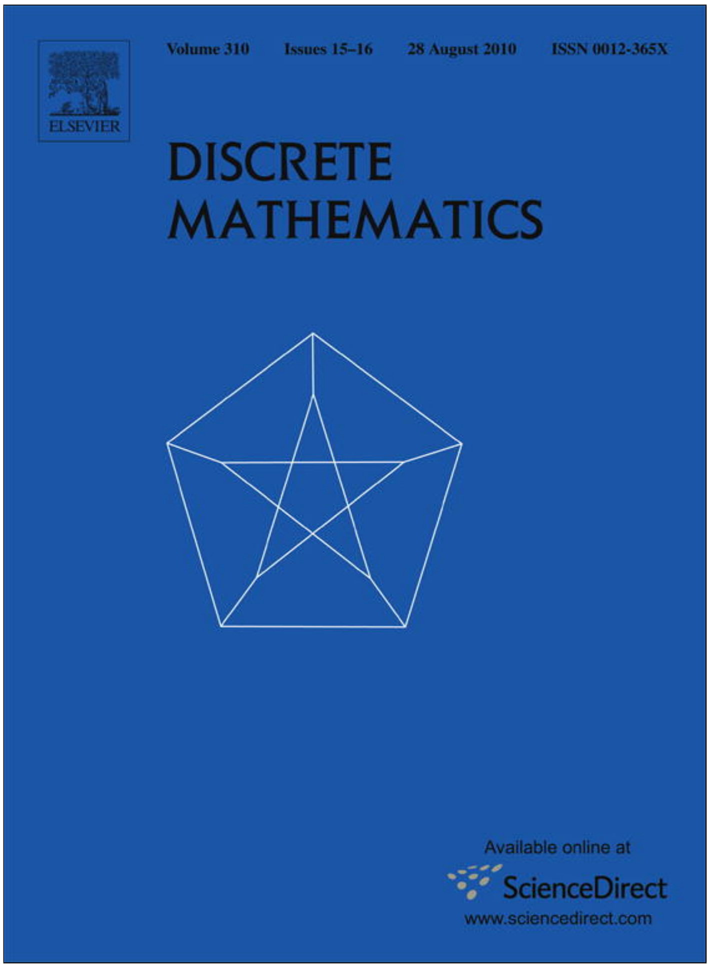
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# The Chvátal–Erdös condition for supereulerian graphs and the Hamiltonian index

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ABSTRACT

A classical result of Chvátal and Erdös says that the graph $G$ with connectivity $\kappa(G)$ not less than its independence number $\alpha(G)$ (i.e. $\kappa(G) \geq \alpha(G)$) is Hamiltonian. In this paper, we show that the 2-connected graph $G$ with $\kappa(G) \geq \alpha(G) - 1$ is one of the following: supereulerian, the Petersen graph, the 2-connected graph with three vertices of degree two obtained from $K_{2,3}$ by replacing a vertex of degree three with a triangle, one of the 2-connected graphs obtained from $K_{2,3}$ by replacing a vertex of degree two with a complete graph of order at least three and by replacing at most one branch of length two in the resulting graph with a branch of length three, or one of the graphs obtained from $K_{2,3}$ by replacing at most two branches of $K_{2,3}$ with a branch of length three. We also show that the Hamiltonian index of the simple 2-connected graph $G$ with $\kappa(G) \geq \alpha(G) - t$ is at most $\lfloor \frac{2t+2}{3} \rfloor$ for every nonnegative integer $t$. The upper bound is sharp.


## 1. Introduction

We denote by $\kappa(G)$ and $\alpha(G)$ the connectivity and the independence number of $G$ respectively, considering only the simple graph $G$ with $\kappa(G) \geq 2$ in the following. Chvátal and Erdös gave the following well-known sufficient condition for a graph to be Hamiltonian.

**Theorem 1** ([7]). *If $\kappa(G) \geq \alpha(G)$, then $G$ is Hamiltonian.*

There are many extensions of Theorem 1, one of which aims at showing what properties a graph has when it satisfies a stronger Chvátal–Erdös condition.

**Theorem 2** ([9]). *Let $t$ be a nonnegative integer. If $\kappa(G) \geq \alpha(G) + t$, then any system of disjoint paths of total length at most $t$ is contained in a Hamiltonian cycle of $G$.*

An immediate consequence of Theorem 2 is that $G$ is Hamilton connected if $\kappa(G) \geq \alpha(G)+1$. A graph $G$ is called *trivial* if it has only one vertex, and is called *even* if each vertex of $G$ has even degree. A graph is called *eulerian* if it is connected and even.

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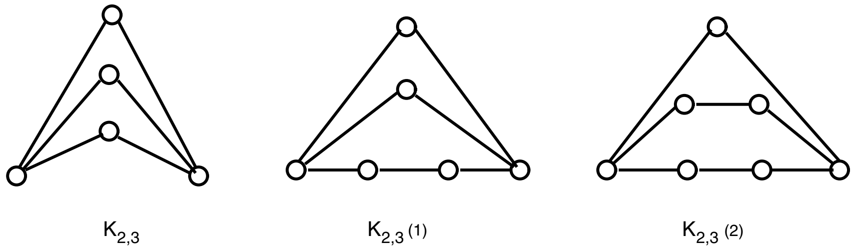


**Fig. 1.** Three graphs $K_{2,3}$, $K_{2,3}(1)$, $K_{2,3}(2)$.

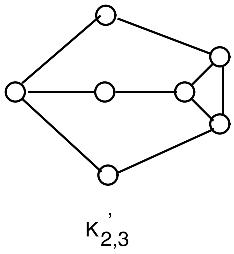


**Fig. 2.** The graph $K'_{2,3}$.

A graph is called *supereulerian* if it contains a spanning eulerian subgraph. Obviously, a Hamiltonian graph is supereulerian, whereas the converse is not generally true. The first aim of this paper is to consider whether a graph with a slightly weaker Chvátal–Erdös condition is supereulerian.

**Theorem 3.** *Let $G$ be a 2-connected simple graph. If $\kappa(G) \geq \alpha(G) - 1$, then exactly one of the following holds.*

(a) *$G$ is supereulerian;*
(b) *$G \in \{$the Petersen graph, $K_{2,3}$, $K_{2,3}(1)$, $K_{2,3}(2)$, $K'_{2,3}\}$: see* Figs. 1 and 2*;*
(c) *$G$ is one of the two 2-connected graphs obtained from $K_{2,3}$ and $K_{2,3}(1)$ by replacing a vertex whose neighbors have degree three in $K_{2,3}$ and $K_{2,3}(1)$ with a complete graph of order at least three.*

The *line graph* $L(G)$ of $G = (V(G), E(G))$ has $E(G)$ as its vertex set, where two vertices are adjacent in $L(G)$ if and only if the corresponding edges are incident in $G$. The *m-iterated line graph* $L^m(G)$ is defined recursively by $L^0(G) = G$, $L^1(G) = L(G)$, and $L^m(G) = L(L^{m-1}(G))$. The *Hamiltonian index* of a graph $G$, denoted by $h(G)$, is the smallest integer $m$ such that $L^m(G)$ is Hamiltonian. In Section 2, we will show that the following known result is a consequence of Theorem 3.

**Theorem 4** ([2])**.** *If $\kappa(G) \geq \alpha(G) - 1$, then $L(G)$ is Hamiltonian, i.e., $h(G) \leq 1$.*

The following result extends Theorem 1 for triangle-free graphs, showing that a graph with a weaker Chvátal–Erdös condition has a weaker result.

**Theorem 5** ([8])**.** *If $G$ is a 2-connected triangle-free graph of order $n$ with $\kappa(G) \geq \frac{\alpha(G)+2}{2}$, then every longest cycle $C$ in $G$ is dominating (i.e., every edge of $G$ has at least one end vertex in $C$), and $G$ has a cycle of length at least* $\min\{n - \alpha(G) + \kappa(G), n\}$.

There are many other ways to extend Theorem 1 (see the survey paper [11] and update papers [1,13,14]). Our second aim in this paper is to extend Theorems 1 and 4 by presenting an upper bound for $h(G)$.

**Theorem 6.** *Let $G$ be a 2-connected simple graph and let $t$ be a nonnegative integer. If $\kappa(G) \geq \alpha(G) - t$, then $h(G) \leq \lfloor \frac{2t+2}{3} \rfloor$.*

In Section 2, we will provide some auxiliary results that are applied in Sections 3 and 4 to prove our main results. The sharpness of Theorem 6 is presented in the last section.

## 2. Preliminaries

### *2.1. Catlin's reduction method for supereulerian graphs*

For a simple graph $G$, let $O(G)$ denote the set of odd degree vertices in $G$. In [3] Catlin defined collapsible graphs. Given a subset $R \subseteq V(G)$, with $|R|$ being even, a subgraph $\Gamma$ of $G$ is an *$R$-subgraph* if $O(\Gamma) = R$ and $G - E(\Gamma)$ is connected. A graph $G$ is *collapsible* if, for any even subset $R$ of $V(G)$, $G$ has an $R$-subgraph. Catlin showed in [3] that every vertex of $G$ lies in a unique maximal collapsible subgraph of $G$. If the graph $G$ itself is collapsible, then we let $R = O(G)$. Obviously, $|R|$ is even. So, there exists an $R$-subgraph $\Gamma$ of $G$ such that $O(\Gamma) = R$ and $G - E(\Gamma)$ is connected. Hence, $G - E(\Gamma)$ must be a spanning eulerian subgraph. Note that $C_4$ is supereulerian but not collapsible. Thus every collapsible graph is supereulerian whereas the converse is not generally true. If $H$ is a connected subgraph of a graph $G$, then $G/H$ denotes the graph by contracting $H$ obtained from $G$ by replacing $H$ by a vertex $v_H$ such that the number of edges in $G/H$ joining any $v \in V(G) - V(H)$ to $v_H$ in $G/H$ equals the number of edges joining $v$ in $G$ to $H$. In [3], Catlin showed that any graph $G$ has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$ such that $\bigcup_{i=1}^{c} V(H_i) = V(G)$. The *reduction* of $G$, denoted by $G'$, is the graph obtained from $G$ by contracting each $H_i$ $(1 \le i \le c)$ to a single vertex $v_{H_i}$. We call $H_i$ the pre-image of $v_{H_i}$, denoted by $R^{-}_{G'}(v_{H_i})$. If $H_i$ contains only one vertex, then $v_{H_i}$ is called trivial. If the reduction of $G$ is itself, then $G$ is called a *reduced graph.*

**Theorem 7** ([3]). *Let $G$ be a connected graph and $G'$ the reduction of $G$. Then, each of the following holds.*

(a) *$G$ is supereulerian if and only if $G'$ is supereulerian.*
(b) *$G'$ is triangle-free with $\delta(G') \le 3$.*

**Theorem 8** ([12]). *If $G$ is a 4-edge-connected graph, then $G$ is supereulerian.*

The following theorem is obvious, where $\kappa'(G)$ denotes the edge-connectivity of $G$.

**Theorem 9.** *Let $G'$ be the reduction of $G$. Then, each of the following holds.*

(a) $\alpha(G') \le \alpha(G)$.
(b) *If $G'$ is nontrivial, then $\kappa'(G') \ge \kappa'(G)$.*

**Theorem 10** ([5]). *If $G$ is a reduced graph and $\alpha(G) \ge 4$, then we have $(\delta(G)\alpha(G) + 4)/2 \le |V(G)| \le 4\alpha(G) - 5$.*

**Theorem 11** ([6]). *Let $G$ be the graph of order $n \le 11$. If $\kappa'(G) \ge 3$, then $G$ is either collapsible or the Petersen graph.*

### 2.2. Hamiltonian iterated line graphs

Let $G$ be a graph. For any two subgraphs $H_1$ and $H_2$ of $G$, the *distance* $d_G(H_1, H_2)$ between $H_1$ and $H_2$ is defined to be the minimum of the distances $d_G(v_1, v_2)$ over all pairs with $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$. If $d_G(e, H) = 0$ for an edge $e$ of $G$, then we say that $H$ *dominates* $e$. A subgraph $H$ of $G$ is called *dominating* if it dominates all edges of $G$. There is a characterization of a graph $G$ with $h(G) \le 1$ that involves the existence of a dominating eulerian subgraph in $G$.

**Theorem 12** ([10]). *Let $G$ be a graph with at least three edges. Then $h(G) \le 1$ if and only if $G$ has a dominating eulerian subgraph.*

By Theorem 12, all supereulerian graphs and the Petersen graph have Hamiltonian line graphs, and those graphs obtained from $K_{2,3}$ described in Theorem 3 have Hamiltonian line graphs, which shows that Theorem 4 is a consequence of Theorem 3.

For each integer $i \ge 0$, define $V_i(G) = \{v \in V(G) : d_G(v) = i\}$. *A branch* in $G$ is a nontrivial path in which each inner vertex lies in $V_2(G)$ and neither end vertex lies in $V_2(G)$. We denote by $\mathcal{B}(G)$ the set of branches of $G$ and by $\mathcal{B}_1(G)$ the subset of $\mathcal{B}(G)$ in which every branch has an end vertex in $V_1(G)$. For any subgraph $H$ of $G$, we denote by $\mathcal{B}_H(G)$ the set of branches of $G$ whose edges are all in $H$.

The following theorem can be considered as an analogue of Theorem 12 for $L^k(G)$.

**Theorem 13** ([16]). *Let $G$ be a connected graph that is not a path, and let $k \ge 2$ be an integer. Then, $h(G) \le k$ if and only if $EU_k(G) \ne \emptyset$ where $EU_k(G)$ denotes the set of those subgraphs $H$ of $G$ which satisfy the following five conditions.*

(I) *$H$ is an even subgraph.*
(II) $V_0(H) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$.
(III) *$d_G(H_1, H - H_1) \le k - 1$ for every induced subgraph $H_1$ of $H$ with $\emptyset \ne V(H_1) \subsetneq V(H)$.*
(IV) *$|E(B)| \le k + 1$ for every branch $B \in \mathcal{B}(G) \setminus \mathcal{B}_H(G)$.*
(V) *$|E(B)| \le k$ for every branch $B$ in $\mathcal{B}_1(G)$.*

Note that, if we only consider 2-connected graphs, then Condition (V) in the definition of $EU_k(G)$ is superfluous.

## 3. Proof of Theorem 3

Before presenting our proof, we start by proving three useful lemmas.

**Lemma 14.** *Let $G'$ be a reduced graph such that $\kappa'(G') \ge 2$ and $\alpha(G') \le 3$, then $\kappa(G') \ge 2$.*

**Proof of Lemma 14.** Assume, in contrast, that $v$ is a cut vertex of $G'$, then $d_{G'}(v) \geq 4$ by $\kappa'(G') \geq 2$. Since $G'$ is triangle-free, $v$ has at least four neighbors in $G'$ which form an independent set of $G'$, a contradiction. □

**Lemma 15.** *Let $G'$ be a reduced graph such that $\kappa(G') = 2$ and $\alpha(G') = 3$. Then, exactly one of the following holds.*

(a) *$G'$ is supereulerian.*
(b) *$G'$ is isomorphic to one of the three graphs depicted in Fig. 1.*

**Proof of Lemma 15.** Notice that a loop graph with one vertex is collapsible, and a two-vertex graph with parallel edges is also collapsible. Since $G'$ is reduced, $G'$ is a simple graph.

Let $k = \kappa(G')$, and $C = u_1u_2 \cdots u_1$ be a closed trail with a maximal number of vertices in $C$ (i.e., $|V(G') \setminus V(C)|$ is minimal). Then, $C$ is a cycle (otherwise there is a vertex $v$ of $C$ with $d_C(v) \geq 4$ and hence the neighbors of $v$ consist of an independent set of at least four vertices of $G'$ by Theorem 7, contradicting that $\alpha(G') = 3$). We denote by $u^+$ ($u^-$, respectively) the successor (the predecessor, respectively) of $u$ on $C$, and let $u^{++} = (u^+)^+$, $u^{--} = (u^-)^-$ and so on; a portion of $C$ from $u$ to $v$ means $uu^+u^{++} \ldots v^-v$. Since $G'$ is triangle-free, we obtain that $u^-u^+ \notin E(G)$ for any vertex $u$ in $C$. Since $\delta(G') \geq \kappa(G')$, and every graph with $\delta(G') \geq 2$ has a cycle of length at least $\delta(G') + 1$, $C$ has at least $k + 1$ vertices.

If $C$ is not a spanning cycle of $G'$, then there is a component $H$ of $G' - V(C)$. We will prove that $G'$ is isomorphic to one of those graphs depicted in Fig. 1.

Since $\kappa(G') = 2$, $C$ has at least two vertices having neighbors in $V(H)$. Let $x, y$ be any two vertices of $C$ having neighbors in $V(H)$.

We claim that none of $\{x^-, x^+, y^-, y^+\}$ has a neighbor in $H$; otherwise, say, $x^+$ has a neighbor in $H$, then we can obtain a cycle from $C$ by adding an $x, x^+$-path through $H$ and by deleting the edge $xx^+$, which contains more vertices than $C$, contradicting the maximality of $|V(C)|$. We have the following two facts.

**Claim 1.** *$x^+y^+ \notin E(G')$ and $x^-y^- \notin E(G')$.*

**Proof of Claim 1.** By symmetry, we only need to prove that $x^+y^+ \notin E(G')$. Since $C$ is a cycle and $x \neq y^+$, $x^+y^+ \notin E(C)$. Suppose $x^+y^+ \in E(G') \setminus E(C)$, then we can obtain a cycle that contains more vertices than $C$ by including $x^+y^+$, the portions of $C$ from $x^+$ to $y$ and from $y^+$ to $x$, and an $x, y$-path through $H$, contradicting the maximality of $|V(C)|$. □

**Claim 2.** *Either $x^+ = y^-$ or $x^+y^- \in E(C)$ and either $x^- = y^+$ or $x^-y^+ \in E(C)$.*

**Proof of Claim 2.** By symmetry, we only need to prove the second part of the claim. We will prove this by contradiction. If possible, suppose that neither $x^- = y^+$ nor $x^-y^+ \in E(C)$, then there are at least five vertices in the portion of $C$ from $y$ to $x$. Suppose first that $x^-y^+ \notin E(G')$; then, by Claim 1, $S = \{x^+, x^-, y^+\}$ is an independent set such that none of $S$ is adjacent to $H$; hence we can get an independent set of four vertices by adding one vertex of $H$ to $S$, contradicting $\alpha(G') = 3$. Now, suppose that $x^-y^+ \in E(G')$; then, by the fact that $G'$ is triangle-free and by the assumption that $x^-y^+ \notin E(C)$, $x^{--}y^+ \notin E(G')$ and there are at least six vertices in the portion of $C$ from $y$ to $x$. Note that $x^{--}$ is not adjacent to any vertex of $H$; otherwise, we can obtain a cycle that contains more vertices than $C$ by including $x^-y^+$, the portions of $C$ from $y^+$ to $x^{--}$ and from $x^-$ to $y$, and an $x^{--}, y$-path through $H$, contradicting $\alpha(G') = 3$. Hence, $x^{--}x^+ \in E(G')$; otherwise $\{x^+, x^{--}, y^+, h\}$ is an independent set of four vertices, where $h \in V(H)$, contradicting $\alpha(G') = 3$. Note that $x^{--}x^+ \notin E(C)$. Hence the cycle obtained from $C$ by deleting three edges $xx^+, x^{--}x^-, yy^+$, and by adding two edges $x^+x^{--}, y^+x^-$ and an $x, y$-path through $H$, contains more vertices than $C$, contradicting the maximality of $|V(C)|$. This proves Claim 2. □

By Claim 2, $C$ is a cycle of length at most six and any vertex in $V(C) \setminus \{x, y\}$ is not adjacent to $H$. This implies that $\{x, y\}$ is a 2-vertex cut-set of $G'$. By the choice of $C$, $H$ has at most two vertices; otherwise, suppose $H$ has at least three vertices, then $x$ and $y$ must have different neighbors $h_x$ and $h_y$ in $H$, respectively, since $\kappa(G') = 2$. No matter whether $h_x$ and $h_y$ are adjacent or not, there must be a path $P(h_x, h_y)$ of $H$ between $h_x$ and $h_x$ with length at least two, since $\kappa(G') = 2$. Hence, we will obtain a longer cycle than $C$ by including the path $P(h_x, h_y)$, $xh_x$, $yh_y$ and a longest portion of $C$ between $x$ and $y$, contradicting the maximality of $|V(C)|$. So $|V(H)| \leq 2$.

**Claim 3.** *$|V(H)| = 1$ and $G' - V(C)$ does not have more than one component.*

**Proof of Claim 3.** Noticing that $4 \leq |V(C)| \leq 6$, we only need to discuss the following two cases.
**Case I**: $|V(C)| = 6$.

By Claim 2, we may assume $C = xx^+y^-yy^+x^-x$. Then $|V(H)| = 1$; otherwise, since $\kappa(G') = 2$, $|V(H)| = 2$ and $G'$ is triangle-free, we may assume that $V(H) = \{h_x, h_y\}$ with $xh_x, yh_y \in E(G')$ and $xh_y \notin E(G')$, and hence $\{x, y^-, y^+, h_y\}$ is an independent set of $G'$, contradicting $\alpha(G') = 3$. Hence, $x$ and $y$ share the same neighbor $v_H$ in $H$. It follows that $xy \notin E(G')$ since $G'$ is triangle-free. Therefore $C$ is an induced cycle of $G'$ by the fact that $G'$ is triangle-free and by Claims 1 and 2.

If $G' - V(C)$ has one more component $K$, then $|V(K)| = 1$ by the same argument on $H$. Moreover, the only vertex $v_K$ in $K$ is only adjacent to two vertices of $C$ and the two vertices must be $\{x, y\}$ or $\{x^-, y^-\}$ or $\{x^+, y^+\}$ by the maximality of $|V(C)|$. If the two vertices are $\{x, y\}$, then $\{v_H, v_K, x^-, x^+\}$ is an independent set of order 4, a contradiction. If the two vertices are $\{x^-, y^-\}$, then $\{v_H, v_K, x^+, y^+\}$ is an independent set of order 4, a contradiction. If the two vertices are $\{x^+, y^+\}$, then $\{v_H, v_K, x^-, y^-\}$

is an independent set of order 4, a contradiction. Hence, $G' - V(C)$ does not have more than one component in this situation and $G'$ is the graph $K_{2,3}(2)$ depicted in Fig. 1.

**Case II**: $|V(C)| = 4$ or 5.

We claim $|V(H)| = 1$ (otherwise, since $\kappa(G') = 2$ and $|V(H)| = 2$, we may assume that $V(H) = \{h_x, h_y\}$ with $xh_x, yh_y, h_xh_y \in E(G')$ and obtain a longer cycle than $C$ by including the path $xh_xh_yy$ and a longest portion of $C$ between $x$ and $y$, contradicting the maximality of $|V(C)|$). Since $G'$ is triangle-free, $xy \notin E(G')$. It follows that $C$ is an induced cycle of $G'$, by Claims 1 and 2. Note that the vertex $v_H$ of $H$ is not adjacent to any vertex of $V(C) \setminus \{x, y\}$.

If $G' - V(C)$ has one more component $K$, then $|V(K)| = 1$ by the same argument on $H$. Moreover, the only vertex $v_K$ in $K$ is only adjacent to two nonadjacent vertices of $C$.

If $|V(C)| = 5$, then $v_K$ is only adjacent to the two vertices $\{x, y\}$ or $\{x, y^-\}$ or $\{x^-, y^-\}$ or $\{x^-, x^+\}$ or $\{x^+, y\}$. Notice that $x^- = y^+$. By symmetry, we only need to discuss the following three cases: if the two vertices are $\{x, y\}$, then $\{v_H, v_K, x^-, y^-\}$ is an independent set of order 4, a contradiction; if the two vertices are $\{x, y^-\}$, then $\{v_H, v_K, x^-, x^+\}$ is an independent set of order 4, a contradiction; if the two vertices are $\{x^-, x^+\}$, then $v_Hxx^-v_Kx^+y^-yv_H$ is a longer cycle than $C$, a contradiction. Hence $G' - V(C)$ does not have more than one component in this situation; and then $G'$ is the graph $K_{2,3}(1)$ depicted in Fig. 1.

If $|V(C)| = 4$, then $v_K$ is only adjacent to the two vertices $\{x, y\}$ or $\{x^-, y^-\}$. If the two vertices are $x, y$, then $\{v_H, v_K, x^-, y^-\}$ is an independent set of order 4, a contradiction; if the two vertices are $\{x^-, y^-\}$, then $v_Hxx^-v_Ky^-yv_H$ is a longer cycle than $C$, a contradiction. Hence $G' - V(C)$ does not have more than one component in this situation, and then $G' = K_{2,3}$. So we finish the proof of Claim 3. □

Above all, $G'$ is isomorphic to one of those graphs depicted in Fig. 1. This completes the proof of Lemma 15. □

**Lemma 16.** *Let $G'$ be the reduction of a simple graph $G$. Then, the following statements hold.*

(a) *If $v$ is a nontrivial vertex of $G'$ with $d_{G'}(v) = 2$, then there exists at least one vertex $v'$ of $R^-_{G'}(v)$ such that $v'$ is not adjacent to any vertex of $\bigcup_{s \in V(G')\setminus\{v\}} R^-_{G'}(s)$.*

(b) *if $u$ is a nontrivial vertex of $G'$ with $d_{G'}(u) = 3$, then for any pair of vertices $\{x, y\} \subseteq N_{G'}(u)$, there exists at least one vertex $u'$ of $R^-_{G'}(u)$ such that $u'$ is not adjacent to any vertex of $R^-_{G'}(x) \cup R^-_{G'}(y)$.*

**Proof of Lemma 16.** Note that, by the definition of contraction, $R^-_{G'}(s)$ is collapsible for any nontrivial vertex $s$ of $V(G')$; hence, since $G$ is simple, $R^-_{G'}(s)$ has at least three vertices. Therefore, since $d_{G'}(v) = 2$, there are at most two vertices $v''$, $v'''$ of $R^-_{G'}(v)$ that are adjacent to one or two vertices of $\bigcup_{s \in N_{G'}\setminus\{v\}} R^-_{G'}(s)$ by the definition of contraction. Hence, there exists at least one vertex $v' \in R^-_{G'}(v) \setminus \{u'', u'''\}$ that is not adjacent to any vertex of $\bigcup_{s \in V(G')\setminus\{v\}} R^-_{G'}(s)$. This proves the first statement. For a nontrivial vertex $u$ of $G'$ with $d_{G'}(u) = 3$, there is exactly one vertex of $R^-_{G'}(u)$ that is adjacent to some vertex of $R^-_{G'}(s)$ for any $s \in N_{G'}(u)$ by the definition of contraction. Therefore, since $|V(R^-_{G'}(s))| \geq 3$ for any nontrivial vertex $s$ of $G'$, the second statement holds. This completes the proof of Lemma 16. □

Now we finish the proof of Theorem 3.

**Proof of Theorem 3.** Let $G'$ be the reduction of $G$. If $G' = K_1$, then $G$ is supereulerian by Theorem 7. So, we only need to consider the case when $G'$ is not $K_1$.

If $\kappa(G) \geq 4$, then $G$ is supereulerian by Theorem 8.

If $\kappa(G) = 3$, then, by hypothesis, we have that $\alpha(G) \leq 4$. We let $\alpha(G) = 4$ since, otherwise, $G$ is Hamiltonian by Theorem 1. Theorem 9 gives that $\alpha(G') \leq 4$ and $\kappa'(G') \geq \kappa'(G) \geq \kappa(G) \geq 3$. First, suppose that $\alpha(G') = 4$, then from Theorem 10 we deduce that $|V(G')| \leq 11$. Hence, $G'$ is either collapsible or the Petersen graph by Theorem 11. If $G'$ is collapsible, then $G'$ is supereulerian and $G$ is hence supereulerian by Theorem 7. If $G'$ is the Petersen graph, then we claim that every vertex of $G'$ is trivial. Assume, in contrast, that there exists a nontrivial vertex $w$ of $G'$. Let $N_{G'}(w) = \{x, y, z\}$. By Lemma 16(b), there exists at least one vertex $w' \in R^-_{G'}(w)$ such that $w'$ is not adjacent to any vertex of $\cup_{v\in\{x,y\}} R^-_{G'}(v)$. Let $N_{G'}(z) \setminus \{w\} = \{u, v\}$. Taking $x' \in R^-_{G'}(x)$, $y' \in R^-_{G'}(y)$, $u' \in R^-_{G'}(u)$ and $v' \in R^-_{G'}(v)$, we obtain an independent set $\{w', x', y', u', v'\}$ of $G$, contradicting that $\alpha(G) = 4$. This shows that $G'$ has no nontrivial vertex. Hence, $G = G'$, i.e., $G$ is the Petersen graph. Now, suppose $\alpha(G') \leq 3$. Since $\kappa'(G') \geq 3$, $\kappa(G') \geq 2$ by Lemma 14. In the case when $\kappa(G') \geq \alpha(G')$, we have that $G'$ is Hamiltonian (hence supereulerian) by Theorem 1. So $G$ is supereulerian by Theorem 7. It remains the case when $\kappa(G') = 2$ and $\alpha(G') = 3$. By Lemma 15, $G'$ is isomorphic to one of the three graphs in Fig. 1, which contradicts that $\kappa'(G') \geq 3$.

It remains to consider the case that $\kappa(G) = 2$ and $\alpha(G) = 3$. Theorem 9 gives that $\alpha(G') \leq 3$ and $\kappa'(G') \geq \kappa'(G) \geq \kappa(G) = 2$. Hence, $\kappa(G') \geq 2$ by Lemma 14. We only need to consider the case that $\kappa(G') = 2$ and $\alpha(G') = 3$ since, in all other cases, $G'$ is Hamiltonian and $G$ is hence supereulerian. By Lemma 15, $G'$ is either supereulerian or isomorphic to one of $\{K_{2,3}, K_{2,3}(1), K_{2,3}(2)\}$. If $G'$ is supereulerian, then so is $G$ by Theorem 7.

Now, suppose that $G'$ is isomorphic to one of $\{K_{2,3}, K_{2,3}(1), K_{2,3}(2)\}$. Then, we claim that $G'$ has at most one nontrivial vertex. Assume, in contrast, that $G'$ has two nontrivial vertices $u$ and $v$. Note that $G$ is a simple graph. Firstly, suppose $d_{G'}(u) = d_{G'}(v) = 2$. Then, by Lemma 16(a), there exist at least two vertices $u'$ and $v'$ in $R^-_{G'}(u)$ and $R^-_{G'}(v)$, respectively, such that they are not adjacent to any vertex of $\bigcup_{x\in V(G')\setminus\{u,v\}} R^-_{G'}(x)$. Taking any pair of vertices $y'$ in $R^-_{G'}(y)$ and $z'$ in $R^-_{G'}(z)$ for the two vertices $y$ and $z$ of degree 3 in $G'$, we obtain that $\{u', v', x', y'\}$ is an independent set of $G$, contradicting $\alpha(G) = 3$. Secondly, suppose $d_{G'}(u) = d_{G'}(v) = 3$, and $x, y$ are vertices with degree two in $G'$ and $xy \notin E(G')$. Then, by Lemma 16(b), there exist at least two vertices $x'$ and $y'$ in $R^-_{G'}(x)$ and $R^-_{G'}(y)$, respectively, such that they are not adjacent to any of $R^-_{G'}(u) \cup R^-_{G'}(v)$. It follows that $\{x', y', u', v'\}$ is an independent set of $G$ (where $u' \in R^-_{G'}(u)$ and $v' \in R^-_{G'}(v)$), contradicting $\alpha(G) = 3$. Finally, suppose that $d_{G'}(u) \neq d_{G'}(v)$, say, $d_{G'}(u) = 3$, $d_{G'}(v) = 2$. Take a pair of vertices $\{x, y\} \subseteq V_2(G') \setminus \{v\}$ such that $xy \notin E(G')$. Hence, by Lemma 16(b), there exists at least one vertex $u'$(say) of $R^-_{G'}(u)$ that is not adjacent to any vertex of $R^-_{G'}(x) \cup R^-_{G'}(y)$, and there is a vertex $v'$ of $R^-_{G'}(v)$ that is not adjacent to any vertex of $\bigcup_{s\in V(G')\setminus\{v\}} R^-_{G'}(s)$. Taking $x' \in R^-_{G'}(x)$ and $y' \in R^-_{G'}(y)$, we obtain an independent set $\{x', y', v', u'\}$ of $G$, contradicting $\alpha(G) = 3$. This proves that $G'$ has at most one nontrivial vertex if $G' \in \{K_{2,3}, K_{2,3}(1), K_{2,3}(2)\}$.

If $G'$ has no nontrivial vertices, then $G = G'$ and $G$ is isomorphic to one of those graphs shown in Fig. 1. Now, suppose that $G'$ has exactly one nontrivial vertex $u$. Note that $G$ is a simple graph. Hence, if $d_{G'}(u) = 2$, then, by Lemma 16(a), there is at least one vertex $u'$ of $R^-_{G'}(u)$ such that $u'$ is not adjacent to any vertex in $\bigcup_{x\in V(G')\setminus\{u\}} R^-_{G'}(x) = V(G') \setminus \{u\}$; if $d_{G'}(u) = 3$ then, by Lemma 16(b), for any pair of vertices in $N_{G'}(u)$, say $s$ and $t$, there exists a vertex in $R^-_{G'}(u)$ which is adjacent to neither $s$ nor $t$, since $|V(R^-_{G'}(u))| \geq 3$. Hence, $G' = K_{2,3}(2)$ has no nontrivial vertices. For $G' \in \{K_{2,3}, K_{2,3}(1)\}$, $R^-_{G'}(u)$ must be a complete graph. Suppose, in contrast, that there exist two nonadjacent vertices $s, t$ in $R^-_{G'}(u)$. If $d_{G'}(u) = 2$, then the vertices $s, t$ and the other two trivial vertices (that are not adjacent to $u$ in $G'$) of degree two in $G'$ induce an independent set in $G$, contradicting $\alpha(G) = 3$. If $d_{G'}(u) = 3$, then there exists one vertex in $R^-_{G'}(u)$ that is not adjacent to any one of the three independent trivial vertices of degree two in $G'$, contradicting $\alpha(G) = 3$. Hence, $R^-_{G'}(u)$ is a complete graph. If $G' = K_{2,3}$ and $d_{G'}(u) = 3$, then $R^-_{G'}(u)$ is a triangle and the original graph $G$ is isomorphic to $K'_{2,3}$ (otherwise, by a way similar to the proof of Lemma 16(b), we can prove that there exists a vertex in $R^-_{G'}(u)$ that is not adjacent to any one of the three nonadjacent vertices of degree two in $G'$, contradicting $\alpha(G) = 3$). If $G' = K_{2,3}(1)$, then $d_{G'}(u) = 2$ (otherwise, by Lemma 16(b) and by the fact that there exists a vertex of degree two that is not adjacent to $u$, there exists a vertex in $R^-_{G'}(u)$ that is not adjacent to any one of the three independent vertices of degree two in $G'$, contradicting $\alpha(G) = 3$). Note that there exists a vertex $w$ of $R^-_{G'}(u)$ such that it is not adjacent to any vertex of $V(G') \setminus \{u\}(= \bigcup_{z\in V(G')\setminus\{u\}} R^-_{G'}(z))$. We have that any vertex of $N_{G'}(u)$ having degree three in $G'$ (otherwise $\{w\} \bigcup (V_2(G') \setminus \{u\}) = \{w\} \bigcup (\bigcup_{z\in V_2(G')\setminus\{u\}} R^-_{G'}(z))$ is an independent set of four vertices in $G$, contradicting $\alpha(G) = 3$). This completes the proof of Theorem 3. □

## 4. Proof of Theorem 6

Before presenting our proof of Theorem 6, we need some additional terminologies and notation. A subset $S$ of $\mathcal{B}(G)$ is called a *branch cut* if the deletion of all edges and all inner vertices (of degree two) in any branch of $S$ will induce more components than $G$ has. A minimal branch cut is called a *branch-bond*. It is easily shown that, for a connected graph $G$, a subset $S$ of $\mathcal{B}(G)$ is a branch-bond if and only if the deletion of all edges and all inner vertices (of degree two) in any branch of $S$ will induce exactly two components. We denote by $\mathcal{BB}(G)$ the set of branch-bonds of $G$. A branch-bond $S \in \mathcal{BB}(G)$ is called *odd* if $S$ consists of an odd number of branches. The *length* of a branch-bond $S \in \mathcal{BB}(G)$, denoted by $l(S)$, is the length of a shortest branch in it. Define $\mathcal{BB}_3(G) = \{S \in \mathcal{BB}(G) : |S| \geq 3 \text{ and } S \text{ is odd}\}$. Define $h_3(G) = \max\{l(S) : S \in \mathcal{BB}_3(G)\}$ if $\mathcal{BB}_3(G)$ is not empty; 0, otherwise. By $S \triangle T$ we denote the symmetric difference $(S \setminus T) \cup (T \setminus S)$. Note that, if $H$ is an even subgraph of $G$ and $C$ is a cycle of $G$, then $G[E(H) \triangle E(C)]$ is also an even subgraph of $G$, but $G[E(H) \triangle E(C)]$ may have more components than $H$.

From a result of [15], one can easily obtain the following.

**Theorem 17** ([17]). *Let $G$ be a 2-connected graph. Then,*

$$h_3(G) - 1 \leq h(G) \leq h_3(G) + 1.$$

The following characterization of eulerian graphs involves branch-bonds.

**Theorem 18** ([15]). *A connected graph is eulerian if and only if each branch-bond contains an even number of branches.*

The following two results will be used in the proof of Theorem 6.

**Theorem 19** ([4]). *If $G$ is a connected graph such that $\delta(G) \geq 3$, then*

$$h(G) \leq 2.$$

**Lemma 20.** *If $H$ is an even subgraph of $G$, then every branch of $\mathcal{B}_H(G)$ lies in a cycle of $H$.*

**Proof.** It immediately follows from the definitions of a branch and an even subgraph. □

Now, we present the proof of our second main result.

**Proof of Theorem 6.** By Theorems 1 and 4, we only need to consider the case when $t \geq 2$. Let $m = \lfloor \frac{2t+2}{3} \rfloor$. Then, $m \geq 2$. Since $h(G) \leq 2$ for the graph $G$ with $\delta(G) \geq \kappa(G) \geq 3$ by Theorem 19, we can assume that $\kappa(G) = 2$. If $h_3(G) \leq m-1$, then $h(G) \leq m$ by Theorem 17. So, it remains to consider the case when $h_3(G) \geq m$. Let $\mathcal{B}_0$ be a branch-bond in $\mathcal{B}\mathcal{B}_3(G)$ such that $h_3(G) = \min\{|E(B)| : B \in \mathcal{B}_0\}$. Note that there are at least $\lfloor \frac{3h_3(G)}{2} \rfloor$ vertices in $G[\cup_{B\in\mathcal{B}_0} V(B)]$ forming an independent set of $G$, which contains at most one end vertex of $\mathcal{B}_0$, and hence $\alpha(G) \geq \lfloor \frac{3h_3(G)}{2} \rfloor$. We claim that $h_3(G) \leq m+1$ since otherwise $\alpha(G) \geq \lfloor \frac{3h_3(G)}{2} \rfloor \geq \frac{3h_3(G)-1}{2} \geq \frac{3m+5}{2}$, which contradicts the fact that

$$\alpha(G) \leq \kappa(G) + t = 2 + t < 2 + \frac{3m+3-2}{2} = \frac{3m+5}{2}. \tag{4.1}$$

We will prove that there is a subgraph $H \in EU_m(G)$ that implies that $h(G) \leq m = \lfloor \frac{2t+2}{3} \rfloor$ by Theorem 13.

Since the edgeless graph with vertex set $\bigcup_{i=3}^{\Delta(G)} V_i(G)$ satisfies (I) and (II), we may let $H$ be a subgraph such that

(1) $H$ satisfies (I) and (II);
(2) subject to (1), $H$ contains a maximum number of branches of length at least $m+2$;
(3) subject to (1) and (2), $\max_{\emptyset\neq V(H_1)\subsetneq V(H)} d_G(H_1, H-H_1)$ is minimized;
(4) subject to the above, $H$ contains a minimum number of induced subgraphs $F$ for which $d_G(F, H-F) = \max_{\emptyset\neq V(H_1)\subsetneq V(H)} d_G(H_1, H-H_1)$.

We claim that $H \in EU_m(G)$. It suffices to prove that $H$ satisfies (III) and (IV) since $G$ is 2-connected. It suffices to prove the following two claims.

**Claim 4.** *$d_G(H_1, H-H_1) \leq m-1$ for any subgraph $H_1$ of $H$.*

**Proof of Claim 4.** Assume, in contrast, that there is an induced subgraph $F$ of $H$ with $d_G(F, H-F) \geq m \geq 2$, then any shortest path between $F$ and $H-F$ is a branch of $G$ and hence both $F$ and $H-F$ are even subgraphs of $G$ (by the condition (II) of the definition of $H$ and by the fact that the shortest path has length at least two). Since $G$ is 2-connected, there are at least two branches $P_1, P_2$ between $F$ and $H-F$ and we may hence choose a cycle $C$ containing both $P_1$ and $P_2$ such that the length of a longest branch in $\mathcal{B}_C(G) \bigcap \mathcal{B}_H(G)$ is as small as possible. Hence, $|E(P_i)| \geq m$. By the choice of $C$, we have the following facts.

**Fact 4.1.** If $B$ is a longest branch in $\mathcal{B}_C(G) \bigcap \mathcal{B}_H(G)$, then $|E(B)| \geq m$ and there exists at least one branch $B' \in \mathcal{B}_H(G) \setminus \mathcal{B}_C(G)$ of length at least $|E(B)|$ such that the end vertices of both $B$ and $B'$ are all in the same one of $\{H, H-F\}$.

**Proof of Fact 4.1.** We prove the first statement. Assume, in contrast, that $|E(B)| < m$. Then the subgraph $H'$ obtained from the subgraph $G[E(H) \bigtriangleup E(C)]$ by adding the remaining vertices of $\bigcup_{i=3}^{\Delta(G)} V_i(G)$ as isolated vertices of $H'$ is a subgraph satisfying (1) and (2) such that $\max_{\emptyset\neq V(H_1)\subsetneq V(H')} d_G(H_1, H'-H_1)$ is less than $\max_{\emptyset\neq V(H_1)\subsetneq V(H)} d_G(H_1, H-H_1)$, contradicting the condition (3) of the choice of $C$.

We then prove the second statement. Let $B$ be a longest branch of $\mathcal{B}(G) \cap \mathcal{B}_H(G)$. By symmetry, say, $B \subseteq F$. By Lemma 20, there exists a cycle $C'$ of $F$ such that $B \subseteq C'$ and $C'$ contains a branch $B' \in \mathcal{B}_F(G) \setminus \mathcal{B}_C(G)$ of length at least $|E(B)|$, since otherwise the subgraph $H'$ obtained from $G[E(H) \bigtriangleup E(C')]$ by adding the remaining vertices of $\bigcup_{i=3}^{\Delta(G)} V_i(G)$ as isolated vertices of $H'$ is a subgraph satisfying (1) and (2) such that $\max_{\emptyset\neq V(H_1)\subsetneq V(H')} d_G(H_1, H'-H_1)$ is less than $\max_{\emptyset\neq V(H_1)\subsetneq V(H)} d_G(H_1, H-H_1)$, contradicting the condition (3) of the choice of $C$. This completes the proof of Fact 4.1. □

**Fact 4.2.** There are no pair of branches of length at least $m+2$ such that all of their end vertices belong to the same one of $\{F, H-F\}$.

**Proof of Fact 4.2.** Assume, in contrast, that $B_1, B_2$ are two branches of length at least $m+2$ such that all of their end vertices belong to the same one of $\{F, H-F\}$. Then there exist an independent set containing at least $\lfloor \frac{m+2}{2} \rfloor + 2\lfloor \frac{m}{2} \rfloor$ inner vertices in the four branches $B_1, B_2, P_1, P_2$ and hence $\alpha(G) \geq 2\lfloor \frac{m+2}{2} \rfloor + 2\lfloor \frac{m}{2} \rfloor + 1 \geq \frac{3m+5}{2}$ by Fact 4.1, which contradicts (4.1). This completes the proof of Fact 4.2. □

Then we have the following fact.

**Fact 4.3.** There are at least two branches $B_1, B_2$ of length $m$ or $m+1$ that are in $\mathcal{B}(G) \setminus \{P_1, P_2\}$ such that $E(B_i) \in E(C)$ and all end vertices of both $B_1$ and $B_2$ are in the same one of $\{F, H-F\}$.

**Proof of Fact 4.3.** Otherwise, by Facts 4.1 and 4.2, there is at most one branch $B_1$ of length $m$ or $m+1$ that is in $\mathcal{B}(G) \setminus \{P_1, P_2\}$ such that $E(B_1) \in E(C)$ and both endvertices of $B_1$ are in the same one of $\{F, H-F\}$. Then the subgraph $H'$ obtained from the subgraph $G[E(H) \bigtriangleup E(C)]$ by adding the remaining vertices of $\bigcup_{i=3}^{\Delta(G)} V_i(G)$ as isolated vertices of $H'$ is a subgraph satisfying (1) and (2) such that either $\max_{\emptyset\neq V(H_1)\subsetneq V(H')} d_G(H_1, H'-H_1)$ is fewer than $\max_{\emptyset\neq V(H_1)\subsetneq V(H)} d_G(H_1, H-H_1)$ or $H'$ has a fewer number of induced subgraphs $F$ for which $d_G(F, H'-F) = \max_{\emptyset \neq V(H_1) \subsetneq V(H')} d_G(H_1, H'-H_1)$ than $H$ (in this case, $H'$ satisfies (3)), which shows that $H$ does not satisfy the condition (4) of the choice of $C$, a contradiction. This completes the proof of Fact 4.3. □

Since $G$ is 2-connected, every component of $H$ has at least one vertex which is not adjacent to any inner vertex of branches of length at least $m$ in $\mathcal{B}_C(G)$. Hence, by the definition of odd branch-bonds, there are at least two branches of length $m$ or $m+1$ in $\mathcal{B}(G) \setminus \mathcal{B}_0$. Hence, by Fact 4.3, there is an independent set containing at least $2\lfloor \frac{m}{2} \rfloor + 1 \geq 3$ vertices, any element of which is not adjacent to any inner vertex of any branch of $\mathcal{B}_0$, and so $\alpha(G) \geq \lfloor \frac{3h_3(G)}{2} \rfloor + 3 \geq \frac{3m-1}{2} + 3 \geq \frac{3m+5}{2}$, which contradicts (4.1). This proves Claim 4. □

**Claim 5.** *$|E(B)| \leq m+1$ for any branch $B \in \mathcal{B}(G) \setminus \mathcal{B}_H(G)$.*

**Proof of Claim 5.** Assume, in contrast, that there is a branch $B_0 \in \mathcal{B}(G) \setminus \mathcal{B}_H(G)$ with $|E(B_0)| \geq m+2$. Let $u$ and $v$ be two end vertices of $B_0$ and $S(u, B_0)$ be a branch-bond containing $B_0$ such that any branch of $S(u, B_0)$ has $u$ as an end vertex. Since $G$ is 2-connected, $|S(u, B_0)| \geq 2$.

By the following algorithm, we first find a cycle of $G$ that contains $B_0$ and then obtain a contradiction.

**Algorithm $B_0$.** 1. If $|S(u, B_0)|$ is even, then select a branch $B_1 \in S(u, B_0) \setminus (\mathcal{B}_H(G) \cup \{B_0\})$ by Theorem 18. Otherwise, since $|E(B_0)| \geq m+2$, (we can) select a branch $B_1 \in S(u, B_0)$ with

$$|E(B_1)| = l(S(u, B_0)) \leq h_3(G) \leq m+1$$

(obviously $B_1 \neq B_0$) and let $u_1 (\neq u)$ be the other end vertex of $B_1$. If $u_1 = v$, then set $t := 1$ and stop. Otherwise $i := 1$.

2. Select a branch-bond $S(u, u_i, B_0)$ in $G$ which contains $B_0$ but not any of $\{B_1, B_2, \ldots, B_i\}$ such that any branch in $S(u, u_i, B_0)$ has exactly one end vertex in $\{u, u_1, u_2, \ldots, u_i\}$. If $|S(u, u_i, B_0)|$ is even, then, by Theorem 18, (we can) select a branch

$$B_{i+1} \in S(u, u_i, B_0) \setminus (\mathcal{B}_H(G) \cup \{B_0\}).$$

Otherwise, since $|E(B_0)| \geq h_3(G) + 1$, (we can) select a branch $B_{i+1} \in S(u, u_i, B_0)$ such that

$$|E(B_{i+1})| = l(S(u, u_i, B_0)) \leq h_3(G)$$

(obviously $B_{i+1} \neq B_0$), and let $u_{i+1}$ be the end vertex of $B_{i+1}$ that is not in $\{u, u_1, u_2, \ldots, u_i\}$.

3. If $u_{i+1} = v$, then set $t := i+1$ and stop. Otherwise replace $i$ by $i+1$ and return to step 2.

Note that $|\mathcal{B}(G)|$ is finite, and $d_G(v) \geq 2$ implies that Algorithm $B_0$ will stop after a finite number of steps. Note that $G[\bigcup_{i=0}^{t} E(B_i)]$ is connected. Furthermore, since $u_t = v$ and $|S(u, u_i, B_0)| \geq 2$, $G[\bigcup_{i=0}^{t} E(B_i)]$ has a cycle $C_0$ of $G$ which contains $B_0$. Let $H'$ be the subgraph of $G$ obtained from $G[E(H) \triangle E(C_0)]$ by adding the remaining vertices of $\bigcup_{i=3}^{\Delta(G)} V_i(G)$ as isolated vertices in $H'$. By the choice of $B_i$, $|E(B)| \leq h_3(G) \leq m+1$ for $B \in \mathcal{B}_H(G) \cap \{B_1, B_2, \ldots, B_t\}$. Hence, since $B_0$ is in $\mathcal{B}_{C_0}(G)$, $H'$ is a subgraph with the conditions (I) and (II) that has fewer branches of length $m+2$ than $H$, which shows that $H$ does not satisfy (2), a contradiction. This proves Claim 5. □

Thus we, in fact, have proved that $G$ has a subgraph in $EU_m(G)$ by Claims 4 and 5. Hence, $h(G) \leq m = \lfloor \frac{2t+2}{3} \rfloor$ by Theorem 13. This also completes the proof of Theorem 6. □

## 5. Sharpness

In this section, we discuss sharpness. We show that Theorem 6 is sharp by describing a graph satisfying the conditions of Theorem 6 and having the Hamiltonian index of exactly $\lfloor \frac{2t+2}{3} \rfloor$. By Theorems 1, 3 and 12, we only need to show the sharpness in the case when $t \geq 2$.

Let $H_1, H_2$ be two vertex-disjoint complete graphs of order at least four and $P_1, P_2, P_3$ be three vertex-disjoint paths with length $m+1 \geq 3$. Now, obtain the graph $G_0$ by identifying three end vertices of $P_1, P_2, P_3$ with three vertices of $H_1$, and identifying the other end vertex of $P_i$ with three vertices of $H_2$, respectively; see Fig. 3.

Let $C$ be a longest cycle of $G_0$. Then, $C \in EU_m(G_0)$. So $h(G_0) \leq m$ by Theorem 13, and $h(G_0) \geq h_3(G_0) - 1 = m$ by Theorem 17. Hence, $h(G_0) = m$. Note that if $m$ is even, then $t = \alpha(G_0) - 2 = \lfloor \frac{3m}{2} \rfloor$ and $h(G_0) = m = \lfloor \frac{2t+2}{3} \rfloor$. For the case when $m$ is odd, consider the reduction $G_0'$ of $G_0$, which is obtained from $G_0$ by contracting $H_1$ and $H_2$, respectively. Then $t = \alpha(G_0') - 2 = \lfloor \frac{3m}{2} \rfloor$ and $h(G_0') = m = \lfloor \frac{2t+2}{3} \rfloor$ for $m$ is odd.

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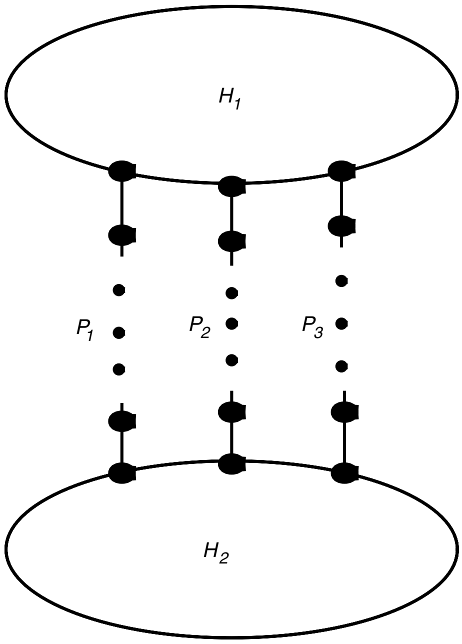


**Fig. 3.** The graph $G_0$.

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