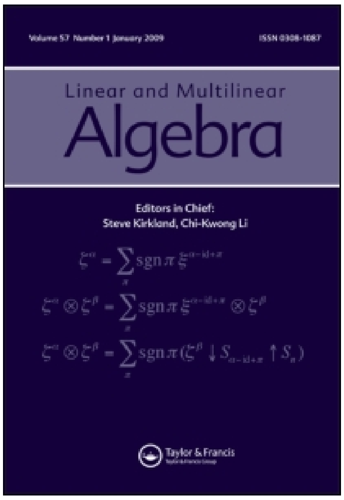

Linear and Multilinear Algebra
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713644116

Multi-*g* base index of primitive anti-symmetric sign pattern matrices

Yanting Liang [a]; Bolian Liu [b]; Hong-Jian Lai [a]
[a] Department of Mathematics, West Virginia University, Morgantown, WV, USA [b] Department of Mathematics, South China Normal University, Guangzhou, P.R. China

To cite this Article Liang, Yanting, Liu, Bolian and Lai, Hong-Jian'Multi-*g* base index of primitive anti-symmetric sign pattern matrices', Linear and Multilinear Algebra, 57: 6, 535 — 546
To link to this Article: DOI: 10.1080/03081080701861849
URL: http://dx.doi.org/10.1080/03081080701861849

Linear and Multilinear Algebra
Vol. 57, No. 6, September 2009, 535–546

Taylor & Francis
Taylor & Francis Group

Multi-*g* base index of primitive anti-symmetric sign pattern matrices

Yanting Liang[a], Bolian Liu[b*] and Hong-Jian Lai[a]

[a]*Department of Mathematics, West Virginia University, Morgantown, WV, USA;* [b]*Department of Mathematics, South China Normal University, Guangzhou, P.R. China*

Communicated by J.-Y. Shao

(*Received 20 June 2006; final version received 10 December 2007*)

In [Z. Li, F. Hall, and C. Eschenbach, *On the period and base of a sign pattern matrix*, Lin. Alg. Appl., 212/213 (1994), pp. 101–120], extended the concepts of base and period from non-negative matrices to powerful sign pattern matrices. In [L. You, J. Shao, and H. Shan, *Bounds on the bases of irreducible generalized sign pattern matrices*, Lin. Alg. Appl., 427 (2007), pp. 285–300], extended the concept of base from powerful sign pattern matrices to non-powerful generalized sign pattern matrices. In [Q. Li and B. Liu, *Multi-g base index of non-powerful generalized sign pattern matrices*, Ln. Multilin. Alg. (to appear)], extended the concept of kth multi-g base index from non-negative matrices to non-powerful generalized sign pattern matrices. In this article, we mainly study the bounds on kth multi-g base index, extremal graphs for the generalized base index for primitive anti-symmetric sign pattern matrices.

Keywords: kth multi-g base index; sign pattern matrices; extremal graph

AMS Subject Classification: 15A18; 15A48

1. Introduction

The *sign* of a real number a, denoted by sgn(a), is defined to be 1, -1 or 0, if $a > 0$, $a < 0$, or $a = 0$ respectively. The sign pattern matrix of a real matrix A, denoted by sgn(A), is the $(0, 1, -1)$-matrix obtained from A by replacing each entry by its sign.

The powers (especially the sign patterns of the powers) of square sign pattern matrices have been studied to some extent [4,5,9,10]. Notice that in the computation of (the signs of) the entries of the power A^k, an ambiguous sign may arise when we add a positive sign to a negative sign, so a new symbol '#' has been introduced to denote the ambiguous sign in [5]. The set $\Gamma = \{0, 1, -1, \#\}$ is called generalized sign set, where '#' denotes the ambiguous sign. We define addition and multiplication involving the symbol '#' as follows: for all $a \in \Gamma$ and for all $b \in \Gamma\backslash\{0\}$,

$$(-1) + 1 = 1 + (-1) = \#; \quad a + \# = \# + a = \#;$$
$$0 \cdot \# = \# \cdot 0 = 0; \quad b \cdot \# = \# \cdot b = \#.$$

*Corresponding author. Email: liubl@scnu.edu.cn

ISSN 0308–1087 print/ISSN 1563–5139 online

DOI: 10.1080/03081080701861849
http://www.informaworld.com

In [10], the matrices with entries in the set Γ are called generalized sign pattern matrices. The addition and multiplication of generalized sign pattern matrices are defined in the usual way, so that the sum and product of the generalized sign pattern matrices are still generalized sign pattern matrices.

It is well-known that graphical methods are often used in the study of the powers of square matrices [3,6,8], so we now introduce some graphical concepts.

A signed digraph S is a digraph where each arc of S is assigned with a value 1 or -1. A walk W is a sequence of arcs: $e_1, e_2, \ldots, e_k$ such that the terminal vertex of e_i is the same as the initial vertex of e_{i+1} for $i=1, 2, \ldots, k-1$. The number k is called the length of walk W, denoted by $l(W)$. The sign of walk W, denoted by $\mathrm{sgn}(W)$, is defined to be $\Pi_{i=1}^{k}\mathrm{sgn}(e_i)$.

Two walks W_1 and W_2 in a signed digraph are called a pair of *SSSD* walks, if they have the same initial vertex, the same terminal vertex and the same length, but different signs.

Let $A=(a_{ij})$ be a square generalized sign pattern matrix of order n. The associated digraph $D(A)$ of A is defined to be the digraph with vertex set $V=\{1, 2, \ldots, n\}$ and arc set $E=\{(i,j)|a_{ij}\neq 0\}$. The associated signed digraph $S(A)$ of A is obtained from $D(A)$ by assigning the sign of a_{ij} to each arc (i,j) in $D(A)$; A is called the associated matrix of $S(A)$.

We now use the associated signed digraph $S(A)$ to determine the generalized sign of the entry $(A^k)_{ij}$ of the power A^k of a square sign pattern matrix A. Notice that we have the following formula for $(A^k)_{ij}$:

$$(A^k)_{ij} = \Sigma_{W\in W_k(i,j)}\mathrm{sgn}(W),$$

where $W_k(i,j)$ denotes the set of walks of length k from vertex i to vertex j in $S(A)$.

From the expression of $(A^k)_{ij}$, we have these observations:

(i) $(A^k)_{ij}=0$ if and only if there is no walk of length k from i to j in $S(A)$ (i.e. $W_k(i,j)=\emptyset$).
(ii) $(A^k)_{ij}=1$ (or -1) if and only if $W_k(i,j)\neq\emptyset$ and all walks of length k from i to j have the same sign 1 (or -1).
(iii) $(A^k)_{ij}=\#$ if and only if there is a pair of *SSSD* walks of length k from i to j.

A square generalized sign pattern matrix A is called *powerful* if each power of A contains no '#' entry [5].

Let S be a signed digraph of order n. Then there is a sign pattern matrix A of order n whose associated signed digraph $S(A)$ is S. We say that S is powerful if A is powerful.

By the observation (iii) above, a sign pattern matrix A is powerful if and only if the associated signed digraph $S(A)$ contains no pair of *SSSD* walks.

Definition 1.1 [9] Let A be a square generalized sign pattern matrix of order n and $A, A^2, A^3, \ldots$ be the sequence of powers of A. (Since there are only a finite number of generalized sign pattern matrices of order n, there must be repetitions in the sequence.) If l is the least positive integer such that there is a positive integer p such that $A^l=A^{l+p}$, then l is called the generalized base of A, denoted by $l(A)$. Let S be the associated signed digraph of A. We define $l(S)=l(A)$.

Let A be a $(0,1)$-matrix. Following [1,7], we define $\exp_A(k)$ $(k\leq n)$ to be the least positive integer p such that there exist k rows with all entries 1 in A^p, if such integer p exists. Let D be the associated graph of matrix A. We also define $\exp_D(k)=\exp_A(k)$, if $\exp_A(k)$ exists.

Definition 1.2 [4] Let A be a non-powerful primitive square sign pattern matrix of order n, and k be a positive integer with $1 \leq k \leq n$. Then we define $l_A(k)$ to be the least positive integer l such that there exist k rows with all entries '#' in A^l. The number $l_A(k)$ is called the kth multiple generalized base index of A, simply called multi-g base index. Let S be the associated signed digraph of A. We define $l_S(k) = l_A(k)$.

By Definitions 1.1 and 1.2, if S is the non-powerful primitive associated signed digraph of an $n \times n$ sign pattern matrix A, then $l_A(n) = l(A) = l(S) = l_S(n)$.

The symmetric digraphs are digraphs with the property that for any distinct vertices i and j there is an arc from i to j if and only if there is an arc from j to i. It is well-known that, a symmetric digraph is primitive if and only if it is connected and has an odd cycle (Theorem 3.3.3 in [7]).

Let v be a vertex of a primitive digraph D (see Definition 3.3.1 of [7]). The vertex exponent of v, denoted by $\exp_D(v)$, is defined to be the least integer k such that for each vertex u in D, there is a walk of length k from v to u. Let S be a non-powerful primitive signed digraph of order n and $d(S)$ denote the diameter of S. The ambiguous index of S, denoted by $r(S)$, is defined to be the least positive integer r such that there is a pair of *SSSD* walks of length r in S. Let u and v be vertices of S, $r_{u,v}$ be the least positive integer r such that there is a pair of *SSSD* walks of length r from u to v, $d(u, v)$ be the length of a shortest path from u to v, and $l_S(u, v)$ be the least positive integer l such that for any integer $t \geq l$, there exists a pair of *SSSD* walks from u to v of length t.

THEOREM 1.1 (Li and Liu, [4]) *Let S be a primitive non-powerful signed digraph, u and v be vertices of S. Then*

$$l_S(k) \leq \min\{d(S), k-1\} + r_{u,v} + \exp_S(v). \tag{1}$$

For each positive odd integer $r \leq n$, let $S_n(r)$ denote the set of all connected graphs with vertex set $\{1, \ldots, n\}$ each of which has at least one cycle of length r but no cycle with odd length less than r.

THEOREM 1.2 (Brualdi and Shao [2]) *Let r be an odd integer with $1 \leq r \leq n$, and let $\exp_{S_n(r)}(k) = \max\{\exp_G(k) | G \in S_n(r)\}$. Then*

$$\begin{aligned}\exp_{S_n(r)}(k) &= \exp_{G_{n,r}}(k)\\ &= \begin{cases} (n-1) + (k-r) & \text{if } r \leq k \leq n \\ \max\left\{n - \dfrac{r+1}{2} + \left\lfloor \dfrac{k+1}{2} \right\rfloor - 1, r-1\right\} & \text{if } 1 \leq k \leq r-1 \end{cases}\end{aligned}$$

where $G_{n,r}$ is the symmetric digraph obtained by adding a cycle of length r at one of the end vertices of a path of order $n-r$.

We call a matrix in $\mathcal{A} = \{(a_{ij})_{n\times n} | a_{ij} = -a_{ji}, 1 \leq i < j \leq n\}$ an anti-symmetric matrix. In this article, we mainly study the kth multi-g base index and the extremal graphs with given generalized base index for a class of non-powerful generalized sign pattern matrices $\mathcal{A}_{n,r} = \{A = (a_{ij})_{n\times n} | a_{ij} = -a_{ji}, 1 \leq i < j \leq n$ and the length of the shortest odd cycle of the associated signed digraph of A is $r\}$. Let $S_{n,r}$ denote the set of all associated signed digraphs of $\mathcal{A}_{n,r}$. For any $S \in S_{n,r}$, the arcs (i,j) and (j,i) $(i \neq j)$ are called a pair of anti-signed arcs. If $W = v_1 v_2 \ldots v_k$ is a walk of S, then W^{-1} denotes the walk $v_k \ldots v_2 v_1$ obtained by reversing W. Let r be an odd integer with $3 \leq r \leq n$ and $C_r = u_1 u_2 \ldots u_r u_1$ be an odd cycle of S in its given order. Then C_r^{-1} denotes the cycle $u_1u_r \dots u_2u_1$ obtained by reversing C_r. If S only has two cycles C_r and C_r^{-1}, then we call S a uniquely cyclic signed digraph. If C is a cycle of odd length in S and x, y are two vertices of C, then the shorter and longer paths from x to y in C are called minor and major sections of C from x to y, respectively. If $P = v_0v_1v_2 \dots v_k$ is a (u, v)-walk and $Q = u_0u_1u_2 \dots u_t$ is a (v, w)-walk, then $P + Q$, called the sum of P and Q, denotes the (u, w)-walk $v_0v_1v_2 \dots v_{k-1}u_0u_1u_2 \dots u_t$. If P is a closed walk, let $2 \cdot P = P + P$. For an integer $s > 2$, define $s \cdot P = P + (s-1) \cdot P$. A walk may have several pairs of anti-signed arcs. For notational convenience, we sometimes will represent a walk W as a sum of some paths or cycles $P_1, P_2, \dots, P_k$ and a multiset of pairs of anti-signed arcs not in the P_i's. For example, $W = v_1v_2v_3v_2v_3v_2v_3v_4 = v_1v_2v_3v_4 + A$, where $A = \{v_3v_2, v_2v_3, v_3v_2, v_2v_3\}$ is the set of anti-signed arcs of W not in $v_1v_2v_3v_4$. For a (u, v)-walk $W = v_0v_1v_2 \dots v_k$ of length k, we define $F(W)$ to be the set of all (u, v)-walks of length k such that every arc has the same number of occurrence as it is in W.

Let S be an associated signed digraph of an anti-symmetric matrix. Then S can be obtained from a graph, which may have a loop at some of its vertices, by replacing each edge by a pair of anti-signed arcs, and replacing each loop by one signed arc. In this article, such digraphs will be represented as graphs with undirected edges.

2. The *k*th multi-*g* base index

Let r be an odd integer with $3 \le r \le n$. For convenience, $S_1^{(n)}$, $S_r^{(n)}$, $S_2^{(n)}$ and $S_2'^{(n)}$ denote the sets of signed digraphs depicted in Figure 1 below such that loops (v_1, v_1) and (v_n, v_n) have the same sign for $S \in S_2^{(n)}$ and different signs for $S' \in S_2'^{(n)}$, and let $\mathcal{A}_1^{(n)}$, $\mathcal{A}_r^{(n)}$, $\mathcal{A}_2^{(n)}$ and $\mathcal{A}_2'^{(n)}$ be the sets of anti-symmetric matrices that are the associated matrices of signed digraphs in $S_1^{(n)}$, $S_r^{(n)}$, $S_2^{(n)}$ and $S_2'^{(n)}$, respectively. For convenience, we sometimes also refer a vertex at which a loop is attached as a loop-vertex. Let $S \in \{S_1^{(n)}, S_r^{(n)}, S_2^{(n)}, S_2'^{(n)}\}$. For any two vertices v_i and v_j in S, $P(v_i, v_j)$ denotes the shortest path in S from v_i to v_j.

LEMMA 2.1 *For any* $S \in S_1^{(n)}$, $l_S(k) \ge n + k$.

Proof Let

$$X = \begin{cases} V \setminus \{v_1, v_2, \dots, v_{k-1}\} & \text{if } k \ge 2, \\ V & \text{if } k = 1. \end{cases}$$

By the definition of $S_1^{(n)}$, for any vertex $v_\lambda \in X$, S has a unique path Q from v_λ to v_n of length $n - \lambda$. By the definition of X, $|X| = n - k + 1$. We will show that S does not have a pair of $SSSD$ walks from v_λ to v_n of length $l = n + k - 1$.

Suppose

$$W_1 \text{ and } W_2 \text{ are a pair of } SSSD \text{ walks of length } l \text{ from } v_\lambda \text{ to } v_n. \tag{2}$$

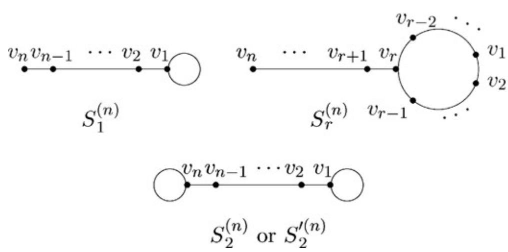

Figure 1. The graphs $S_1^{(n)}$, $S_r^{(n)}$, $S_2^{(n)}$ and $S_2'^{(n)}$.

Then W_i consists of a path Q, several loops (v_1, v_1) and some pairs of anti-signed arcs, for $i=1,2$. So the number of loops contained in W_1 has the same parity with the number of loops contained in W_2. We claim that at least one of W_1 and W_2 contains (v_1, v_1). Otherwise, if $l-(n-\lambda)$ is odd, then both W_1 and W_2 contain (v_1, v_1), since the number of anti-signed arcs is even and $l-(n-\lambda)$ is the number of loops and anti-signed arcs, contrary to the fact that neither W_1 nor W_2 contains a loop. If $l-(n-\lambda)$ is even, $\text{sgn}(W_1)=\text{sgn}(Q)\cdot(-1)^{(l-(n-\lambda))/2}=\text{sgn}(W_2)$ since the sign of a pair of anti-signed arcs is -1, contrary to (2).

Let W be a walk from v_λ to v_n of length l and contain s loops (v_1, v_1). Then $W=P(v_\lambda, v_1)+s\cdot(v_1, v_1)+P(v_1, v_n)+A$, where A is the set of pairs of anti-signed arcs. Letting $|A|=2a$, we have $l(W)=(\lambda-1)+s+(n-1)+2a=n+k-1$. Therefore, $2a+s=k+1-\lambda$. Since $v_\lambda\in X$, $\lambda\geq k$. So $2a+s\leq 1$. Thus $a=0$, and either $s=0$ or $s=1$, and $W=P(v_\lambda, v_1)+s\cdot(v_1, v_1)+P(v_1, v_n)$.

Without loss of generality, we assume the number of loops (v_1, v_1) contained in W_1 is not less than that of loops (v_1, v_1) contained in W_2. Since at least one of W_1 and W_2 contains (v_1, v_1) and $s\leq 1$, W_1 contains exactly one loop (v_1, v_1). Then $W_1=P(v_\lambda, v_1)+(v_1, v_1)+P(v_1, v_n)$ and $l=(\lambda-1)+(n-1)+1=n+\lambda-1$. Therefore, $l-(n-\lambda)=l-n+\lambda=2\lambda-1$ is odd which implies that W_2 also contains a loop. Thus both W_1 and W_2 contain the loop (v_1, v_1), and so $W_1=W_2$, contrary to (2). Hence, $l_S(k)\geq n+k$. ■

Lemma 2.2 *For any $S\in S_2^{(n)}$, $l_S(1)\geq n+1$.*

Proof Let $X=V$. By the definition of $S_2^{(n)}$, for any integer $1\leq\lambda\leq n$, S has a unique path from v_λ to $v_{n-\lambda+1}$. We claim that there is no pair of *SSSD* walks from v_λ to $v_{n-\lambda+1}$ of length n.

By contradiction, suppose

$$W_1 \text{ and } W_2 \text{ are a pair of } SSSD \text{ walks of length } n \text{ from } v_\lambda \text{ to } v_{n-\lambda+1}. \tag{3}$$

Then for $i=1,2$, W_i is a union of path Q, loops (v_1, v_1) and (v_n, v_n), some pairs of anti-signed arcs. Without loss of generality, we assume the number of loops contained in W_1 is not less than that of loops contained in W_2 and $\lambda\leq n-\lambda+1$. Similar to the proof of Lemma 2.1, at least one of W_1 and W_2 contains a loop, and so we may assume $W_1=P(v_\lambda, v_1)+(v_1, v_1)+P(v_1, v_\lambda)+Q$ or $W_1=Q+P(v_{n-\lambda+1}, v_n)+(v_n, v_n)+P(v_n, v_{n-\lambda+1})$. Since the number of loops contained in W_1 and W_2 have the same parity, W_2 contains one loop, too. It follows that either $W_2=P(v_\lambda, v_1)+(v_1, v_1)+P(v_1, v_\lambda)+Q$ or $W_2=Q+P(v_{n-\lambda+1}, v_n)+(v_n, v_n)+P(v_n, v_{n-\lambda+1})$. But

$$\begin{aligned}
&\text{sgn}(P(v_\lambda, v_1)+(v_1, v_1)+P(v_1, v_\lambda)+Q)\\
&\quad=\text{sgn}(Q)\cdot\text{sgn}(P(v_\lambda, v_1)+P(v_1, v_\lambda))\cdot\text{sgn}(v_1, v_1)\\
&\quad=\text{sgn}(Q)\cdot(-1)^{\lambda-1}\cdot\text{sgn}(v_1, v_1)\\
&\quad=\text{sgn}(Q)\cdot(-1)^{\lambda-1}\cdot\text{sgn}(v_n, v_n)\\
&\quad=\text{sgn}(P(v_{n-\lambda+1}, v_n)+(v_n, v_n)+P(v_n, v_{n-\lambda+1})+Q).
\end{aligned}$$

So $\text{sgn}(W_1)=\text{sgn}(W_2)$, contrary to (3). Hence $l_S(1)\geq n+1$. ■

LEMMA 2.3 *For any* $S \in S_2^{(n)} \bigcup S_2'^{(n)}$ *and* $k \geq 2$, *each of the following holds*:

(i) *For any* $u, v \in V(S)$, *let* $d_0(v)$ *and* $d_0(u)$ *be the shortest distances from a loop-vertex to* v *and* u, *respectively. Then*

$$l_S(v, u) \leq 2\min\{d_0(v), d_0(u)\} + 2 + d(v, u). \tag{4}$$

(ii) *The kth multiple generalized base index* $l_S(k)$ *satisfies*

$$l_S(k) \leq n + \left\lceil \frac{k}{2} \right\rceil < n + k. \tag{5}$$

Proof Without loss of generality, we suppose $d_0(v) \leq d_0(u)$. Let P be a shortest path from v to u, (w, w) be a loop closest to v and x be a vertex adjacent to w. Then $d_0(v) = d(w, v)$. Note that for any integer $t \geq 2\min\{d_0(v), d_0(u)\} + 2 + d(v, u)$, there is a pair of *SSSD* walks from v to u of length t, namely, $W_1 = P(v, w) + s \cdot (w, w) + P(w, v) + P$ and $W_2 = P(v, w) + (w, x) + (x, w) + (s - 2) \cdot (w, w) + P(w, v) + P$, where $s = t - 2d_0(v) - d(v, u) \geq 2$ since $\mathrm{sgn}(W_1) = (-1)^{d(w,v)} \cdot (\mathrm{sgn}(w, w))^s \cdot \mathrm{sgn}(P) = -\mathrm{sgn}(W_2) = (-1)^{d(w,v)} \cdot (-1) \cdot (\mathrm{sgn}(w, w))^{s-2} \cdot \mathrm{sgn}(P)$. So $l_S(v, u) \leq 2\min\{d_0(v), d_0(u)\} + 2 + d(v, u)$, which proves (4).

Suppose $S \in S_2^{(n)}$. For any integer $1 \leq i \leq (n + 1/2)$, it follows by (4) that we have both

$$\begin{aligned} l_S(v_i) &= \max_{1\leq j\leq n}\{l_S(v_i, v_j)\} \\ &\leq \max_{1\leq j\leq n}\{2\min\{d_0(v_i), d_0(v_j)\} + 2 + d(v_i, v_j)\} \\ &\leq \max_{1\leq j\leq n}\{2(i-1) + 2 + d(v_i, v_j)\} \\ &= 2i + \max_{1\leq j\leq n} d(v_i, v_j) = 2i + n - i = n + i \end{aligned}$$

and

$$\begin{aligned} l_S(v_{n-i+1}) &= \max_{1\leq j\leq n}\{l_S(v_{n-i+1}, v_j)\} \\ &\leq \max_{1\leq j\leq n}\{2\min\{d_0(v_{n-i+1}), d_0(v_j)\} + 2 + d(v_{n-i+1}, v_j)\} \\ &\leq \max_{1\leq j\leq n}\{2(i-1) + 2 + d(v_{n-i+1}, v_j)\} \\ &= 2i + \max_{1\leq j\leq n} d(v_{n-i+1}, v_j) = 2i + (n - i + 1 - 1) = n + i. \end{aligned}$$

Therefore, $l_S(k) \leq n + \lceil (k/2) \rceil < n + k$. The proof for the case when $S \in S_2'^{(n)}$ is similar. ■

LEMMA 2.4 *For any* $S \in S_2'^{(n)}$, $l_S(1) \leq n$.

Proof For any integer $1 \leq j \leq n - 1$, $l_S(v_1, v_j) \leq 2 \times 0 + 2 + d(v_1, v_j) \leq 2 + j - 1 \leq n$. So we only need to show that $l_S(v_1, v_n) \leq n$.

For any integer $t \geq n$, if $t - n \equiv 0 \pmod 2$, then let $W_1 = (v_1, v_1) + P(v_1, v_n) + ((t-n)/2)((v_n, v_{n-1}) + (v_{n-1}, v_n))$ and $W_2 = P(v_1, v_n) + (v_n, v_n) + ((t-n)/2)((v_n, v_{n-1}) + (v_{n-1}, v_n))$. Since $\mathrm{sgn}(v_1, v_1) = -\mathrm{sgn}(v_n, v_n)$, W_1 and W_2 are a pair of *SSSD* walks from v_1 to v_n of length t. If $t - n \equiv 1 \pmod 2$, let $W_3 = 2 \cdot (v_1, v_1) + P(v_1, v_n) + ((t - n - 1)/2)((v_n, v_{n-1}) + (v_{n-1}, v_n))$ and $W_4 = P(v_1, v_n) + ((t - n + 1)/2)((v_n, v_{n-1}) + (v_{n-1}, v_n))$. Since $\mathrm{sgn}(2 \cdot (v_1, v_1)) = 1 = -\mathrm{sgn}((v_n, v_{n\text{-}1}) + (v_{n\text{-}1}\}, v_n))$, W_3 and W_4 are a pair of *SSSD* walks from v_1 to v_n of length t.

Thus $l_S(v_1) = \max_{1\leq j\leq n}\{l_S(v_1, v_j)\} \leq n < n + 1$. ■

THEOREM 2.5 *For any $S \in S_{n,1}$, $l_S(k) \leq n+k$ $(1 \leq k \leq n)$. Moreover, equality holds if and only if either $k=1$ and S is isomorphic to a member in $S_1^{(n)} \bigcup S_2^{(n)}$ or $k \geq 2$ and S is isomorphic to a member in $S_1^{(n)}$.*

Proof Let v be a vertex of S with a loop and u be adjacent to v. Let $W_1 = (v,v)+(v,v)$ and $W_2 = (v,u)+(u,v)$. Since the associated matrix of S is an anti-symmetric matrix, $\mathrm{sgn}(v,u) = -\mathrm{sgn}(u,v)$. Therefore, $\mathrm{sgn}(W_2) = -1$. By definition, $\mathrm{sgn}(W_1) = 1$. Thus W_1 and W_2 are a pair of *SSSD* walks from v to v, and so $r_{v,v} = 2$.

Note that $d(S) \leq n-1$. Thus $\exp_S(v) \leq n-1$, since for any integer $t \geq n-1$ and for any vertex w of S, there is a walk $T = (t-p)\cdot(v,v)+P$ of length t, where P is the shortest path from v to w of length p $(p \leq n-1 \leq t)$. Further, equality holds if and only if S is a path and v is one of its end vertices. By (1),

$$l_S(k) \leq \begin{cases} (k-1)+2+(n-1) = n+k & \text{if } k-1 \leq d(S), \\ d(S)+2+(n-1) < (k-1)+2+(n-1) = n+k & \text{if } k-1 > d(S). \end{cases}$$

By Lemmas 2.1 and 2.2, if S is isomorphic to a member in $S_1^{(n)} \bigcup S_2^{(n)}$, then $l_S(k) = n+k$ for $k=1$; if S is isomorphic to a member in $S_1^{(n)}$, then $l_S(k) = n+k$ for $2 \leq k \leq n$.

If S is not a path with loops, then $\exp_S(v) < n-1$. Therefore, $l_S(k) < n+k$. If there is a vertex x with a loop which is not an end vertex, then $\exp_S(x) \leq n-2$. Substituting x for v in the above discussion, we have $l_S(k) \leq n+k-1 < n+k$. So if $l_S(k) = n+k$, then S must be a path and v is an end vertex with a loop. Therefore, S is isomorphic to a member in $S_1^{(n)} \bigcup S_2^{(n)} \bigcup S_2^{\prime(n)}$. By Lemmas 2.1–2.4, $l_S(k) = n+k$ only if S is isomorphic to a member in $S_1^{(n)} \bigcup S_2^{(n)}$ for $k=1$ and S is isomorphic to a member in $S_1^{(n)}$ for $k \geq 2$. ■

COROLLARY 1 *For any $S \in S_{n,1}$,*

$$l(S) = l_S(n) \leq 2n,$$

where equality holds if and only if S is isomorphic to a member in $S_1^{(n)}$.

LEMMA 2.6 *Let r be an odd integer with $3 \leq r \leq n$. Then for any $S \in S_{n,r}$,*

$$\begin{aligned} l_S(k) &\leq \exp_{S_n(r)}(k) + r \\ &= \begin{cases} n-1+k & \text{if } r \leq k \leq n, \\ \max\left\{n + \dfrac{r-1}{2} + \left\lfloor \dfrac{k+1}{2} \right\rfloor - 1, 2r-1\right\} & \text{if } 1 \leq k \leq r-1. \end{cases} \end{aligned}$$

Proof Suppose $\exp(v_1) \leq \exp(v_2) \leq \cdots \leq \exp(v_n)$. For any integer $t \geq \exp_{S_n(r)}(k)+r$, and for any vertices v_i, v_j with $1 \leq i \leq k$ and $1 \leq j \leq n$, we will show that S has a pair of *SSSD* walks from v_i to v_j of length t. If S is not uniquely cyclic, then S contains a spanning subgraph S' such that S' is uniquely cyclic and $l_S(k) \leq l_{S'}(k)$. Therefore, we may assume that S is a uniquely cyclic signed digraph with an r-cycle C_r and only need to prove $l_S(k) \leq \exp_{S_n(r)}(k)+r$. Note that $\mathrm{sgn}(C_r) = -\mathrm{sgn}(C_r^{-1})$. We consider two cases:

Case 1: $v_i \in C_r$ or $v_j \in C_r$.

Since $t - r \geq \exp_{S_n(r)}(k)$, S has a walk W of length $t-r$ from v_i to v_j. Let $W_1 = W + C_r$ and $W_2 = W + C_r^{-1}$. Then W_1 and W_2 are a pair of *SSSD* walks of length t from v_i to v_j.

Case 2: $v_i \notin C_r$ and $v_j \notin C_r$.

Since $t-r, t-r+1 \geq \exp_{S_n(r)}(k)$, S has walks W_1 and W_2 from v_i to v_j of length $t-r$ and $t-r+1$, respectively. Then $W = W_1 + W_2^{-1}$ is a closed walk from v_i to v_i of length $2(t-r)+1$. Therefore, W contains the odd cycle C_r.

Subcase 2.1: $\exists v_s \in C_r \bigcap W_1$.

Let P_1 be the walk from v_i to v_s and P_2 be the walk from v_s to v_j such that $W_1 = P_1 + P_2$. Then $W_3 = P_1 + C_r + P_2$ and $W_4 = P_1 + C_r^{-1} + P_2$ are a pair of *SSSD* walks of length t from v_i to v_j.

Subcase 2.2: $C_r \bigcap W_1 = \emptyset$.

Then $C_r \subseteq W_2$. Suppose the first vertex of C_r contained in W_2 is v_s. Let P_1 be the walk from v_i to v_s and P_2 be the walk from v_s to v_j such that $W_2 = P_1 + C_r + P_2$. Let u be a vertex adjacent to v_j. Then $W_3 = P_1 + C_r + P_2 + ((r-1)/2) \cdot ((v_j, u) + (u, v_j))$ and $W_4 = P_1 + C_r^{-1} + P_2 + ((r-1)/2) \cdot ((v_j, u) + (u, v_j))$ are a pair of *SSSD* walks of length t from v_i to v_j.

Thus $l_S(k) \leq \exp_{S_n(r)}(k) + r$. ■

LEMMA 2.7 *Let r be an odd integer with $3 \leq r \leq n$, u and v be two vertices of $S \in S_{n,r}$. Then $l(u, v) = l(v, u)$.*

Proof For any integer $t \geq l(v, u)$, let P_1 and P_2 be a pair of *SSSD* walks from v to u of length t. Then $\mathrm{sgn}(P_1) = -\mathrm{sgn}(P_2)$. Since $\mathrm{sgn}(P_1^{-1}) = (-1)^t \mathrm{sgn}(P_1) = (-1)^t(-\mathrm{sgn}(P_2)) = -\mathrm{sgn}(P_2^{-1})$, P_1^{-1} and P_2^{-1} are a pair of *SSSD* walks from u to v of length t. Thus $l(u, v) \leq l(v, u)$.

An analogous argument shows that $l(u, v) \geq l(v, u)$. Hence $l(u, v) = l(v, u)$. ■

LEMMA 2.8 *Let r be an odd integer with $3 \leq r \leq n$. For any $S \in S_r^{(n)}$, let C_r denote the r-cycle $v_r v_{r-2} \ldots v_1 v_2 v_4 \ldots v_{r-1} v_r$ of S, then each of the following holds:*

(i) *For any $v \in C_r$,*

$$l_S(v) \geq 2r - 1. \tag{6}$$

(ii) *The kth multiple generalized base index $l_S(k)$ satisfies*

$$l_S(k) \geq \exp_{S_n(r)}(k) + r = \begin{cases} n - 1 + k & \text{if } r \leq k \leq n, \\ \max\left\{n + \dfrac{r-1}{2} + \left\lfloor \dfrac{k+1}{2} \right\rfloor - 1, 2r - 1\right\} & \text{if } 1 \leq k \leq r - 1. \end{cases}$$

Proof For any vertex $v \in V(S)$, we consider two cases:

Case 1: $v \in C_r = v_r v_{r-2} \ldots v_1 v_2 v_4 \ldots v_{r-1} v_r$.

Let W be a walk from v to v of length $2r-2$. Then for some non-negative integers x and y, $W \in F(x \cdot C_r + y \cdot C_r^{-1} + A)$ where A is the set of z pairs of anti-signed arcs. Therefore, $2r - 2 = (x + y) \cdot r + 2z$. Clearly, $0 \leq x + y \leq 1$. If $x + y = 1$, then $2z = r - 2$. But $r - 2$ is odd, a contradiction. So $x + y = 0$, $\mathrm{sgn}(W) = (-1)^{r-1}$. Hence there is no pair of *SSSD* walks from v to v of length $2r-2$. Thus $l_S(v) \geq 2r - 1$, which proves (6).

Suppose $v = v_k$ $(1 \leq k \leq r)$. If $k = r$, let $A_1 = \emptyset$ and $A_2 = C_r$. Otherwise, let A_1 and A_2 be the minor and major sections of C_r from v_k to v_r, respectively.

Then $l(A_1) = ((r+1)/2 - \lfloor (k+1)/2 \rfloor$ and $l(A_2) = ((r-1)/2 + \lfloor (k+1)/2 \rfloor$. Let W be a walk from v_k to v_n of length $n + ((r-1)/2) + \lfloor ((k+1)/2 \rfloor - 2$. Then $W \in F(A_1 + x \cdot C_r + y \cdot C_r^{-1} + P(v_r, v_n) + A)$ or $W \in F(A_2 + x \cdot C_r + y_r^{-1} + \ P(v_r, v_n) + A)$ where A is the set of z pairs of anti-signed arcs.

We first assume that $W \in F(A_2 + x \cdot C_r + y \cdot C_r^{-1} + P(v_r, v_n) + A)$. Then $n + ((r-1)/2 + \lfloor ((k+1)/2) \rfloor - 2 = (((r-1)/2) + \lfloor ((k+1)/2) \rfloor) + (x+y) \cdot r + (n-r) + 2z$. Therefore, $2z + (x+y)r = r-2$. So $x+y=0$ and $2z = r-2$, contrary to the assumption that r is odd.

Thus we may assume that $W \in F(A_1 + x \cdot C_r + y \cdot C_r^{-1} + P(v_r, v_n) + A)$. Then $n + ((r-1)/2) + \lfloor ((k+1)/2) \rfloor - 2 = (((r+1)/2) - \lfloor ((k+1)/2) \rfloor) + (x+y) \cdot r + (n-r) + 2z$, from which $(x+y) \cdot r + 2z = r - 3 + 2\lfloor ((k+1)/2) \rfloor \leq 2r-2$. Therefore, $0 \leq x+y \leq 1$. If $x+y=1$, then $2z = 2\lfloor ((k+1)/2) \rfloor - 3$, and so $3 \equiv 0 \pmod 2$, a contradiction. So $x+y=0$, $\mathrm{sgn}(W) = (-1)^{(r-3+2\lfloor (k+1/2) \rfloor)/2} \cdot \mathrm{sgn}(A_1) \cdot \mathrm{sgn}(P(v_r, v_n))$

Thus S cannot have a pair of *SSSD* walks from v_k to v_n of length $n + ((r-1)/2) + \lfloor ((k+1)/2) \rfloor - 2$. Hence

$$l_S(v_k) \geq \max\left\{ n + \frac{r-1}{2} + \left\lfloor \frac{k+1}{2} \right\rfloor - 1, 2r-1 \right\}, \quad 1 \leq k \leq r.$$

Case 2: $v \notin C_r$.

In this case, there is an integer k with $r+1 \leq k \leq n$ such that $v = v_k$. Let W be a walk from v_k to v_n of length $n+k-2$. Then for some non-negative integers t, x and y, $W \in F(P(v_k, v_n) + x \cdot C_r + y \cdot C_r^{-1} + t(P(v_k, v_r) + P(v_r, v_k)) + A)$, where A is the set of z pairs of anti-signed arcs. Clearly, if $x+y \neq 0$, then $t \neq 0$. Then $n+k-2 = (n-k) + (x+y)r + 2t(k-r) + 2z$, from which $(x+y)r + 2t(k-r) + 2z = 2k-2$. If $x+y \neq 0$, then $t \geq 1$, $(x+y)r + 2z = 2k - 2 - 2t(k-r) \leq 2k - 2 - 2k + 2r = 2r-2$. So $x+y=1$ and $2z = 2k - 2 - 2t(k-r) - r$, contrary to the assumption that r is odd. Hence $x+y=0$, W is a union of $P(v_k, v_n)$ and $k-1$ pairs of anti-signed arcs. Then $\mathrm{sgn}(W) = (-1)^{k-1}\mathrm{sgn}(P(v_k, v_n))$. Thus S does not have a pair of *SSSD* walks from v_k to v_n of length $n+k-2$. Hence,

$$l_S(v_k) \geq n+k-1 \geq n+r \text{ and } r+1 \leq k \leq n.$$

Notice that if $k=r$, then $n + ((r-1)/2 + \lfloor (k+1)/2 \rfloor - 1 = n+r-1$ and if $k \geq r, n+k-1 \geq 2r-1$, so the statement (ii) holds. ∎

By Lemmas 2.6 and 2.8, we have

THEOREM 2.9 *Let r be an odd integer with $3 \leq r \leq n$, and let $l_{S_{n,r}}(k) = \max\{l_S(k) | S \in S_{n,r}\}$. Then*

$$l_{S_{n,r}}(k) = \begin{cases} n-1+k & \text{if } r \leq k \leq n, \\ \max\left\{ n + \dfrac{r-1}{2} + \left\lfloor \dfrac{k+1}{2} \right\rfloor - 1, 2r-1 \right\} & \text{if } 1 \leq k \leq r-1. \end{cases}$$

COROLLARY 2 *Let r be an odd integer with $3 \leq r \leq n$. Then for any $S \in S_{n,r}$,*

$$l(S) = l_S(n) \leq 2n-1.$$

It is natural to consider the question when the equality holds in Corollary 2. In the next section, we will study the extremal graph S satisfying $l(S) = l_S(n) = 2n - 1$.

3. The extremal graphs for the generalized base index

LEMMA 3.1 *Let P_1 and P_2, Q_1 and Q_2 be two pairs of SSSD walks from vertex u to vertex v of length i and j, respectively, such that i and j have different parity. Then* $l(u, v) \leq \max\{i-1, j-1\}$.

Proof Without loss of generality, suppose $i < j$. Let w be a vertex adjacent to v. For $s = 1$, 2 and any integer $k \geq j-1$, if k has the same parity with i, let $W_s = P_s + ((k-i)/2) \cdot ((v, w) + (w, v))$. Otherwise, let $W_s = Q_s + ((k-j)/2)((v, w) + (w, v))$. Then W_1 and W_2 are a pair of $SSSD$ walks from u to v of length k. Therefore, $l(u, v) \leq j - 1$. ■

LEMMA 3.2 *Let $S \in S_{n,1} \backslash S_1^{(n)}$ with $n \geq 3$. Then $l(S) \leq 2n-2$.*

Proof First, we suppose S is the union of a tree T and loops. We consider two cases:

Case 1: T is a path.

Suppose both of the end vertices have loops. If $n = 3$, it is easy to prove that $l(S_2^{(n)}) = 4 = 2n-2$ and $l(S_2^{\prime(n)}) = 3 < 2n-2$. If $n \geq 4$, by (5), $l(S) = l_S(n) \leq n + \lceil n/2 \rceil$. Therefore, $l(S) \leq 2n-2$.

Suppose there is an internal vertex with a loop. Let v be the internal vertex with loop and w be a vertex adjacent to v. For any vertices $u_1, u_2 \in V(S)$ and any integer $t \geq 2n-2$, let P_1 and P_2 be the unique paths from u_1 to v of length p_1 and from v to u_2 of length p_2, respectively. Define $W_1 = P(u_1, v) + (t - p_1 - p_2) \cdot (v, v) + P(v, u_2)$ and $W_2 = P(u_1, v) + (v, w) + (w, v) + (t - p_1 - p_2 - 2) \cdot (v, v) + P(v, u_2)$. Then W_1 and W_2 are a pair of $SSSD$ walks from u_1 to u_2 of length t since $p_1, p_2 \leq n-2$. Therefore, $l(S) = \max\{l(u_1, u_2) | u_1, u_2 \in V(S)\} \leq 2n-2$.

Case 2: T is not a path.

Since S is connected and not a path, $d(S) \leq n-2$. Let v be the vertex with loop. Then $\exp_S(v) \leq n-2$. By (1), $l(S) \leq (n-2) + 2 + (n-2) = 2n-2$.

If S is not the union of a tree and loops, then S has a spanning subgraph S' such that S' is the union of a tree and loops. Therefore, $l(S) \leq l(S') \leq 2n-2$. ■

Remark 1 There is no signed digraph $S \in S_{n,1}$ with $n \geq 3$ such that $l(S) = 2n-1$.

Let

$$S_n = \bigcup_{1 \leq r \leq n, r \text{ is odd}} S_{n,r}.$$

Parts (i) and (ii) of Theorem 3.3 below follow from Corollaries 1 and 2 in section 2.

THEOREM 3.3 *Let r be an odd integer with $3 \leq r \leq n$ $(n \geq 3)$. Then*

(i) $l(S) \leq 2n$ *for any* $S \in S_n$;
(ii) $l(S) = 2n$ *if and only if S is isomorphic to a member in* $S_1^{(n)}$;
(iii) $l(S) = 2n-1$ *if and only if S is isomorphic to a member in $S_r^{(n)}$ or C_n (if n is odd)*;
(iv) *For any integer $3 \leq t \leq 2n-2$, there exists signed digraph $S \in S_n$ such that* $l(S) = t$.

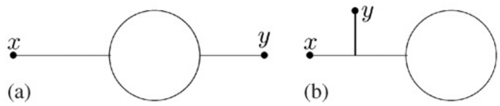

Figure 2. The graphs in the proof of case 2.

Proof We only need to prove (iii) and (iv). Let x and y be any vertices of $S \in S_{n,r}$. By the definition of $S_{n,r}$, S has an r-cycle. We consider two cases:

Case 1: There exists an r-cycle C_r such that $x \in C_r$ or $y \in C_r$.
Subcase 1.1: $x, y \in C_r$.

If $x = y$, let $P_1 = \emptyset$ and $P_2 = C_r$. Otherwise, let P_1 and P_2 be the minor and major sections of C_r from x to y of length p_1 and p_2, respectively. Then $W_1 = P_1 + C_r$ and $W_2 = P_1 + C_r^{-1}$ are a pair of *SSSD* walks from x to y of length $p_1 + r$. And $Q_1 = P_2 + C_r$ and $Q_2 = P_2 + C_r^{-1}$ are a pair of *SSSD* walks from x to y of length $p_2 + r$. Since $p_1 + p_2 = r, p_1$ and p_2 have different parity. By Lemma 3.1, $l(x, y) \le p_2 + r - 1 \le 2r - 1$. Furthermore, by (6), the equality holds only if $x = y$.

Subcase 1.2: $x \in C_r, y \notin C_r$ or $y \in C_r, x \notin C_r$.

Since $l(x, y) = l(y, x)$, we suppose $y \in C_r$ and $x \notin C_r$. Let P_1 be the shortest path from x to C_r of length p_1 and $\{v\} = V(P_1) \bigcap V(C_r)$. If $y = v$, let $P_2 = \emptyset$ and $P_3 = C_r$. Otherwise, let P_2 and P_3 be the minor and major sections of C_r from v to y of length p_2 and p_3, respectively. Then $W_1 = P_1 + P_2 + C_r$ and $W_2 = P_1 + P_2 + C_r^{-1}$ are a pair of *SSSD* walks from x to y of length $p_1 + p_2 + r$. And $Q_1 = P_1 + P_3 + C_r$, $Q_2 = P_1 + P_3 + C_r^{-1}$ are a pair of *SSSD* walks from x to y of length $p_1 + p_3 + r$. By Lemma 3.1, $l(x, y) \le p_1 + p_3 + r - 1 \le (n - r) + r + r - 1 = n + r - 1 < 2n - 1$.

Case 2: x and y are not contained in any r-cycle of S.

Then S contains at least one r-cycle C_r.

Subcase 2.1: For any path P from x to y in S, $V(P) \bigcap V(C_r) \neq \emptyset$ (Figure 2(a)).

Using a similar argument of Subcase 1.2, we can prove that $l(x, y) < 2n - 1$.

Subcase 2.2: S has a path P from x to C_r such that $y \in P$ or from y to C_r such that $x \in P$.

Since $l(x, y) = l(y, x)$, we may assume that P is a path from x to C_r such that $y \in P$. Let v be the vertex such that $\{v\} = V(P) \bigcap V(C_r)$, P_1 and P_2 be the paths from x to y of length p_1 and from y to v of length p_2, respectively, such that $P = P_1 + P_2$. Then $W_1 = P_1 + P_2 + C_r + P_2^{-1}$ and $W_2 = P_1 + P_2 + C_r^{-1} + P_2^{-1}$ are a pair of *SSSD* walks from x to y of length $p_1 + 2p_2 + r$. And $Q_1 = P_1 + P_2 + 2C_r + P_2^{-1}$, $Q_2 = P_1 + P_2 + C_r + C_r^{-1} + P_2^{-1}$ are a pair of *SSSD* walks from x to y of length $p_1 + 2p_2 + 2r$. By Lemma 3.1, $l(x, y) \le p_1 + 2p_2 + 2r - 1 \le (n - r) + p_2 + 2r - 1 = n + r + p_2 - 1 \le 2n - 1$.

Now suppose that $l(x, y) = 2n-1$. Then both $p_1 + p_2 = n - r$ and $p_2 + r = n$. Therefore, $p_1 = 0$. So the equality holds only if $x = y = v_n$ and S is isomorphic to a member in $S_r^{(n)}$.

Subcase 2.3: S has a path P from x to y such that $V(P) \bigcap V(C_r) = \emptyset$, but there is no path P_2 from x to C_r such that $y \in P_2$ and no path P_3 from y to C_r such that $x \in P_3$ Figure 2(b).

Similar to Subcase 2.2, $l(x, y) < 2n - 1$.

Thus $l(S) = 2n - 1$ if and only if S is isomorphic to a member in $S_r^{(n)}$ or C_n (if n is odd).

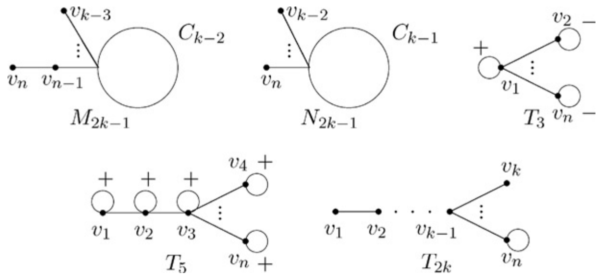

Figure 3. The graphs in the proof for Theorem 3.3(iv).

It is routine to check that $l(T_3) = 3$ and $l(T_5) = 5$ (Figure 3). Using Lemma 2.8 and similar proof of Theorem 3.3(iii), we can prove $l(M_{2k-1}) = 2k-1$ with $3 < k \leq n-1$ and k odd, $l(N_{2k-1}) = 2k-1$ with $3 < k \leq n-1$ and k even (Figure 3). Using Lemma 2.1 and similar argument of Theorem 2.5, we have $l(T_{2k}) = 2k$ with $2 \leq k \leq n-1$ (Figure 3).

Hence (iv) holds.

Acknowledgements

The authors wish to thank the referees for their many helpful suggestions to improve the presentation of this article. This work is supported by NNSF of China(10771080).

References

[1] R.A. Brualdi and B. Liu, *Generalized exponents of primitive directed graphs*, J. Graph Theory, 14 (1990), pp. 483–499.

[2] R.A. Brualdi and J. Shao, *Generalized exponents of primitive symmetric digraphs*, Discrete Appl. Math., 74 (1997), pp. 275–293.

[3] Y. Gao and Y. Shao, *The k-th upper generalized exponents of primitive symmetric digraphs*, Acta Math. Appl. Sinica, 21 (1998), pp. 161–164.

[4] Q. Li and B. Liu, *Multi-g base index of non-powerful generalized sign pattern matrices*, Lin. Multilin. Alg. (to appear).

[5] Z. Li, F. Hall, and C. Eschenbach, *On the period and base of a sign pattern matrix*, Lin. Alg. Appl., 212/213 (1994), pp. 101–120.

[6] B. Liu, *Generalized exponents of boolean matrices*, Lin. Alg. Appl., 373 (2003), pp. 169–182.

[7] B. Liu and H.-J. Lai, *Matrices in Combinatorics and Graph Theory*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[8] J. Shao and S. Hwang, *Generalized exponents of non-primitive graphs*, Lin. Alg. Appl., 279 (1998), pp. 207–225.

[9] L. You, J. Shao, and H. Shan, *Bounds on the bases of irreducible generalized sign pattern matrices*, Lin. Alg. Appl., 427 (2007), pp. 285–300.

[10] J. Stuart, C. Eschenbach, and S. Kirkland, *Irreducible k-potent sign pattern matrices*, Lin. Alg. Appl., 294 (1999), pp. 85–92.