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Note

On (s,t)-supereulerian graphs in locally highly connected graphs[☆]

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ABSTRACT

Given two nonnegative integers s and t, a graph G is (s, t)-*supereulerian* if for any disjoint sets $X, Y \subset E(G)$ with $|X| \le s$ and $|Y| \le t$, there is a spanning eulerian subgraph H of G that contains X and avoids Y. We prove that if G is connected and locally k-edge-connected, then G is (s, t)-supereulerian, for any pair of nonnegative integers s and t with $s+t \le k-1$. We further show that if $s + t \le k$ and G is a connected, locally k-edge-connected graph, then for any disjoint sets $X, Y \subset E(G)$ with $|X| \le s$ and $|Y \le t$, there is a spanning eulerian subgraph H that contains X and avoids Y, if and only if $G - Y$ is not contractible to K_2 or to $K_{2,l}$ with l odd.

1. Introduction

Graphs in this note are simple, nontrivial, and finite. We follow the notations of Bondy and Murty [2] unless otherwise stated. For a graph G, $O(G)$ denotes the set of all vertices of odd degree in G. A graph G with $O(G) = \emptyset$ is an *even graph*, and a connected even graph is an *eulerian graph*. A graph is *supereulerian* if it has a spanning eulerian subgraph. The collection of all supereulerian graphs will be denoted by SL. For a graph G with a connected subgraph H, the contraction G/H is the graph obtained from G by replacing H by a vertex v_H, such that the number of edges in G/H joining any $v \in V(G) - V(H)$ to v_H in G/H equals the number of edges joining v to v_H in G. A graph H is *nontrivial* if $E(H) \neq \emptyset$. As in [2], the connectivity, the edge-connectivity, the minimum degree, and the maximum degree of G are denoted by $\kappa(G)$, $\kappa'(G)$, $\delta(G)$, and $\Delta(G)$, respectively.

For an integer $i \ge 1$, define $D_i(G) = \{v \in V(G) | d_G(v) = i\}$. For a vertex $v \in V(G)$, $N_G(v)$ denotes the set of all vertices adjacent to v in G. When the graph G is understood from the context, we also use $N(v)$ for $N_G(v)$. A vertex v is a *locally connected* vertex if $G[N_G(v)]$, the subgraph induced by $N_G(v)$, is connected. A graph is *locally connected* if every $v \in V(G)$ is locally connected. For disjoint nonempty subsets $A, B \subset V(G)$, $[A, B]_G$ denotes the set of edges with one end in A and the other end in B. When G is understood from the context, we also use $[A, B]$ for $[A, B]_G$. In particular, for $v \in V(G)$, we define $E_G(v) = [\{v\}, V(G) - \{v\}]$.

The problem of supereulerian graphs was initiated in [1], and it has been intensively studied by many authors (see [3–5, 7], among others). Given two nonnegative integers s and t, a graph G is (s, t)-*supereulerian* if for any disjoint sets $X, Y \subset E(G)$ with $|X| \le s$ and $|Y| \le t$, there is a spanning eulerian subgraph H of G that contains X and avoids Y. Clearly, G is supereulerian if and only if G is $(0, 0)$-supereulerian. Since every supereulerian graph must be 2-edge-connected, it follows that any (s, t)-supereulerian graph must be $(t + 2)$-edge-connected. In [3], Catlin obtained the following theorem.

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Theorem 1.1. *If G is connected and locally connected, then $G \in SL$.*

In order to extend Theorem 1.1, we here introduce some definitions and notations. A graph is *collapsible* if for every set $R \subset V(G)$ with $|R|$ even, there is a spanning connected subgraph H_R of G, such that $O(H_R) = R$. Thus K_1 is both supereulerian and collapsible. Denote the family of collapsible graphs by CL. Let G be a collapsible graph and let $R = \emptyset$. By definition, G has a spanning connected subgraph H with $O(H) = \emptyset$, and so G is supereulerian. Therefore, we have $CL \subset SL$.

In [3], Catlin showed that every graph G has a unique collection of pairwise disjoint maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$. The contraction of G obtained from G by contracting each H_i into a single vertex ($1 \le i \le c$), is called the *reduction* of G. A graph is *reduced* if it is its own reduction.

Let G be a graph and let $e = uv$ be an edge of G. An *elementary subdivision* of G at e is a graph G_e obtained from $G - e$ by adding a new vertex $v(e)$ and by adding two new edges $uv(e)$ and $v(e)v$. We also say that we obtained G_e by *subdividing* the edge e. For a subset $X \subseteq E(G)$, define G_X to be the graph obtained from G by applying elementary subdivision to each edge of X (subdividing every edge in X).

Theorem 1.1 has been extended to (s, t)-supereulerian graphs by Lei et al. [8], for the special case when $s \le 2$.

Theorem 1.2. *Let s and t be nonnegative integers with $s \le 2$. Suppose that G is a $(t + 2)$-edge-connected locally connected graph on n vertices. For any disjoint sets $X, Y \subset E(G)$ with $|X| \le s$ and $|Y| \le t$, exactly one of the following holds:*

(i) *G has a spanning eulerian subgraph H such that $X \subset E(H)$ and $Y \cap E(H) = \emptyset$.*
(ii) *The reduction of $(G - Y)_X$ is a member of $\{K_1, K_2, K_{2,t}\ (t \ge 1)\}$.*

The main purpose of this paper is to improve further these results on $(0, 0)$-supereulerian and $(2, t)$-supereulerian locally connected graphs to (s, t)-supereulerian in locally k-edge-connected graphs. A graph is *locally k-edge-connected* if for every $v \in V(G)$, $G[N(v)]$ is k-edge-connected. Our first result is as follows.

Theorem 1.3. *Let $k \ge 1$ be an integer. If G is a connected, locally k-edge-connected graph, then G is (s, t)-supereulerian for all pairs of nonnegative integers s and t with $s + t \le k - 1$.*

Consider a connected, locally k-edge-connected graph G with an edge cut D of $k + 1$ edges (for example, $G \cong K_{k+2}$). Let $Y \subset D$ with $|Y| = k$. The graph G cannot have a spanning eulerian subgraph that avoids the edges of Y. Thus Theorem 1.3 is best possible in the sense that the bound $s + t \le k - 1$ cannot be relaxed. However, this example motivates the following theorem.

Theorem 1.4. *For $k \ge 1$, let s and t be nonnegative integers such that $s + t \le k$. Let G be a connected, locally k-edge-connected graph, then for any disjoint sets $X, Y \subset E(G)$ with $|X| \le s$ and $|Y| \le t$, there is a spanning eulerian subgraph H that contains X and avoids Y if and only if $G - Y$ is not contractible to K_2 or to $K_{2,l}$ with l odd.*

Corollary 1.5. *Let G be a connected, locally k-edge-connected graph. Let s and t be nonnegative integers such that $s + t \le k$.*

(i) *If $t < k$ and $k \ge 3$, then G is (s, t)-supereulerian.*
(ii) *If $\kappa'(G) \ge k + 2$ and $k \ge 3$, then G is (s, t)-supereulerian.*

The rest of this note is organized as follows: several lemmas that will be used in the subsequent section are established in Section 2, and the proofs of all the main results are deferred to Section 3.

2. Preliminary results

The following well-known result comes from Catlin [3].

Lemma 2.1. *Let H be a collapsible subgraph of a graph G, then*

(i) *G is collapsible if and only if G/H is collapsible;*
(ii) *G is supereulerian if and only if G/H is supereulerian.*

The next lemma follows immediately from Theorem 8 and Lemma 5 in Caltin [3].

Lemma 2.2. *If G is reduced, then G is simple and contains no K_3. Moreover, if $\kappa'(G) \ge 2$, then $\sum_{i=2}^{3} |D_i(G)| \ge 4$, and if $\sum_{i=2}^{3} |D_i(G)| = 4$, then G is eulerian.*

The following lemma follows from the definition and Lemma 2.1 immediately.

Lemma 2.3. *Let G be a graph and let $X, Y \subseteq E(G)$ be disjoint subsets such that $|X| \le s$ and $|Y| \le t$. Then the following are equivalent.*

(i) *G has a spanning eulerian subgraph H that contains X and avoids Y.*
(ii) *$(G - Y)_X$ has a spanning eulerian subgraph.*
(iii) *The reduction of $(G - Y)_X$ is supereulerian.*

Note that every supereulerian graph must be 2-edge-connected. The next result follows from Lemma 2.3.

Lemma 2.4. *If G has a spanning eulerian subgraph H that contains X and avoids Y, then $\kappa'(G - Y) \ge 2$.*

3. Highly locally connected graphs

Before we prove the main results, note that G is a connected and locally k-edge-connected graph in this section, so we need to obtain the relative properties of G. It suffices to prove the following lemmas.

Lemma 3.1. *Let $k \geq 1$ be an integer. If G is connected and locally k-edge-connected, then G is 2-connected and $k+1$-edge-connected.*

Proof. If G is connected and has a cut-vertex u, then the neighborhood of u in G is disconnected, contradicting the assumption that G is locally k-edge-connected.

Let X be an edge cut of G, and let $e = uv \in X$. We will show that $|X| \geq k+1$. Since G is locally k-edge-connected, $\kappa'(G[N(v)]) \geq k$, and so $|V(G[N(v)])| \geq k+1$. If $E_G(v) \subseteq X$, then $|X| \geq |V(G[N(v)])| \geq k+1$. Thus, we assume that $E_G(v) \nsubseteq X$, and so there must be a vertex $w \in N(v) - \{u\}$ such that v and w are in the same component of $G-X$. As $G[N(v)]$ is connected, $G[N(v)]$ has a (u, w)-path P, and so

$$X' = E(G[N(v)]) \cap X \supseteq E(P) \cap X \neq \emptyset.$$

Since u and w are in different components of $G-X$, u and w must be in different components of $G[N(v)] - X$. It follows that $X' - uv$ is an edge cut of $G[N(v)]$, whence $|X| \geq |X'| \geq \kappa'(G[N(v)]) \geq k$, then $|X| \geq k+1$. □

A graph G is *k-triangulated* if every edge of G lies in at least k different triangles of G.

Lemma 3.2. *Let G be a simple nontrivial graph, and let v be an arbitrary vertex in $V(G)$. If $\delta(G[N(v)]) \geq k$, then G is k-triangulated. In particular, every connected, locally k-edge-connected graph is k-triangulated.*

Proof. Let $e = uv$. In $H = G[N(v)]$, since $\delta(H) \geq k$, u is adjacent to at least k different vertices. Thus u and v have at least k common vertices as neighbors, and $e = uv$ lies in at least k triangles.

Suppose that G is connected and locally k-edge-connected. For each $v \in V(G)$, $\delta(G[N(v)]) \geq \kappa'(G[N(v)]) \geq k$. Thus G must be k-triangulated. □

Let G be a simple graph. For disjoint subsets X and Y of $E(G)$ with $|X \cup Y| \leq k$, we define $X' = \{uv(e), v(e)v : uv = e \in X, v(e)$ is a newly added vertex at $e\}$ and denote G' the reduction of $(G-Y)_X$.

Lemma 3.3. *Let $k > 0$ be an integer. If G is a connected, locally k-edge-connected graph, X and Y are disjoint subsets of $E(G)$ with $|X \cup Y| \leq k$, and X' is as defined above, then*

(i) *If $|X \cup Y| \leq k-1$, then $E(G') \subseteq X'$.*
(ii) *If $|X \cup Y| = k \geq 3$, then either $\kappa'(G') = 1$, or $E(G') \subseteq X'$.*
(iii) *If $|X \cup Y| = k \geq 3$, and $\kappa'(G') = 1$, then $|E(G') - X'| \leq 1$.*

Proof. We argue by contradiction and assume that $E(G') - X'$ has an edge e, and so $e \in E(G) - (X \cup Y)$. By Lemma 3.2, G is k-triangulated, and so G has at least k triangles, denoted as $L_1, \ldots, L_k$, all containing e.

(i) Since $|X \cup Y| \leq k-1$ and since $e \notin X \cup Y$, there is at least one triangle that is disjoint from $X \cup Y$. Therefore, by Lemma 2.2, e lies in a collapsible subgraph of G'. It follows that the reduced graph G' has a triangle, contrary to Lemma 2.2.

(ii) and (iii) . Now note that $k > 2$ and suppose that $|E(G') - X'| \geq 1$. □

Claim 1. $|E(G') - X'| \leq 1$.

If not, then $G' - X'$ has two edges e_1 and e_2. By Lemma 3.2, G is k-triangulated, and so G has 3-cycles $L_1^j, \ldots, L_k^j$, all containing e_j, for $j \in \{1, 2\}$. Since G is simple, for each $i \neq i'$ and $j \in \{1, 2\}$ we have

$$E(L_i^j) \cap E(L_{i'}^j) = \{e_j\}.$$

Since $e_1 \neq e_2$, we have

$$|\{L_i^1 : i = 1, 2, \ldots, k\} \cap \{L_i^2 : i = 1, 2, \ldots, k\}| \leq 1.$$

We assume that when the equality above holds, then $L_1^1 = L_1^2$.

Since G' is reduced, by Lemma 2.2, none of these 3-cycles are contained in G': they are broken by deleting Y or subdividing X. It follows by $|X \cup Y| = s + t \leq k$ that for $i \in \{1, \ldots, k\}$,

$$|(X \cup Y) \cap E(L_i^1) \cap E(L_i^2)| = 1, \text{ and so } s + t = k. \quad (1)$$

First we assume that $L_1^1 \neq L_1^2$. Since $e_1 \neq e_2$, Lemma 3.2 implies that every edge of G lies in at least k triangles. Thus we need $2k$ edges to be subdivided or deleted. It follows that e_1 and e_2 are adjacent. By (1), we may assume, relabelling if necessary, that $e_1 = v_0v_1$ and $e_2 = v_0v_2$, and there exist $z_1, z_2, z_3, \ldots, z_k$ such that $e_i' = v_0z_i \in X$ for $1 \leq i \leq s$, and $e_i' = v_0z_i \in Y$ for $s+1 \leq i \leq s+t$. Note that $v(e_i')$ is a newly added vertex at e_i', thus $v_0v(e_i'), v(e_i')z_i \in X'$, for $1 \leq i \leq s$. See Fig. 1.

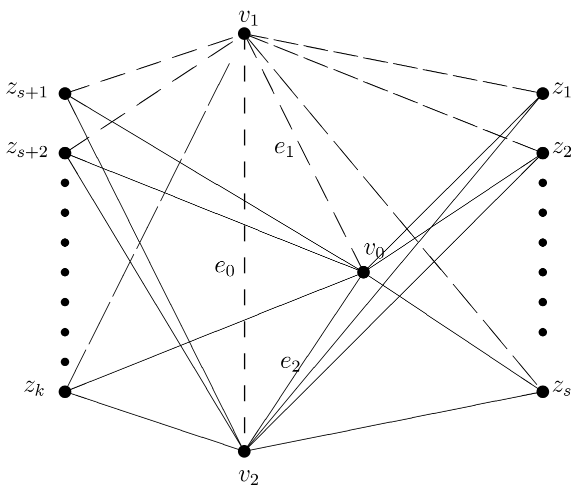

Fig. 1. $G[N_G(v_1)]$ and $G[N_G(v_2)]$.

Lemma 1A. *$v_jz_i \in E(G')$ for $j = 1, 2$ and $i = 1, 2, \ldots, k$.*

In fact, if one of them, say v_1z_1, is not in $E(G')$, then by the definition of G', v_1z_1 must be in a collapsible subgraph H' of $(G-Y)_X$. It is clear that $H' \cup (G-Y)_X[\{v_1z_1, v_2z_1, e_1, e_2\}]/H' \cong K_3$, which consists of the edges e_1, e_2, v_2z_1. Since K_3 is collapsible, it follows by Lemma 2.1(i) that $H' \cup (G-Y)_X[\{v_1z_1, v_2z_1, e_1, e_2\}]$ is also a collapsible subgraph of $(G-Y)_X$. Hence both e_1 and e_2 are in a collapsible subgraph of $(G-Y)_X$, which is contrary to the assumption that $e_1, e_2 \in E(G')$. This proves Lemma 1A.

Lemma 2A. $N_G(v_1) = N_G(v_2) = \{v_0, z_1, z_2, \ldots, z_k\}$.

If not, then we may assume, without loss of generality, that there exists $z \in N_G(v_1) - \{v_0, z_1, z_2, \ldots, z_k\}$. Since G is locally connected, we may further assume that this vertex z is adjacent to z_1 in G, and so in $(G-Y)_X$ as well. Now $G[\{v_1, z_1, z\}] \cong K_3$, and K_3 is collapsible. Hence the definition of G' yields v_1z_1, contrary to Lemma 1A. This proves Lemma 2A.

By its definition, G' is reduced. By Lemma 2.2, G' cannot have any 3-cycles, and so $\{z_1, z_2, \ldots, z_k\}$ must be an independent set in G'. It follows by the definition of G' that $\{z_1, z_2, \ldots, z_k\}$ must also be an independent set in G. If $k > 2$, then by Lemma 1A, $G[N(v_1)]$ cannot be k-edge-connected, and so we must have $k = 1$ or $k = 2$, contrary to the assumption that $k > 2$. This contradiction establishes Claim 1 under the assumption that $L_1^1 \neq L_1^2$.

If suppose that $L_1^1 = L_1^2$, then $e_0 = v_1v_2 \in E(G)$, as $L_1^1 = L_1^2 = G[\{v_0, v_1, v_2\}]$. By Lemma 2.2, G' cannot have any 3-cycles, and so $v_1v_2 \in X \cup Y$. With an argument like that used in the proof of Lemma 1A, we conclude that $v_jz_i \in E(G')$ for $j \in \{1, 2\}$ and $i \in \{2, \ldots, k\}$, and when $e_0 \in X$, $v_1v(e_0), v_2v(e_0) \in E(G')$. With a similar argument used in the proof of Lemma 2A, we further conclude that $N_G(v_1) = \{v_0, v_2, z_2, \ldots, z_k\}$ and $N_G(v_2) = \{v_0, v_1, z_2, \ldots, z_k\}$. Since $k > 2$, and the facts above imply that $G[N_G(v_1)]$ cannot be k-edge-connected, contrary to the assumption that G is locally k-edge-connected. This contradiction establishes Claim 1 under the assumption that $L_1^1 = L_1^2$.

We now prove (ii) and (iii). Suppose that $k > 2$ and $e \in E(G') - X'$. By Lemma 3.2, G has at least k triangles, denoted by $L_1, \ldots L_k$, all containing e. It follows that $|E(\bigcup_{i=1}^{k} E(L_i))| = 2k+1$. Since $|X \cup Y| = k > 2$, each L_i must have exactly one edge in $X \cup Y$. Moreover, by Lemma 2.2, G' contains no 3-cycles, and so if L_i in G contains an edge in X, then $E(L_i) - X \subseteq E(G')$. It follows that if $s > 0$, then G' contains at least two edges in $E(G') - X'$, contrary to Claim 1. Thus we must have $s = 0$, and so $|Y| = k$. Using Claim 1 and the facts that $s = 0$ and $t = |Y| = k$, we conclude that $|E(G')| = 1$, and so $G' \cong K_2$. This proves that if $|E(G') - X'| = 1$, then $G' \cong K_2$, and so $\kappa'(G') = 1$. Otherwise, Claim 1 yields $|E(G') - X'| = 0$ and hence $E(G') \subseteq X'$, thus (ii) holds. □

We now turn to the proof of Theorem 1.3. We assume that G is a connected, locally k-edge-connected graph, and X and Y are disjoint subsets of $E(G)$ with $|X| \le s$ and $|Y| \le t$, and that G' is the reduction of $(G-Y)_X$.

Proof of Theorem 1.3. By Lemma 3.1, $\kappa'(G) \ge k+1$, further $G' \cong K_1$ or $\kappa'(G') \ge 2$ from the hypothesis $|X| + |Y| \le k-1$. In the first case, G' is supereulerian, and we are done. In the second case, let $v \in V(G')$ be an arbitrary vertex; note that $d_{G'}(v) \neq 0$. Let H_v be the preimage of v, and let D denote the edge cut in G consisting of the edges with exactly one end in $V(H_v)$. Lemma 3.3(i) implies $E(G') \subseteq X'$, and so $D \subseteq X \cup Y$. Now $k+1 \le \kappa'(G) \le |D| \le |X| + |Y| \le k-1$, a contradiction. This proves Theorem 1.3. □

Lemma 3.4 (*Catlin, Lemma 1 of [6]*). *$K_{3,3} - e$ is collapsible.*

Lemma 3.5. *Let k be a positive integer, let s and t be nonnegative integers with $s + t = k$, and let G be a connected, locally k-edge-connected graph.*

(i) *If $k = 2$, then G is (s, t)-supereulerian or $G' \in \{K_2, K_{2,3}\}$;*
(ii) *If $k = 1$, then G is (s, t)-supereulerian or $G' \in \{K_2, K_{2,l} : l \text{ is odd}\}$.*

Proof. (i) Assume that G' is nontrivial; otherwise, there is nothing to prove. If $E(G') - X' = \emptyset$, i.e., $E(G') \subseteq X'$, since $s \leq 2$, then G' is a path whose length is 2 or 4. So $\kappa'(G')$ is 1. However, if G' is a path with length 2, then $t \leq 1$. Now Lemma 2.4 implies that $\kappa'(G - Y) \geq 2$, which contradicts G' being a path. If G' is a path with length 4, then $t = 0$; now Lemma 3.1 yields $\kappa'(G - Y) \geq 3$, again a contradiction.

So we can assume there is an edge $e = uv \in E(G') - X'$; that is, $E(G') - X' \neq \emptyset$. Note that $e \in E(G)$. By Lemma 3.2, every edge of G lies in at least two triangles. By Lemma 2.2, none of these 3-cycles are contained in G', which is the reduction of $(G - Y)_X$. If e lies in at least three triangles in G, we can find a contradiction. Because when $s + t = 2$, this edge e will be contracted even though two triangles disappear by deleting Y or subdividing X. Since $e \in E(G') - X'$, this is a contradiction. Hence we may assume that e lies in exactly two triangles. Let L_1 and L_2 be triangles containing e. Denote the third vertex of L_i by z_i, for $i \in \{1, 2\}$.

Case 1. $t = 2$.

Since $e = uv \in G'$, we must delete one edge respectively from triangle L_1 and L_2.

Case 1.1. $e' = z_1z_2 \in E(G)$.

By symmetry, if we delete the two edges $\{uz_1, uz_2\}$, then $G' \cong K_2$. If we delete the two edges $\{uz_1, vz_2\}$, then there is no edge lies in some other triangle since $e \in E(G') - X'$, thus the G' is a C_4, so that G is $(0, 2)$-supereulerian.

Case 1.2. $e' = z_1z_2 \notin E(G)$.

Each edge of G lies in at least two triangles by Lemma 3.2. If we delete the two edges $\{uz_1, uz_2\}$ or $\{uz_1, vz_2\}$, the left edges of $G - e$ will be contracted and therefore $G' \cong K_2$.

Case 2. $t = 0$, $t = 1$.

In this case, $e' = z_1z_2 \in E(G)$. If not, for $s = 2$ or $s = 1$, we must subdivide one edge from triangle L_1 or L_2. Without loss of generality, we may assume that the edge uz_1 is subdivided from triangle L_1 by adding a new vertex w at uz_1. The edge z_1v of L_1 lies in some other triangle by Lemma 3.2, so that $\{uw, wz_1, z_1v, uv\}$ will be contracted to K_1, which contradicts to $e = uv \in G'$.

Case 2.1. $t = 1$.

Without loss of generality, we suppose that one edge in L_1 is subdivided and one edge in L_2 is deleted. If we subdivide uz_1 and delete uz_2, then $G' \cong K_1$, a contradiction. If we subdivide uz_1 and delete vz_2, then there is no edge lies in some other triangle since $e \in E(G') - X'$, thus $G' \cong K_{2,3}$. If we subdivide uz_2 and delete vz_1, then the same result can be obtained in the similar manner.

Case 2.2. $t = 0$.

Clearly, if we subdivide the two edges $\{uz_1, uz_2\}$ or $\{vz_1, vz_2\}$, then $G' \cong K_1$, a contradiction. If we subdivide the two edges $\{uz_2, vz_1\}$ or $\{uz_1, vz_2\}$, then G' is $K_{3,3} - e$. By Lemma 3.4, $G' \in CL \subseteq SL$. Therefore G is $(2, 0)$-supereulerian. So (i) holds.

(ii) By Lemma 3.2, each edge of G lies in at least one triangle. Since $k = 1$, let $e_1 = X \bigcup Y$, which is subdivided if $e_1 \in X$ or deleted if $e_1 \in Y$.

If e_1 lies in exactly one triangle in G, then let L denote this triangle. First, we assume $s = 0$ and $t = 1$. Consider the other two edges of L after deleting e_1; we have three cases. If each edge lies in some triangle, then G is $(0, 1)$-supereulerian. If only one edge lies in the other triangle, then $G' \cong K_2$. If each edge only lies in L, then $G' \cong K_{2,1}$.

Now, we consider $s = 1$ and $t = 0$. After we subdivide the edge e_1, if one of the other two edges of L lies in some triangle, then $G' \cong K_1$. Otherwise, G' is a 4-cycle, which is supereulerian. In these two cases, G is $(1, 0)$-supereulerian.

If e_1 lies in at least two triangles in G, then let L denote one of the triangles. Suppose $s = 0$ and $t = 1$. If some edge of L other than e_1 lies in some other triangle, then $G' \cong K_1$ and G is $(0, 1)$-supereulerian. Otherwise, G is (s, t)-supereulerian if l is even, and $G' \cong K_{2,l}$ if l is odd.

Again we consider $s = 1$ and $t = 0$. After we subdivide the edge e_1, if some edge of L lies in some other triangle, then $G' \cong K_1$ and G is $(0, 1)$-supereulerian. Otherwise, when l is odd, G' is supereulerian and G is $(1, 0)$-supereulerian; when l is even, $G' \cong K_{2,l+1}$. This proves Lemma 3.5(ii). □

Proof of Theorem 1.4. *Necessity*. If G is connected, then G is (s, t)-supereulerian, so $G - Y$ cannot be contracted to K_2 or to $K_{2,l}$ for odd l.

Sufficiency. Let G be a connected, locally k-edge-connected graph and let $X, Y \subset E(G)$ be disjoint sets with $|X| \leq s$ and $|Y| \leq t$.

If $s + t < k$, then Theorem 1.3 implies that G is (s, t)-supereulerian, and so G has a spanning eulerian subgraph H with $X \subseteq E(H)$ and with $E(H) \cap Y = \emptyset$. Hence we assume that $s + t = k$.

Case 1. $k \geq 3$.

Note that a connected graph G cannot be contracted to K_2 if and only if $\kappa'(G) \geq 2$. Now we assume that $\kappa'(G - Y) \geq 2$ to show that G has a spanning eulerian subgraph H with $X \subseteq E(H)$ and with $E(H) \cap Y = \emptyset$.

Again we use G' to denote the reduction of $(G-Y)_X$, and we will show that G' is supereulerian. Since $\kappa'(G-Y)\geq 2$, we have either $G'\cong K_1$ or $\kappa'(G')\geq 2$. In the first case, G' is supereulerian, and we are finished. In the second case, since $k\geq 3$, Lemma 3.2 implies that G is k-triangulated. By Lemma 3.3(ii), since $k\geq 3$, the assumption that G' is 2-edge-connected implies that $E(G')\subseteq X'$. Let $w\in V(G')$ be an arbitrary vertex; note that $d_{G'}(w)\neq 0$. If $D'=E_{G'}(w)$, then D' corresponds to an edge cut $D\subseteq X$ of $G-Y$, and so $D\cup Y$ contains an edge cut of G. By Lemma 3.1, $\kappa'(G)\geq k+1$. We now have

$$k+1\leq \kappa'(G)\leq |D|+|Y|\leq |X\cup Y|=s+t=k,$$

a contradiction, which implies that G' must be a cycle, and so G' is supereulerian.

Case 2. $k=1$, $k=2$.

By Lemma 3.5, if $k=1$, $k=2$, and G cannot be contracted to K_2 or to $K_{2,l}$ for odd l, then G is (s,t)-supereulerian. Thus the proof is complete. □

Proof of Corollary 1.5. Let X and Y be disjoint subsets of $E(G)$ such that $|X|\leq s$ and $|Y|\leq t$.

By Lemma 3.1, $\kappa'(G)\geq k+1$. If $|Y|\leq t<k$ and $k\geq 3$, then $\kappa'(G-Y)\geq 2$, and so $G-Y$ cannot be contracted to K_2. It follows by the proof of Theorem 1.4 that G has a spanning eulerian subgraph H with $X\subseteq E(H)$ and with $E(H)\cap Y=\emptyset$, and so by definition, G is (s,t)-supereulerian. This proves Corollary 1.5(i).

If $\kappa'(G)\geq k+2$, then since $|Y|\leq t\leq k$ and $k\geq 3$, we conclude that $\kappa'(G-Y)\geq 2$, and so $G-Y$ cannot be contracted to K_2. Again by the proof of Theorem 1.4, there is a spanning eulerian subgraph H that contains X and avoids Y, and so by definition, G is (s,t)-supereulerian. This proves Corollary 1.5(ii). □

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