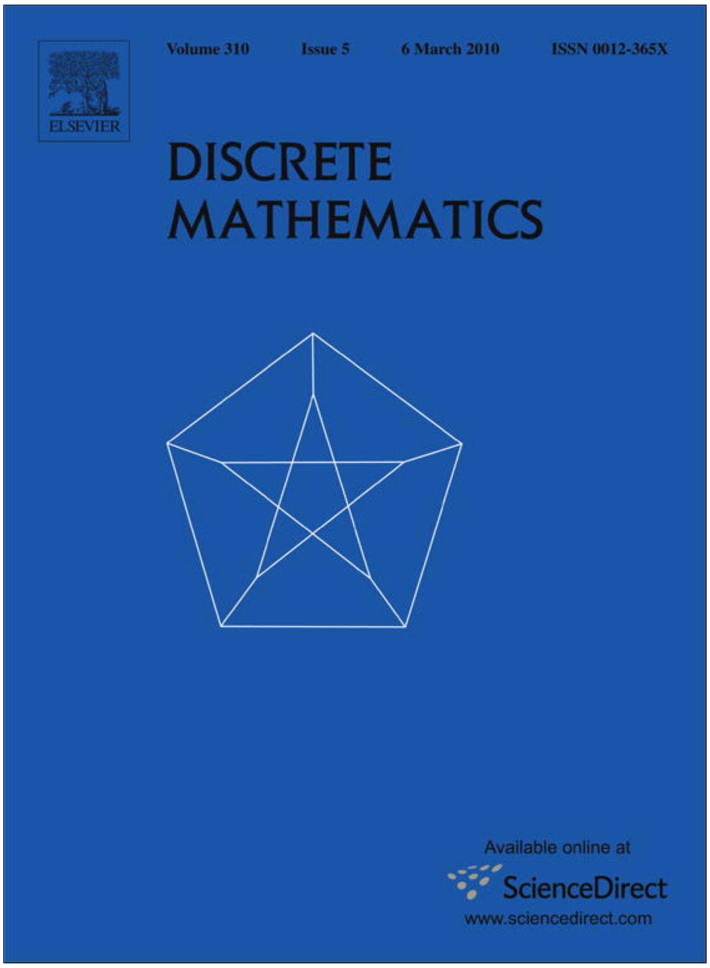
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Degree conditions for group connectivity

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A R T I C L E I N F O

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Let G be a 2-edge-connected simple graph on $n \geq 13$ vertices and A an (additive) abelian group with $|A| \geq 4$. In this paper, we prove that if for every $uv \notin E(G)$, $\max\{d(u), d(v)\} \geq n/4$, then either G is A-connected or G can be reduced to one of $K_{2,3}$, C_4 and C_5 by repeatedly contracting proper A-connected subgraphs, where C_k is a cycle of length k. We also show that the bound $n \geq 13$ is the best possible.

1. Introduction

The graphs in this paper are finite and may have multiple edges. The terms and notations not defined here are from [1] and [17]. Let G be a graph and let V_1, V_2 be two subsets of $V(G)$ such that $V_1 \cap V_2 = \emptyset$. We define $e(V_1, V_2)$ as the number of edges with one end vertex in V_1 and the other one in V_2. In particular, when $V_1 = X$ and $V_2 = V(G) - X$, we use $\partial(X)$ instead of $e(X, V(G) - X)$. An *n-cycle* is a cycle of length n.

Let $D = D(G)$ be an orientation of a graph G. If an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then let *tail* $(e) = u$ and *head*$(e) = v$. For a vertex $v \in V(G)$, let

$$E_D^-(v) = \{e \in E(D) : v = \text{tail}(e)\}, \quad \text{and} \quad E_D^+(v) = \{e \in E(D) : v = \text{head}(e)\}.$$

We write D for $D(G)$ when its meaning can be understood from the context.

Let A denote an (additive) abelian group where the identity of A is denoted by 0. Let A^* denote the set of nonzero elements of A. We define:

$$F(G, A) = \{f : E(G) \mapsto A\} \quad \text{and} \quad F^*(G, A) = \{f : E(G) \mapsto A^*\}.$$

Given a function $f \in F(G, A)$, define $\partial f : V(G) \mapsto A$ by

$$\partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e),$$

where "$\sum$" refers to the addition in A.

Group connectivity was introduced by Jaeger et al. [6] as a generalization of nowhere-zero flows. For a graph G, a function $b : V(G) \mapsto A$ is called an A-**valued zero sum function** on G if $\sum_{v \in V(G)} b(v) = 0$. The set of all A-valued zero sum functions

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on G is denoted by $Z(G, A)$. Given $b \in Z(G, A)$, a function $f \in F^*(G, A)$ is called **an** (A, b)-**nowhere-zero flow** if G has an orientation $D(G)$ such that $\partial f = b$. A graph G is A-connected if for any $b \in Z(G, A)$, G has an (A, b)-nowhere-zero flow. In particular, G admits **a nowhere-zero** A-**flow** if G has an $(A, 0)$-nowhere-zero flow. G admits **a nowhere-zero** k-**flow** if G admits a nowhere-zero Z_k-flow, where Z_k is a cyclic group of order k. Tutte [16] proved that G admits a nowhere-zero A-flow with $|A| = k$ if and only if G admits a nowhere-zero k-flow. One notes that if a graph G is A-connected and $|A| \geq k$, then G admits a nowhere-zero k-flow. Generally speaking, when G admits a nowhere-zero k-flow, G may not be A-connected with $|A| \geq k$. For example, a n-cycle is A-connected if and only if $|A| \geq n + 1$ given in [6, Lemma 3.3] while for any n, a n-cycle admits a nowhere-zero 2-flow. Thus, group connectivity generalizes nowhere-zero flows.

For an abelian group A, let $\langle A\rangle$ be the family of graphs that are A-connected. It is observed in [6] that the property $G \in \langle A\rangle$ is independent of the orientation of G, and that every graph in $\langle A\rangle$ is 2-edge-connected.

The nowhere-zero flow problems were introduced by Tutte in [14–16] and surveyed by Jaeger in [6] and Zhang in [18]. The following conjecture is due to Tutte. Partial results of this conjecture can be found in [6] and others. However, it is still open.

Conjecture 1.1 (*4-flow Conjecture, [15]*). *Every bridgeless graph containing no subdivision of the Petersen graph admits a nowhere-zero* 4*-flow.*

For a 2-edge-connected graph G, we define the *group connectivity number* of G as follows:

$$\Lambda_g(G) = \min\{k : \text{ if } A \text{ is an abelian group with } |A| \geq k, \text{ then } G \in \langle A\rangle\}.$$

If G is 2-edge-connected, then $\Lambda_g(G)$ exists as a finite number. Recently, there have been some degree conditions adapted to assure the existence of nowhere-zero flows and group connectivity of graphs. Fan and Zhou [5] proved that if G is a simple graph on $n \geq 3$ vertices satisfying for every pair of nonadjacent vertices u and v in G, if $d(u) + d(v) \geq n$, then either G has a nowhere-zero 3-flow or G is one of the six well-classified exceptional graphs. Fan and Zhou's result has been generalized as follows.

Theorem 1.2 (*Luo, Xu, Yin and Yu [11]*). *Let G be a simple graph on $n \geq 3$ vertices. If $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices, then either $\Lambda_g(G) \leq 3$, or G is one of the* 12 *well-classified exceptional graphs.*

Theorem 1.3 (*Sun, Xu and Yin [13]*). *Let G be a simple graph on $n \geq 3$ vertices. If $d(u)+d(v) \geq n$ for every pair of nonadjacent vertices, then either $\Lambda_g(G) \leq 4$, or G^* is a* 4*-cycle.*

A contraction [3] of G is the graph G' obtained from G by contracting a set (possibly empty) of edges and deleting any loops generated in the process. If G' is a contraction of G, then we say that G is contractible to G'. When H is a subgraph of G, the contraction of G obtained from G by contracting each edge of $E(H)$ and deleting resulting loops is denoted as G/H. Note that each component of H is a vertex of G/H.

For a graph G, define $\mathcal{T}$ to be a set of the subgraphs of G, which either has two edge-disjoint spanning trees or is isomorphic to a cycle of length 3. Note that a 2-cycle has two edge-disjoint spanning trees. Let G^* be the graph obtained from G by repeatedly contracting non-trivial subgraphs in $\mathcal{T}$ until no subgraph in $\mathcal{T}$ left. In this case, We say G^* is the $\mathcal{T}$*-reduction* of G. If $v \in V(G^*)$ is obtained by contracting a subgraph $H \in \mathcal{T}$ of G, then H is called the **preimage of** v and v is called an **image of** H. In the rest of this paper, we use G^* to denote the $\mathcal{T}$-reduction of a graph G. Motivated by the results mentioned above, we present the following result in this paper.

Theorem 1.4. *Let A be an abelian group with $|A| \geq 4$, and G a* 2*-edge-connected simple graph on $n \geq 13$ vertices. If for every $uv \notin E(G)$, $\max\{d(u), d(v)\} \geq n/4$, then either G is A-connected, or $G^* \in \{K_{2,3}, C_4, C_5\}$, where C_k is a k-cycle. Moreover, if $G^* \in \{K_{2,3}, C_4\}$, then $\Lambda_g(G) = 5$; and if $G^* = C_5$, then $\Lambda_g(G) = 6$.*

Theorem 1.4 is sharp in the sense that the bound $n \geq 13$ cannot be relaxed. Let P_{10} denote the Petersen graph and let v be a vertex of P_{10} and v_1, v_2, v_3 the three neighbors of v. Let P_{12} denote the graph obtained from $P_{10} - v$ by adding a 3-cycle $u_1u_2u_3u_1$ and then joining u_i to v_i by an edge u_iv_i, $1 \leq i \leq 3$ (See Fig. 1). Then $|V(P_{12})| = 12$ and P_{12} is 3-regular. Thus P_{12} both satisfies the degree condition of Theorem 1.4 and can be contracted to P_{10}. By [10, Theorem 3.2], $\Lambda_g(P_{10}) = 5$ and $\Lambda_g(P_{12}) \geq 5$ given by [6, Proposition 3.2]. This shows that Theorem 1.4 does not hold when $n = 12$.

We organize this paper as follows. In Section 2, we present a reduction method that will be used in the proofs. We deal with the small case when $13 \leq n \leq 16$ in Section 3. We complete the proof of Theorem 1.4 in Section 4.

2. Reduction method

We first summarize some previous results in the following two lemmas which are used in the proof of Theorem 1.4. For a graph G, let $\tau(G)$ be the maximum number of edge-disjoint spanning trees of G.

Lemma 2.1 (*[6–8]*). *Let A be an abelian group and let H be a subgraph of a graph G. Then each of the following statements holds.*

(1) $K_1 \in \langle A\rangle$

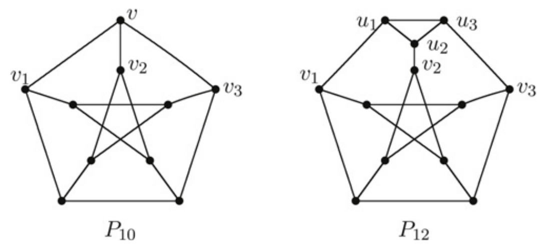

Fig. 1. Graph P_{10} and Graph P_{12}.

(2) *Suppose that* $H \in \langle A\rangle$. *Then* $G/H \in \langle A\rangle$ *if and only if* $G \in \langle A\rangle$.
(3) *If* $\tau(G) \geq 2$, *then* $G \in \langle A\rangle$ *for any A with* $|A| \geq 4$.
(4) $C_n \in \langle A\rangle$ *if and only if* $|A| \geq n+1$, *where* C_n *is a n-cycle.*

Lemma 2.2 ([4]). *Let* $n \geq 3$ *be an integer. Then*

$$\Lambda_g(K_n) = \begin{cases} 4 & \text{if } 3 \leq n \leq 4, \\ 3 & \text{if } n \geq 5. \end{cases}$$

Let $m \geq n \geq 2$ *be integers. Then*

$$\Lambda_g(K_{m,n}) = \begin{cases} 5 & \text{if } n = 2, \\ 4 & \text{if } n = 3, \\ 3 & \text{if } n \geq 4. \end{cases}$$

Let t be a positive integer and let M be a loopless matroid. Define a t-packing of M to be a family $\mathcal{F}$ of bases of M such that each element of M is in at most t bases of $\mathcal{F}$. M_G refers to the cycle matroid of a loopless graph G. Let $\eta_t(G)$ be the cardinality of the largest t-packing of M_G. In review of cycle matroid of a graph G, Nash-Williams [12] proved:

Theorem 2.3. *If G is a connected loopless graph with at least two vertices, then*

$$\eta_t(G) = \min_{F \subseteq E(G)} \left\lfloor \frac{|F|}{\omega(G-F)-1} \right\rfloor,$$

where $\omega(G-F)$ *denotes the number of components of the graph* $G-F$, *and the minimum is taken over all subsets F of* $E(G)$ *for which* $\omega(G-F) > 1$.

Let M be a matroid on set S and r be a rank function of M. The notations of $g(M)$, $g(X)$, $\gamma(M)$ and $\eta(M)$ was defined in [2] as follows. If $r(M) \geq 1$, we define

$$g(M) = \frac{|S|}{r(S)} \quad \text{and} \quad g(X) = \frac{|X|}{r(X)} \quad \text{for any } X \subseteq S \text{ with } r(X) > 0.$$

We define

$$\gamma(M) = \max_{X \subseteq S} g(X), \tag{1}$$

where the maximum is taken over all subsets $X \subseteq S$ for which $r(X) > 0$. Define

$$\eta(M) = \min_{X \subseteq S} \frac{|S \setminus X|}{r(S)-r(X)},$$

where the minimum is taken over all subsets $X \subseteq S$ which $r(X) < r(S)$. For simplicity, we use $g(G)$, $\gamma(G)$, $\eta(G)$ to denote $g(M_G)$, $\gamma(M_G)$, $\eta(M_G)$, respectively. From Theorem 2.3, we obtain the following result.

Theorem 2.4. *Let G be a non-trivial graph and let k be a positive integer. If* $|E(G)|/(|V(G)|-1) \geq k$, *then G has a non-trivial subgraph H with* $\tau(H) \geq k$.

Proof. In terms of cycle matroid of a graph G it follows from (1) that $\gamma(G) \geq |E(G)|/(|V(G)|-1)$.

By the definition of $\gamma(G)$, there is an edge subset X, such that $g(X) = \gamma(G)$. Let $H = G[X]$. Since $\gamma(G) = g(X) \leq \gamma(H) \leq \gamma(G)$, we must have $\gamma(H) = g(X)$, and so by [2, Theorem 6], $\eta(H) = g(X) = \gamma(H) \geq |E(H)|/(|V(H)|-1)$. If $|E(H)|/(|V(H)|-1) \geq k$, then $\eta(H) \geq k$. By [2, Corollary 5], $\eta_1(H) = \lfloor \eta(H) \rfloor \geq k$. It follows by Theorem 2.3 that H must have at least k edge-disjoint spanning trees. ■

Lemma 2.5. *If* G^* *is non-trivial, then* $2|V(G^*)| - |E(G^*)| \geq 3$.

Proof. Applying Theorem 2.4 to G^*, $|E(G^*)|/(|V(G^*)|-1)<2$, which implies that $2|V(G^*)|-|E(G^*)|>2$. We conclude that $2|V(G^*)|-|E(G^*)|\geq 3$ since $|V(G^*)|$ and $|E(G^*)|$ are both integers. ■

Define $D_i(G^*)=\{v\in V(G^*): d_{G^*}(v)=i\}$. Throughout this paper, we write D_i for $D_i(G^*)$. We use $\delta(G)$, $\Delta(G)$ and $\kappa'(G)$ to denote the minimum and the maximum degrees of the vertices of a graph G, and the edge connectivity of G, respectively.

Theorem 2.6. *If G^* is non-trivial, then each of the following holds.*
(i) *G^* is simple and contains no 3-cycles and no non-trivial subgraphs H with $\tau(H)\geq 2$.*
(ii) *$\delta(G^*)\leq 3$ and*

$$3|D_1|+2|D_2|+|D_3|\geq 6+\sum_{i\geq 5}(i-4)|D_i|.$$

Moreover, if $\kappa'(G^*)\geq 2$, then

$$2|D_2|+|D_3|\geq 6+\sum_{i\geq 5}(i-4)|D_i|. \quad (2)$$

Proof. (i) It follows immediately from the definition of $\mathcal{T}$-reduction.
(ii) Applying Theorem 2.4 to G^*, $|E(G^*)|/(|V(G^*)|-1)<2$. Thus,

$$\delta(G^*)|V(G^*)|\leq \sum_{v\in V(G^*)} d_{G^*}(v)=2|E(G^*)|<4|V(G^*)|-4,$$

which implies that $\delta(G^*)\leq 3$.
Since G^* is non-trivial, by Lemma 2.5,

$$4\sum_{i\geq 1}|D_i|-\sum_{i\geq 1}i|D_i|=4|V(G^*)|-2|E(G^*)|=2(2|V(G^*)|-|E(G^*)|)\geq 6.$$

It follows that

$$3|D_1|+2|D_2|+|D_3|\geq 6+\sum_{i\geq 5}(i-4)|D_i|.$$

When $\kappa'(G^*)\geq 2$, $|D_1|=0$ and hence (2) follows. ■

Lemma 2.7. *If G^* is a K_1, then $\Lambda_g(G)\leq 4$.*

Proof. It follows from Lemmas 2.1 and 2.2. ■

Lemma 2.8. *Let G be a simple graph and let H be a subgraph of G. If $d_G(v)\geq q$ for every $v\in V(H)$ and $\partial(H)<q$, then $|V(H)|>q$.*

Proof. Suppose that $|V(H)|=p$. We claim that $p>1$. Otherwise, let $V(H)=\{v_H\}$, then $q\leq d_G(v_H)=\partial(H)<q$, a contradiction. Since G is simple,

$$p(p-1)\geq \sum_{v\in V(H)} d_H(v)=\sum_{v\in V(H)} d_G(v)-\partial(H)\geq pq-\partial(H)>pq-q=q(p-1),$$

which implies that $p>q$ since $p>1$. Thus, $|V(H)|>q$. ■

Lemma 2.9. *Let k, c be positive integers. Suppose that G is a 2-edge-connected simple graph on n vertices such that for every $uv\notin E(G)$,*

$$\max\{d(u),d(v)\}\geq n/c. \quad (3)$$

Define $Y=\{v\in V(G^*): d_{G^*}(v)\leq k\}$. If $n>kc$, then $|Y|\leq c+1$.

Proof. Let $Y=\{v_1,v_2,\ldots,v_l\}$ and let $H_1,H_2,\ldots,H_l$ denote the preimages of $v_1,v_2,\ldots,v_l$, respectively. By the definition of preimages, $H_1,H_2,\ldots,H_l$ are vertex-disjoint.

Let $X=\{x\in V(G): d_G(x)<\frac{n}{c}\}$. We claim that Y contains at most two vertices v_i,v_j such that $V(H_i)\cap X\neq\emptyset$ and $V(H_j)\cap X\neq\emptyset$. Suppose otherwise that G^* contains $v_{i1},v_{i2},\ldots,v_{ip}$, $p\geq 3$, such that $V(H_{ik})\cap X\neq\emptyset$, $1\leq k\leq p$. Take $u_{ik}\in V(H_{ik})\cap X$. By (3), $G[\{u_{i1},u_{i2},\ldots,u_{ip}\}]\cong K_p$. By Lemma 2.2, $G[\{u_{i1},u_{i2},\ldots,u_{ip}\}]$ is a subgraph of some H_t for $t\in\{1,2,\ldots,l\}$, contrary to that $H_1,H_2,\ldots,H_l$ are vertex-disjoint.

Thus, we assume, without losing of generality, that each of the preimages of $v_1,\ldots,v_q$ has a vertex in X, where $0\leq q\leq 2$ and none of the preimages of $v_{q+1},\ldots,v_l$ has a vertex in X. It follows that for each vertex $v\in V(H_i)$, $d_G(v)\geq n/c$, where $q+1\leq i\leq l$. On the other hand, $d_{G^*}(v_i)\leq k$, which is equivalent to $\partial(H_i)\leq k$ for $q+1\leq i\leq l$. Since $k<n/c$, Lemma 2.8 shows that $|V(H_i)|>n/c$ for $q+1\leq i\leq l$. Since $H_1,H_2,\ldots,H_l$ are vertex-disjoint, $n\geq\sum_{i=1}^{l}|V(H_i)|>2+(l-2)n/c$. It follows that $l<c+2-2c/n$. Since l and c are both integers, $l\leq c+1$. ■

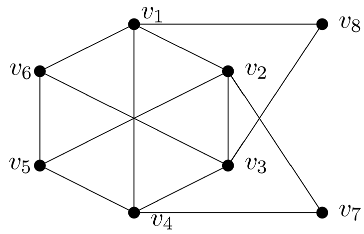

Fig. 2. The graph L

3. Graphs with small orders

In this section, we pay our attention to the case when G is a 2-edge-connected simple graph on $13 \le n \le 16$ vertices. Recall that G^* is the $\mathcal{T}$-reduction of G. For this purpose, we define $W = \{u \in V(G) : d_G(u) < 4\}$. For a vertex $v \in V(G^*)$ with $d_{G^*}(v) < 4$, v is defined to be **a vertex of type 1** if the preimage of v has a vertex in W and **a vertex of type 2** otherwise.

Lemma 3.1. *Let G be a 2-edge-connected simple graph on $13 \le n \le 16$ vertices. If for every $uv \notin E(G)$,*

$$\max\{d(u), d(v)\} \ge n/4, \tag{4}$$

then $n \ge \sum_{i\ge 4} |D_i| + 2 + 5(|D_2| + |D_3| - 2)$. Moreover, if $D_2 \cup D_3$ is an independent set, then $n \ge \sum_{i\ge 4} |D_i| + 1 + 5(|D_2| + |D_3| - 1)$.

Proof. Since G is 2-edge-connected, $|D_1| = 0$. We first claim that G^* contains at most two vertices of type 1. Suppose otherwise that v_1, v_2, v_3 are three vertices of type 1 in G^*. Let H_j be the preimages of v_j where $j = 1, 2, 3$. By the definition, $V(H_j) \cap W \ne \emptyset$ and pick $x_j \in V(H_j) \cap W$ for $j = 1, 2, 3$. Then $d_G(x_j) < 4$. By (4), $x_1x_2, x_2x_3, x_3x_1 \in E(G)$. This means that G^* has a 3-cycle, contrary to Theorem 2.6(i).

Let $v \in V(G^*)$ be a vertex of type 2 and let H be the preimage of v. By the definition, $V(H) \cap W = \emptyset$ and $d_{G^*}(v) < 4$. It follows that $\partial(H) < 4$ and $d(u) \ge 4$ for each $u \in V(H)$. Applying Lemma 2.8 to H, $|V(H)| \ge 5$.

Thus, by the argument above, G^* contains at least $|D_2| + |D_3| - 2$ vertices of type 2. It follows that $n \ge \sum_{i\ge 4} |D_i| + 2 + 5(|D_2| + |D_3| - 2)$. If $D_2 \cup D_3$ is an independent set, then G^* contains at most one vertex of type 1. Thus, we similarly conclude that $n \ge \sum_{i\ge 4} |D_i| + 1 + 5(|D_2| + |D_3| - 1)$. ■

Lemma 3.2. *Let G be a 2-edge-connected simple graph on $13 \le n \le 16$ vertices. If for every $uv \notin E(G)$, $\max\{d(u), d(v)\} \ge n/4$, then either $G^* \cong K_1$ or*

$$3 \le |D_2| + |D_3| \le 4. \tag{5}$$

Proof. If $G^* \cong K_1$, we are done. Thus, we assume that $G^* \ncong K_1$. By Theorem 2.6(i), G^* is simple and hence $|V(G^*)| \ge 3$. Since $n/4 > 3$, by Lemma 2.9, G^* has at most 5 vertices of degree at most 3, that is, $|D_2| + |D_3| \le 5$.

If $|D_2| + |D_3| \le 2$, let $|D_2| + |D_3| = t$ and $\sum_{i\ge 4} |D_i| = n_1$. Then $2|E(G^*)| \ge 4n_1 + 2t$ and $|V(G^*)| = n_1 + t$. Since $t \le 2$, we have $2|V(G^*)| - |E(G^*)| \le 2n_1 + 2t - (2n_1 + t) = t \le 2$, which is contrary to Lemma 2.5. So far, we have proved that $|D_2| + |D_3| \ge 3$.

Suppose that $|D_2| + |D_3| \ge 5$. Applying Lemma 3.1 to $|D_2| + |D_3|$, $n \ge \sum_{i\ge 4} |D_i| + 2 + 5(|D_2| + |D_3| - 2) \ge 3 \times 5 + 2 = 17$, contrary to the condition $13 \le n \le 16$. ■

Theorem 3.3. *Let G be a 2-edge-connected simple graph on $13 \le n \le 16$ vertices. If for every $uv \notin E(G)$, $\max\{d(u), d(v)\} \ge n/4$, then $G^* \in \{K_1, C_4\}$ or G^* is isomorphic to the graph L, where C_4 is a 4-cycle (see Fig. 2).*

Proof. It sufficient to show our theorem for the case when $G^* \ne K_1$. By (2) and (5),

$$|D_2| \ge 2 + \sum_{i\ge 5} (i - 4)|D_i|. \tag{6}$$

In order to complete our proof, we need to show the following claims.

Claim 1. $\Delta(G^*) \le 4$.

If $\Delta(G^*) \ge 7$, then by (6), $|D_2| \ge 2 + (\Delta(G^*) - 4) \ge 2 + 3 = 5$, contrary to (5). If $\Delta(G^*) = 6$, then by (5) and (6),

$$4 \ge |D_2| + |D_3| \ge |D_2| \ge 2 + |D_5| + 2|D_6| \ge 2 + |D_5| + 2 \ge 4, \tag{7}$$

which implies that $|D_6| = 1$, $|D_5| = 0$, $|D_3| = 0$ and $|D_2| = 4$. It follows that $|V(G^*)| = 5$ and $\Delta(G^*) = 6$, which ensure that G^* cannot be simple, contrary to Theorem 2.6(i).

If $\Delta(G^*) = 5$, then by (5) and (6),

$$4 \ge |D_2| + |D_3| \ge |D_2| \ge 2 + |D_5|, \tag{8}$$

which forces that $|D_5| \le 2$.

Suppose first that $|D_5| = 2$. By (8), $|D_3| = 0$ and $|D_2| = 4$. Applying Lemma 3.1 to $W = D_2$, $n \geq |D_4|+|D_5|+2+5(|D_2|-2)$, which implies that $|D_4| \leq n - |D_5| - 2 - 5(|D_2| - 2) \leq 16 - 2 - 2 - 10 = 2$. If $|D_4| = 0$, let $u_1, u_2 \in D_5$. In this case, $|V(G^*)| = 6$. Thus, for $i = 1, 2$, u_i is adjacent to all other vertices of G^*. It follows that G^* contains a 3-cycle, contrary to Theorem 2.6(i). Thus, $|D_4| = 2$ or 1. Let $S = D_4 \cup D_5$. Note that G^* has no cycle of length at most 3. If $|D_4| = 2$, then $|S| = 4$ and $|E(G^*[S])| \leq 4$. Thus, $8 \geq \partial(D_2) = e(D_2, S) = \partial(S) = \sum_{v\in S} d_{G^*}(v) - 2|E(G^*[S])| \geq 10 + 8 - 8 = 10$, a contradiction. If $|D_4| = 1$, then $|S| = 3$ and $|E(G^*[S])| \leq 2$. Thus, $8 \geq \partial(D_2) = e(D_2, S) = \partial(S) \geq 10 + 4 - 4 = 10$, a contradiction.

Then suppose that $|D_5| = 1$. Since the number of the vertices of odd degree is even, by (8), $|D_3| = 1$ and $|D_2| = 3$. Since $\Delta(G^*) = 5$, $|V(G^*)| \geq 6$, which implies that $|D_4| \geq 1$. Applying Lemma 3.1 to $|D_2| + |D_3| = 4$, $n \geq |D_5| + |D_4| + 2 + 5(|D_2| + |D_3| - 2)$, which implies $|D_4| \leq n - |D_5| - 2 - 5(|D_2| + |D_3| - 2) \leq 16 - 1 - 2 - 10 = 3$. If $|D_4| = 1$, then $|V(G^*)| = 6$. It follows that the vertex in D_5 must be adjacent to every other vertex. Since $\delta(G^*) \geq 2$, $|E(G^*[D_2 \cup D_3 \cup D_4])| \geq 1$ and G^* contains a 3-cycle, contrary to Theorem 2.6(i). Thus, $|D_4| = 2$ or 3. Recall that G^* has no cycle of length at most 3. Let $S = D_3 \cup D_4 \cup D_5$. If $|D_4| = 2$, then $|S| = 4$ and $|E(G^*[S])| \leq 4$, thus, $6 \geq \partial(D_2) = e(D_2, S) = \partial(S) \geq 5|D_5|+4|D_4|+3|D_3|-2|E(G^*[S])| \geq 5+3+8-8 = 8$, a contradiction; if $|D_4| = 3$, then $|S| = 5$ and $E(G^*[S])| \leq 6$ given by Turàn Theorem, thus, $6 \geq \partial(D_2) = e(D_2, S) = \partial(S) \geq 5|D_5|+4|D_4|+3|D_3|-2|E(G^*[S])| \geq 5+3+12-12 = 8$, a contradiction.

Claim 2. $\Delta(G^*) \neq 4$.

Suppose otherwise that $\Delta(G^*) = 4$. By (5) and (6),

$$4 \geq |D_2| + |D_3| \geq |D_2| \geq 2. \tag{9}$$

On the other hand, $|D_3|$ is even and hence $|D_3| = 2$ or 0.

Case 1. $|D_3| = 2$.

By (9), $|D_2| = 2$. Applying Lemma 3.1 to $|D_2| + |D_3| = 4$, $n \geq |D_4| + |D_5| + 2 + 5(|D_2| + |D_3| - 2)$, which implies that $|D_4| \leq 16 - 2 - 10 = 4$. If $|D_4| = 1$, then $|V(G^*)| = 5$. Then the vertex in D_4 is adjacent to every other vertex of G^*. Since $\delta(G^*) \geq 2$, $|E(G^*[D_2 \cup D_3])| \geq 1$ and then G^* contains a 3-cycle, contrary to Theorem 2.6(i).

Suppose that $|D_4| = 2$ or 3. Let $S = D_3 \cup D_4$. If $D_4 = 2$, then $|S| = 4$ and $|E(G^*[S])| \leq 4$. Thus, $4 \geq \partial(D_2) = e(D_2, S) = \partial(S) \geq 8 + 6 - 8 = 6$, a contradiction. If $|D_4| = 3$, then $|S| = 5$ and $|E(G^*[S])| \leq 6$. Thus, $4 \geq \partial(D_2) = e(D_2, S) = \partial(S) \geq 12 + 6 - 12 = 6$, a contradiction.

Finally, we assume $|D_4| = 4$. If $|E(G^*[D_4])| \leq 3$, then $10 \geq \partial(D_2\cup D_3) = e(D_2\cup D_3, D_4) = \partial(D_4) \geq 4|D_4|-2|E(G^*[D_4])| \geq 16 - 6 = 10$, which implies that $D_2 \cup D_3$ is an independent set of G^*. Applying Lemma 3.1 to $|D_2| + |D_3| = 4$, $n \geq |D_4| + |D_5| + 1 + 5(|D_2| + |D_3| - 1) \geq 1 + 4 + 15 = 20$, contrary to $n \leq 16$. Thus, $|E(G^*[D_4])| = 4$ and hence $G^*[D_4]$ is a 4-cycle. It follows that $\partial(D_2 \cup D_3) = e(D_2 \cup D_3, D_4) = \partial(D_4) = 16 - 8 = 8$. Thus, $2|E(G^*[D_2 \cup D_3])| = \sum_{v\in D_2\cup D_3} d(v) - \partial(D_2 \cup D_3) = 4 + 6 - 8 = 2$. This implies that $E(G^*[D_2 \cup D_3])$ contains exactly one edge e. If e has one end in D_2, then there exists a vertex v in D_3 with $N(v) \subseteq D_4$ since $|D_3| = 2$. Thus, G^* contains a 3-cycle, which is contrary to Theorem 2.6(i). Therefore, $\{e\} = E(G^*[D_3])$. Since G^* has no 3-cycle, G^* is the graph L in Fig. 2.

Case 2. $|D_3| = 0$.

It follows from (5) that $3 \leq |D_2| \leq 4$. Assume first that $|D_2| = 3$. Since $\Delta(G^*) = 4$, $|V(G^*)| \geq 5$ and $|D_4| \geq 2$. If $|D_4| = 2$, let $v_1, v_2 \in D_4$. In this case, $|V(G^*)| = 5$ and for each $i = 1, 2$, v_i is adjacent to all other vertices of G^*. It follows that G^* contains a 3-cycle, contrary to Theorem 2.6(i). Thus, we may assume that $|D_4| \geq 3$. If $|E(G^*[D_2 \cup D_3])| = 0$, then $D_2 \cup D_3$ is an independent set. Applying Lemma 3.1 to $D_2 \cup D_3$, $n \geq |D_4| + 1 + 5(|D_2| - 1)$ and hence $|D_4| \leq 16 - 10 - 1 = 5$. If $|D_4| = 3$, then $|E(G^*[D_4])| \leq 2$. Thus, $6 \geq \partial(D_2 \cup D_3) = e(D_2 \cup D_3, D_4) = \partial(D_4) \geq 12 - 4 = 8$, a contradiction. If $|D_4| = 4$, then $|D_4| = 4$ and $|E(G^*[D_4])| \leq 4$. Thus, $6 \geq \partial(D_2 \cup D_3) = e(D_2 \cup D_3, D_4) = \partial(D_4) \geq 16 - 8 = 8$, a contradiction. If $|D_4| = 5$, then $|E(G^*[D_4])| \leq 5$. Thus, $6 \geq \partial(D_2 \cup D_3) = e(D_2 \cup D_3, D_4) = \partial(D_4) \geq 20 - 12 = 8$, a contradiction.

Thus, $|E(G^*[D_2 \bigcup D_3])| \geq 1$. It follows that $\partial(D_4) = \partial(D_2 \cup D_3) = \partial(D_2) \leq 4$ since $|D_2| = 3$, which implies that $2|E(G^*[D_4])| \geq 4|D_4| - 4$. Since $|V(G^*[D_4])| = |D_4|$, $|E(G^*[D_4])|/(|V(G^*[D_4])| - 1) \geq 2$. Applying Theorem 2.3 to $G^*[D_4]$, $G^*[D_4]$ contains a subgraph H with $\tau(H) \geq 2$, contrary to that G^* is the reduction of G.

Now, we assume that $|D_2| = 4$. If $|D_4| = 1$, then $|V(G^*)| = 5$. Thus, the vertex in D_4 is adjacent to all other vertices of G^*. It follows from $\delta(G^*) \geq 2$ that $G^*[D_2]$ contains edges and thus G^* contains a 3-cycle, contrary to Theorem 2.6(i). Thus, we have $|D_4| \geq 2$. If $|E(G^*[D_2])| = 0$, then D_2 is an independent set. Applying Lemma 3.1 to D_2, $n \geq |D_4| + 1 + 5(|D_2| - 1)$ and hence $|D_4| \leq 16 - 15 - 1 = 0$, contrary to the hypothesis that $\Delta(G^*) = 4$. Thus, $|E(G^*[D_2])| \geq 1$. Applying Lemma 3.1 to D_2, $|D_4| \leq 16 - 10 - 2 = 4$. If $|D_4| = 4$, then $|E(G^*[D_4])| \leq 4$. In this case, $6 \geq \partial(D_2) = e(D_2, D_4) = \partial(D_4) \geq 16 - 8 = 8$, a contradiction. If $|D_4| = 3$, then $|E(G^*[D_4])| \leq 1$ and $6 \geq \partial(D_2 \cup D_3) = e(D_2 \cup D_3, D_4) = \partial(D_4) \geq 12 - 2 = 10$, a contradiction. Thus. $|D_4| = 2$. Recall that $|E(G^*[D_2])| \geq 1$. If two vertices in D_4 are not adjacent, then each vertex is adjacent to both end vertices of an edge in $E(G^*[D_2])$. Then G^* has a 3-cycle, contrary to Theorem 2.6(i). Thus, two vertices in D_4 are adjacent. In this case, $G^*[D_2]$ has only one edge. Thus, D_2 has a vertex adjacent to both vertices in D_4, which implies that G^* also has a 3-cycle, contrary to Theorem 2.6(i).

We are ready to complete the proof of our theorem. By Claims 1 and 2, $\Delta(G^*) \leq 3$. If $\Delta(G^*) = 3$, then by (5) and (6) $|D_3| = 2$ and $|D_2| = 2$ since $|D_3|$ is even. Then $|V(G^*)| = 4$ and G^* has a 3-cycle, which is contrary to Theorem 2.6(i). If $\Delta(G^*) = 2$, then $|E(G^*)| = |D_2| = |V(G^*)|$. Then G^* is a cycle. By (5), $|D_2| \leq 4$. Since G^* contains neither 2-cycle nor 3-cycles, it is a 4-cycle. ■

4. Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4. Theorem 3.3 tells us that Theorem 1.4 holds or G is isomorphic to the graph L in Fig. 2 for the case when $n \le 16$. Thus, we present here the complete proof of Theorem 1.4.

Lemma 4.1. *$\Lambda_g(L) \le 4$, where L is the graph in Fig. 2.*

Proof. Let L_0 be the subgraph of L induced by $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Then L_0 is isomorphic to a $K_{3,3}$. By Lemma 2.2 or by [9, Theorem 1.5], $\Lambda_g(K_{3,3}) \le 4$. L/L_0 contains 2-cycles. We repeatedly contract these 2-cycles until no 2-cycle left and the resulting graph is K_1. It follows that $\Lambda_g(L/L_0) \le 4$ from Lemma 2.1 and thus $\Lambda_g(L) \le 4$. ■

Theorem 4.2. *Let G be a 2-edge-connected simple graph on $n \ge 17$ vertices. If for every $uv \notin E(G)$, $\max\{d(u), d(v)\} \ge n/4$, then $G^* \in \{K_1, K_{2,3}, C_4, C_5\}$, where C_k is a k-cycle.*

Proof. Since $n \ge 17$, $n/4 > 4$. If $G^* = K_1$, we are done. Thus, assume that $G^* \ne K_1$. Since G^* is 2-edge-connected, by Lemma 2.9,

$$|D_2| + |D_3| + |D_4| \le 5. \tag{10}$$

Utilizing (2) and (10), we have

$$|D_2| \ge 1 + |D_4| + \sum_{i \ge 5} (i-4)|D_i|. \tag{11}$$

In order to complete our proof, we need to establish the following claims.

Claim 1. $\Delta(G^*) \le 6$.

If $\Delta(G^*) \ge 9$, then by (11), $|D_2| \ge 1 + (\Delta(G^*) - 4) \ge 1 + 5 = 6$, contrary to (10). If $\Delta(G^*) = 8$, then $|D_8| \ge 1$. By (10) and (11),

$$5 \ge |D_2| + |D_3| \ge |D_2| \ge 1 + |D_4| + |D_5| + 2|D_6| + 3|D_7| + 4|D_8| \ge 5,$$

which implies that $|D_2| = 5$ and $|D_i| = 0$ for $3 \le i \le 7$. In this case, $|D_8| = 1$. It follows that $|V(G^*)| = |D_2| + |D_8| = 6$. As $\Delta(G^*) = 8$, G^* cannot be simple, contrary to Theorem 2.6(i).

Suppose that $\Delta(G^*) = 7$. By (10) and (11),

$$5 \ge |D_2| + |D_3| \ge |D_2| \ge 1 + |D_4| + |D_5| + 2|D_6| + 3|D_7| \ge 4, \tag{12}$$

which shows that $|D_7| = 1$, $|D_6| = 0$ and $|D_4| + |D_5| \le 1$.

If $|D_5| = 1$, then by (12) $|D_3| = |D_4| = 0$ and $|D_2| = 5$. Thus $|V(G^*)| = |D_7| + |D_5| + |D_2| = 7$. On the other hand, $\Delta(G^*) = 7$. It follows that G^* is not a simple, which is contrary to Theorem 2.6(i). Thus, $|D_5| = 0$. Since the number of all vertices of odd degree in G^* is even, it follows from (10) and (12) that $|D_3| = 1$, $|D_4| = 0$ and $|D_2| = 4$. Thus, $|V(G^*)| = |D_7| + |D_3| + |D_2| = 6$. On the other hand, $\Delta(G^*) = 7$, which also implies that G^* cannot be simple, contrary to Theorem 2.6(i).

Claim 2. $\Delta(G^*) \le 5$.

By Claim 1, $\Delta(G^*) \le 6$. Suppose otherwise that $\Delta(G^*) = 6$. By (10) and (11),

$$5 \ge |D_2| + |D_3| \ge |D_2| \ge 1 + |D_4| + |D_5| + 2|D_6|, \tag{13}$$

which implies that $1 \le |D_6| \le 2$.

If $|D_6| = 2$, then by (13), $5 \ge |D_3| + |D_2| \ge |D_2| \ge 1 + |D_4| + |D_5| + 4 \ge 5$, and thus $|D_3| = |D_4| = |D_5| = 0$, $|D_2| = 5$. Therefore $|V(G^*)| = |D_6| + |D_2| = 7$. Let $D_6 = \{v_1, v_2\}$. Then v_i is adjacent to all other vertices of G^*, for $i = 1, 2$. It follows that G^* contains a 3-cycle, contrary to Theorem 2.6(i).

Thus we may assume that $|D_6| = 1$. By (10) and (11),

$$5 \ge |D_2| + |D_3| \ge |D_2| \ge 1 + |D_4| + |D_5| + 2|D_6| \ge 1 + |D_4| + |D_5| + 2. \tag{14}$$

Then $|D_4| + |D_5| \le 2$. Since $|D_2| \ge |D_4| + |D_5| + 3$, by (10), $5 \ge |D_2| + |D_4| \ge 2|D_4| + |D_5| + 3$ and hence $|D_4| \le 1$.

Let $S = D_4 \cup D_5 \cup D_6$. Then $|S| \le 3$. Assume that $|S| = 3$. By (14), $|D_2| = 5$, $|D_3| = 0$. Since G^* contains neither 3-cycles nor 2-cycles, $|E(G^*[S])| \le 2$. In this case, $\partial(S) = \sum_{v \in S} d_{G^*}(v) - 2|E(G^*[S])| \ge 4 + 5 + 6 - 4 = 11$. On the other hand, since $|D_2| \le 5$, $\partial(D_2) = \sum_{v \in D_2} d_{G^*}(v) - 2|E(G^*[D_2])| \le 10$, which contradicts $\partial(S) = e(S, D_2) = \partial(D_2)$.

Thus, $|S| \le 2$. Since $|D_2| + |D_3| \le 5$, $|V(G^*)| \le 7$. Then the vertex in D_6 is adjacent to all other vertices in G^*. Since $\delta(G^*) \ge 2$, $G^*[D_5 \cup D_4 \cup D_3 \cup D_2]$ contains an edge. Thus, G^* contains a 3-cycle, which is contrary to Theorem 2.6(i).

Claim 3. $\Delta(G^*) \le 4$.

By Claim 2, $\Delta(G^*) \le 5$. Suppose, to the contrary, that $\Delta(G^*) = 5$. In this case, from (10) and (11), we have

$$5 \ge |D_2| + |D_3| \ge |D_2| \ge 1 + |D_4| + |D_5|, \tag{15}$$

which implies that $|D_5| \le 4$.

Assume first that $|D_5| = 4$. By (15), $|D_4| = |D_3| = 0$ and $|D_2| = 5$. Since G^* contains neither 3-cycles nor 2-cycles, $|E(G^*[D_5])| \le 4$ and $\partial(D_5) = \sum_{v\in D_5} d_{G^*}(v) - 2 * |E(G^*[D_5])| \ge 20 - 8 = 12$. On the other hand, $\partial(D_2) \le 10$. This contradicts $\partial(D_5) = e(D_5, D_2) = \partial(D_2)$.

Assume then that $|D_5| = 3$. By (15), $5 \ge |D_2| + |D_3| \ge |D_2| \ge 1 + |D_4| + 3 \ge 4$. Since the number of the vertices of odd degree in G^* is even, $|D_4| = 0$, $|D_3| = 1$ and $|D_2| = 4$. Let $S = D_3 \cup D_5$. Then $|S| = 4$. Since G^* has no 3-cycles nor 2-cycles, $|E(G^*[S])| \le 4$. Thus,

$$8 \ge \partial(D_2) = e(D_2, S) = \partial(S) = \sum_{v\in S} d_{G^*}(v) - 2 * |E(G^*[S])| \ge 15 + 3 - 8 = 10,$$

a contradiction.

Next, assume that $|D_5| = 2$. By (15), $5 \ge |D_2| + |D_3| \ge |D_2| \ge 1 + |D_4| + 2 \ge 3$. Let $S = D_3 \cup D_4 \cup D_5$. Since the number of the vertices of odd degree in G^* is even, $|D_3| = 2$ or 0. In the former case, by (15), $|D_4| = 0$. Thus $|D_2| = 3$ and $|S| = 4$. Since G^* does not have any cycle of length at most 3, $|E(G^*[S])| \le 4$. Thus, $6 \ge \partial(D_2) = e(D_2, S) = \partial(S) \ge 10 + 6 - 8 = 8$, a contradiction. In the latter case, $|D_3| = 0$. By (10) and (15), $5 \ge |D_2| + |D_4| \ge 1 + 2|D_4| + 2$ and thus $|D_4| \le 1$.

If $|D_4| = 1$, then by (15) $|D_2| = 4$ and $|S| = |D_3| + |D_4| + |D_5| = 3$. Since G^* does not have any cycles of length at most 3, $|E(G^*[S])| \le 2$. Thus, $8 \ge \partial(D_2) = e(D_2, S) = \partial(S) \ge 14 - 4 = 10$, a contradiction. Thus, $|D_4| = 0$. In this case, $V(G^*) = D_2 \cup D_5$. Since $\Delta(G^*) = 5$ and G^* is simple, $|V(G^*)| \ge 6$ and hence $|D_2| \ge 6 - 2 = 4$. By (15), $|D_2| \le 5$. If $|D_2| = 4$, then $|V(G^*)| = 6$. Let $D_5 = \{v_1, v_2\}$. For each $i = 1, 2$, v_i is adjacent to all other vertices in G^*. Thus, G^* contains a 3-cycle, contrary to Theorem 2.6(i). Suppose that $|D_2| = 5$. Since G^* does not contain any cycle of length at most 3, $G^* \cong K_{2,5}$. Let $V(G^*) = \{v_1, v_2, \ldots, v_7\}$, where $D_2 = \{v_3, v_4, \ldots, v_7\}$ and $D_5 = \{v_1, v_2\}$, and let H_i denote the preimage of v_i for $i = 1, 2, \ldots, 7$.

Define $X = \{x \in V(G) : d_G(x) < n/4\}$. By the given degree condition, if $x_1, x_2 \in X$, then $x_1x_2 \in E(G)$. Note that D_2 is an independent set of G^*. Then there is at most one vertex, say v_3 in D_2, such that $V(H_3) \cap X \ne \emptyset$, that is, $V(H_j) \cap X = \emptyset$ for $j = 4, 5, 6, 7$. It follows that each vertex in H_j has degree at least $n/4$ for $j = 4, 5, 6, 7$. On the other hand, $d_{G^*}(v_j) = 2 < n/4$, which is equivalent to $\partial(H_j) < n/4$ in G. Applying Lemma 2.8 to H_j for $j = 4, 5, 6, 7$, $|V(H_j)| > n/4$. Then $n = |V(G)| = \sum_{i=1}^{7} |V(H_i)| > 4(n/4) + 3 = n + 3$, a contradiction.

Finally, assume that $|D_5| = 1$. Let $S = D_2 \cup D_3 \cup D_4$. It follows from (10) and $\Delta(G^*) = 5$ that $|S| = 5$. Thus, $v \in D_5$ is adjacent to each vertex in S. On the other hand, since $\delta(G^*) \ge 2$, $G^*[S]$ contains edges. It follows that G^* contains a 3-cycle, contrary to Theorem 2.6(i).

We are ready to complete the proof of Theorem 4.2. By Claim 3, $\Delta(G^*) \le 4$. First, suppose that $\Delta(G^*) = 4$. By (10), $|V(G^*)| \le 5$. If $|D_4| \ge 2$, let $v_1, v_2 \in D_4$. For each $i = 1, 2$, v_i is adjacent to all other vertices of G^*. Thus, G^* has a 3-cycle, contrary to Theorem 2.6(i). If $|D_4| = 1$, $v \in D_4$ is adjacent to all other vertices of G^*. On the other hand, since $\delta(G^*) \ge 2$, $G^*[D_2 \cup D_3]$ contains edges. It follows that G^* contains a 3-cycle, contrary to Theorem 2.6(i).

Next, suppose that $\Delta(G^*) = 3$. It follows from (10) and (11) that:

$$5 \ge |D_2| + |D_3| \ge |D_2| \ge 1 \tag{16}$$

which implies that $|D_3| \le 4$. Since the number of the vertices of odd degree is even, $|D_3| = 4$ or 2. In the former case, by (16), $|D_2| = 1$. Note that G^* does not have any cycle of length at most 3. Then $|E(G^*[D_3])| \le 4$ and hence $2 \ge \partial(D_2) = e(D_2, S) = \partial(D_3) = \sum_{v\in D_3} d(v) - 2|E(G^*[D_3])| \ge 12 - 8 = 4$, which is a contradiction. In the latter case, $|D_2| \le 3$. If $|D_2| = 3$, then $G^* \cong K_{2,3}$. If $|D_2| \le 2$, then $|V(G^*)| \le 4$. Since G^* is 2-edge-connected and $|D_3| = 2$, it is easy to verify that G^* contains a 3-cycle, contrary to Theorem 2.6(i).

Finally, assume that $\Delta(G^*) = 2$. Then $|E(G^*)| = |D_2| = |V(G^*)|$. Since G^* is 2-edge-connected, G^* is a cycle. By (10), $|D_2| \le 5$. If $|D_2| \le 3$, then G^* is a cycle of length at most 3, which is contrary to Theorem 2.6(i). If $|D_2| = 4$, G^* is a 4-cycle. If $|D_2| = 5$, G^* is a 5-cycle. ■

The proof of Theorem 1.4. Let A be an abelian group with $|A| \ge 4$. By Theorems 3.3 and 4.2, $G^* \in \{K_1, C_4, C_5, K_{2,3}\}$, or is the graph L in Fig. 2. In the latter case, G is A-connected by Lemma 4.1. If G^* is K_1, then Lemma 2.7 shows that G is A-connected. If $G^* \in \{K_{2,3}, C_4\}$, then by Lemmas 2.1 and 2.2, $\Lambda_g(G) = 5$. If $G^* = C_5$, then by Lemma 2.1, $\Lambda_g(G) = 6$. ■

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