thus

$$g(H) > g(A) \geq g(A'), \tag{15}$$

so A' is also not 1-balanced.

We consider two cases:

Case 1: $\frac{t}{n} \leq k \leq m't$.

First we show

$$\frac{nk\ell' - (m'-1)t}{m'n - 1} < \frac{e' - (m'-1)t}{\sum_{i=1}^{m'} n_i - 1}. \tag{16}$$

Since $k \geq \frac{t}{n}$, $g(A) \geq g(G_i)$ for each i by Lemma 11, $H \not\subseteq G_i$ for any i, so $m' > 1$.

Recalling that $t = g(G_i) = \frac{e}{n-1}$,

$$\begin{aligned} g(A') &= \frac{nk\ell' + m'e}{m'n - 1} \\ &= \frac{nk\ell' + m'\frac{e}{n-1}(n-1)}{m'n - 1} \\ &= \frac{tm'(n-1) + nk\ell'}{m'n - 1} \\ &= \frac{m'nt - t + t - m't + nk\ell'}{m'n - 1} \\ &= t + \frac{nk\ell' - (m'-1)t}{m'n - 1}. \end{aligned} \tag{17}$$

Also,

$$g(H) = \frac{\sum_{i=1}^{m'} e_i + e'}{\sum_{i=1}^{m'} n_i - 1}.$$

But, with $i \leq m'$, $n_i \geq 1$. Thus, if $e_i \neq 0$, then

$$e_i = \frac{e_i}{n_i - \omega_i}(n_i - \omega_i) \leq g(G_i)(n_i - \omega_i) \leq t(n_i - 1).$$

On the other hand, if $e_i = 0$, then

$$e_i = 0 \leq t(n_i - 1).$$

Thus, from the definitions of the symbols,

$$g(H) = \frac{\sum_{i=1}^{m'} e_i + e'}{\sum_{i=1}^{m'} n_i - 1}$$

$$\le \frac{t\sum_{i=1}^{m'}(n_i-1)+e'}{\sum_{i=1}^{m'} n_i - 1}$$

$$= \frac{t\left(\sum_{i=1}^{m'} n_i - 1\right) + t + e' - \sum_{i=1}^{m'} t}{\sum_{i=1}^{m'} n_i - 1}$$

$$= t + \frac{e' - (m'-1)t}{\sum_{i=1}^{m'} n_i - 1}. \tag{18}$$

Since $g(A') < g(H)$, (16) follows from (17) and (18).

Next we show that

$$\sum_{v_iv_j\in E(L')}\left[\min(n_i,n_j) - \frac{1}{m'}\sum_{r=1}^{m'} n_r\right] > 0 \tag{19}$$

follows from (16), thus leading to a contradiction. But

$$\frac{nk\ell' - (m'-1)t}{m'n-1} = \frac{\frac{k\ell'}{m'} - (m'-1)t}{m'n-1} + \frac{k\ell'}{m'}.$$

Replacing the left-hand side of (16) with this, moving $\frac{k\ell'}{m'}$ to the other side, and using (13),

$$\frac{\frac{k\ell'}{m'} - (m'-1)t}{m'n-1} < \frac{-\frac{k\ell'}{m'}\sum_{i=1}^{m'} n_i + \frac{k\ell'}{m'} + k\sum_{v_iv_j\in E(L')}\min(n_i,n_j) - (m'-1)t}{\sum_{i=1}^{m'} n_i - 1}$$

$$= \frac{\frac{k\ell'}{m'} + k\left(\sum_{v_iv_j\in E(L')}\left[\min(n_i,n_j) - \frac{1}{m'}\sum_{i=1}^{m'} n_i\right]\right) - (m'-1)t}{\sum_{i=1}^{m'} n_i - 1}.$$

Multiplying through by the denominators and canceling like terms, we get

$$\frac{k\ell'}{m'}\sum_{i=1}^{m'} n_i - (m'-1)t\sum_{i=1}^{m'} n_i < \frac{k\ell'}{m'}(m'n) + k(m'n-1)\left(\sum_{v_iv_j\in E(L')}\left[\min(n_i,n_j) - \frac{1}{m'}\sum_{i=1}^{m'} n_i\right]\right) - m'n(m'-1)t.$$

Thus

$$\frac{k\ell'}{m'}\sum_{i=1}^{m'} n_i - \frac{k\ell'}{m'}(m'n) + m'n(m'-1)t - (m'-1)t\sum_{i=1}^{m'} n_i < k(m'n-1)\left(\sum_{v_iv_j\in E(L')}\left[\min(n_i,n_j) - \frac{1}{m'}\sum_{i=1}^{m'} n_i\right]\right).$$

Combining the two terms containing $(m'-1)t$ and then the first two terms of the previous inequality, we get

$$(m'-1)t\sum_{i=1}^{m'}(n-n_i) - \frac{k\ell'}{m'}\left(m'n - \sum_{i=1}^{m'} n_i\right) < k(m'n-1)\left(\sum_{v_iv_j\in E(L')}\left[\min(n_i,n_j) - \frac{1}{m'}\sum_{i=1}^{m'} n_i\right]\right).$$

Combining the terms on the left hand side gives us

$$\left((m'-1)t - \frac{k\ell'}{m'}\right)\sum_{i=1}^{m'}(n-n_i) < k(m'n-1)\left(\sum_{v_iv_j\in E(L')}\left[\min(n_i,n_j) - \frac{1}{m'}\sum_{i=1}^{m'} n_i\right]\right). \tag{20}$$

But $\sum_{i=1}^{m'}(n-n_i) \ge 0$ since $n_i \le n$ for all i. Moreover, since $\ell' = m'-1$, $m' \ge 2$ and $k \le m't$,

$$(m'-1)t - \frac{k\ell'}{m'} = (m'-1)\left[t - \frac{k}{m'}\right] \ge 0.$$

Thus the left hand side of (20) is non-negative. Since $k(m'n-1)$ is positive, the rest of the right hand side must be positive. Hence the inequality (19). But L' is a tree, and so it is 1-balanced and thus balanced. By Theorem 5, the inequality we have just reached is impossible. Thus A' is 1-balanced, so the proposed subgraph H cannot exist.

Case 2: $m't \le k \le mt$.

For this case, we show that $g(A) < g(H)$ and $k \ge m't$ together imply that $k > mt$ which is a contradiction.

Using the similar computations we used in (17), we obtain

$$\begin{aligned} g(A) &= \frac{nk\ell + m'e}{mn-1} \\ &= \frac{nk\ell + m\frac{e}{n-1}(n-1)}{mn-1} \\ &= \frac{tm(n-1)+nk\ell}{mn-1} \\ &= \frac{mnt - t + t - mt + nk\ell}{mn-1} \\ &= t + \frac{nk\ell - (m-1)t}{mn-1}. \end{aligned} \tag{21}$$

From (15), we have $g(A) < g(H)$. Thus by (18) and (21),

$$\frac{nk\ell-(m-1)t}{mn-1} < \frac{e'-(m'-1)t}{\sum_{i=1}^{m'} n_i - 1}. \tag{22}$$

Now, we will get a bound for e'. By (13), we have

$$e' \le k \sum_{v_iv_j \in E(L')} \min(n_i, n_j).$$

By Theorem 5, since L' is a balanced graph,

$$\sum_{v_iv_j \in E(L')} \min(n_i, n_j) \le \frac{\ell'}{m'} \sum_{i=1}^{m'} n_i.$$

Since $\ell' = m' - 1$,

$$e' \le k\left(\sum_{i=1}^{m'} n_i - \frac{1}{m'}\sum_{i=1}^{m'} n_i\right).$$

Substituting this in (22) and adding and subtracting k in the numerator of the left hand side, we have

$$\frac{nk\ell-(m-1)t}{mn-1} < \frac{k\left(\sum_{i=1}^{m'} n_i - 1\right) - k\left(\frac{1}{m'}\sum_{i=1}^{m'} n_i - 1\right) - (m'-1)t}{\sum_{i=1}^{m'} n_i - 1}.$$

Using the fact that $k \ge m't$ and simplifying, we have

$$\begin{aligned} \frac{nk\ell-(m-1)t}{mn-1} &< \frac{k\left(\sum_{i=1}^{m'} n_i - 1\right) - m't\left(\frac{1}{m'}\sum_{i=1}^{m'} n_i - 1\right) - (m'-1)t}{\sum_{i=1}^{m'} n_i - 1} \\ &= \frac{k\left(\sum_{i=1}^{m'} n_i - 1\right) - t\left(\sum_{i=1}^{m'} n_i - m'\right) - (m'-1)t}{\sum_{i=1}^{m'} n_i - 1} \end{aligned}$$

$$= \frac{k\left(\sum_{i=1}^{m'} n_i - 1\right) - t\left(\sum_{i=1}^{m'} n_i - 1\right)}{\sum_{i=1}^{m'} n_i - 1}$$
$$= k - t.$$

Substituting $\ell = m - 1$ and cross-multiplying, we have

$$(m-1)(nk-t) < (mn-1)(k-t),$$

which simplifies to

$$-nk - mt < -k - mnt.$$

Thus, $(n-1)(mt) < (n-1)k$. Since $n > 1$, we have $k > mt$ which is a contradiction.

Hence A is 1-balanced. □

Now, we are ready to show that if L is 1-balanced, then A is 1-balanced. We recall that in the generalized Cartesian product $A = A_k(G_1, \ldots, G_m; L)$, the graphs G_i, $i = 1, 2, \ldots, m$ are 1-balanced and with the same number of vertices and edges; and k is any integer satisfying (12), *i.e.*,

$$\frac{t}{g(L)n} \le k \le \frac{mt}{g(L)},$$

where $t = g(G_i)$, $i = 1, 2, \ldots, m$.

Theorem 13. *If L is 1-balanced, then A is 1-balanced.*

Proof. Let $g(L) = \frac{r}{s}$. Since

$$\frac{t}{g(L)n} \le k \le \frac{mt}{g(L)},$$

substituting $g(L) = \frac{r}{s}$, we have

$$\frac{st}{rn} \le k \le \frac{mst}{r}, \quad \text{or} \quad \frac{st}{n} \le kr \le mst. \tag{23}$$

We first prove that A_{rs} is 1-balanced. By Corollary 4, if A_{rs} is 1-balanced, then A is 1-balanced. To prove A_{rs} is 1-balanced, we will prove that A_{rs} is an edge-disjoint union of r spanning 1-balanced connected subgraphs. Then, Corollary 3 will show that A_{rs} is 1-balanced.

Since L is 1-balanced of density $\frac{r}{s}$, by Theorem 2, there are r spanning trees $T_1, T_2, \ldots, T_r$ in L such that each edge of L appears in exactly s of the trees. Let us denote by B_e the k-regular bipartite graph that replaces the edge $e \in L$ in A. For $1 \le j \le r$, let A_j be the generalized Cartesian product $A_{kr}(G_1^s, \ldots, G_m^s; T_j)$ using the kr-regular graphs B_e^r for each edge e in the tree T_j. Notice that G_i^s is 1-balanced by Corollary 4 and $g(G_i^s) = st$ for $i = 1, \ldots, m$. By (23), we have

$$\frac{g(G_i^s)}{n} \le kr \le mg(G_i^s).$$

By Theorem 12, A_j is 1-balanced for $j = 1, 2, \ldots, r$.

Claim. $A_{rs} = \cup_{j=1}^r A_j$.

Proof of Claim. Each A_j, $1 \le j \le r$ has a copy of G_i^s for each $i \in \{1, 2, \ldots, m\}$. Hence the edges of G_i appear rs times in $\cup_{j=1}^r A_j$.

Now, let $e = (u, v)$ be an edge in L. In A_{rs}, we have B_e^{rs} between G_u and G_v. On the other hand, B_e^r appears in exactly s of $A_1, A_2, \ldots, A_r$ since e appears in exactly s of $T_1, T_2, \ldots, T_r$. Thus we can find B_e^{rs} in $\cup_{j=1}^r A_j$. Hence the claim.

Thus A_{rs} is 1-balanced and the theorem follows. □

Corollary 14. *If connected graphs G_1 and G_2 are both 1-balanced, then the Cartesian product $G_1 \times G_2$ is 1-balanced.*

Proof. There are two ways to view $G_1 \times G_2$ as a generalized Cartesian product. $G_1 \times G_2 = A_1(G_1, G_1, \ldots, G_1; G_2)$ with suitable choices of the bipartite graphs B_{ij}. Similarly, $G_1 \times G_2 = G_2 \times G_1 = A_1(G_2, G_2, \ldots, G_2; G_1)$ with suitable choices of the bipartite graphs B_{ij}.

We first prove that either

$$\frac{g(G_1)}{|V(G_1)|g(G_2)} \le 1, \tag{24}$$

or

$$\frac{g(G_2)}{|V(G_2)|g(G_1)} \leq 1 \tag{25}$$

holds. Suppose both (24) and (25) do not hold. Then we have

$$g(G_1) > |V(G_1)|g(G_2) > |V(G_1)||V(G_2)|g(G_1),$$

a contradiction since $|V(G_1)||V(G_2)| \geq 1$.

Now, if (24) holds, then by Theorem 13 with $k = 1$ (noting that $k = 1$ satisfies (12)), $A_1(G_1, G_1, \ldots, G_1; G_2) = G_1 \times G_2$ is 1-balanced. Similarly, if (25) holds, then by Theorem 13 with $k = 1$, $A_1(G_2, G_2, \ldots, G_2; G_1) = G_1 \times G_2$ is 1-balanced. □

Appendix

(Sufficiency of L strictly balanced) We note that $N_\alpha = 0$ for all constant labelings α. For a contradiction, suppose L is strictly balanced while there is a non-constant, non-negative, integer-valued function α on $V(L)$ with $N_\alpha \geq 0$. Choose non-constant α_0 such that $N_{\alpha_0} \geq 0$ and $s = \max_{1\leq i\leq m} \alpha_0(v_i)$ is as small as possible. Since α_0 is not constant, there is a $j \in \{1, 2, \ldots, m\}$ such that $\alpha_0(v_j) < s$. Then the integer $s \geq 1$ since $\alpha_0(v_j) \geq 0$.

Let $S := \{v_i : \alpha_0(v_i) = s\}$; by definition of s and j, $S \notin \{\emptyset, V\}$. Consider the function α_0' defined by $\alpha_0'(v_i) = \alpha_0(v_i)$ if $v_i \notin S$ and $\alpha_0(v_i) - 1$ if $v_i \in S$. Thus $\max_{1\leq i\leq m} \alpha_0(v_i) < s$.

We claim that $N_{\alpha_0'} > N_{\alpha_0}$, and thus α_0' is non-constant.

Let $L' := L[S]$, and denote $m' := |V(L')| = |S|$ and $\ell' := |E(L')|$. Exactly as in the case of non-strictly balanced,

$$N_{\alpha_0'} - N_{\alpha_0} = m'\left(-\frac{\ell'}{m'} + \frac{\ell}{m}\right).$$

But this is greater than zero either because L is strictly balanced and so $\frac{\ell'}{m'} = b(L') < b(L) = \frac{\ell}{m}$ or because $\ell' = 0$.

Thus $N_{\alpha_0'} > N_{\alpha_0} \geq 0$, so α_0' is not constant. This is a contradiction to the choice of α_0 and the minimality of s. The contradiction proves sufficiency.

(Necessity of L strictly balanced) Suppose $N_\alpha < 0$ for all non-constant labelings α. Let L' be any non-trivial vertex-induced subgraph of L, $L' \neq L$, and suppose L' has m' vertices and ℓ' edges. Define α on $V(L)$ by letting $\alpha(v) = 1$ if $v \in V(L')$ and 0 if $v \notin V(L')$. Then α is not constant, so

$$\frac{1}{m}\sum_{r=1}^{m} \alpha(v_r) = \frac{m'}{m},$$

and

$$\begin{aligned} 0 &> N_\alpha \\ &= \sum_{v_iv_j\in E(L')}\left(1 - \frac{m'}{m}\right) + \sum_{v_iv_j\notin E(L')}\left(-\frac{m'}{m}\right) \\ &= \ell' - \frac{m'\ell'}{m} - \frac{m'\ell}{m} + \frac{m'\ell'}{m} \\ &= \ell' - \frac{m'\ell}{m} \\ &= m'\left(\frac{\ell'}{m'} - \frac{\ell}{m}\right). \end{aligned}$$

Since $m' > 0$, we have $\frac{\ell'}{m'} < \frac{\ell}{m}$ (*i.e.*, $b(L') < b(L)$), so L is strictly balanced.

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