Reinforcing the number of disjoint spanning trees

Damin Liu
Beijing University of Chemical Technology, P. R. China
Hong-Jian Lai
West Virginia University, Morgantown, WV 26506
Zhi-Hong Chen *
Butler University, Indianapolis, IN 46208

Abstract

The **spanning tree packing number** of a connected graph G, denoted by $\tau(G)$, is the maximum number of edge-disjoint spanning trees of G. In this paper, we determine the minimum number of edges that must be added to G so that the resulting graph has spanning tree packing number at least k, for a given value of k.

Key words. Edge-disjoint spanning trees, spanning tree packing numbers, edge arboricity

1. Introduction.

We shall use the notation of Bondy and Murty [1], except defined otherwise. We allow graphs to have multiple edges but not loops. Let G be a graph. The set $E(G^c)$ denotes the collection of edges that are not in $E(G)$ but both ends of each member in $E(G^c)$ are in $V(G)$. A

*Email: Chen@butler.edu

maximal connnected subgraph of G is called a component of G. The number of components of G is denoted by $\omega(G)$. Let L and H be two subgraphs of G with $V(L) \cap V(H) \neq \emptyset$. Define $L \cap H$ to be a subgraph of G with $V(L \cap H) = V(L) \cap V(H)$ and $E(L \cap H) = E(L) \cap E(H)$. For a set of edges $E \subseteq E(G)$, we define the **contraction** G/E to be the graph obtained from G by contracting the edges in E and deleting all resulting loops. If H is a connected subgraph of G, then G/H denotes $G/E(H)$. The maximum number of edge-disjoint spanning trees in G is called the **spanning tree packing number** of G (a recent survey on spanning tree packing number can be found in [7]), and is denoted by $\tau(G)$. For convenience, we define $G/\emptyset = G$ and define $\tau(K_1) = \infty$. The set of all positive integers is denoted by $\mathbf{N}$.

In [6], Payan considered the following problem: Find an edge $e \in E(G)$ and an edge $e' \in E(G^c)$ such that $G - e + e'$ is closer to having k edge-disjoint spanning trees than G does. A partial solution of this problem has been found in [3], and the general case remains open.

In this paper, we consider a problem with a similar nature: for a graph G, and a given integer $k > \tau(G)$, find the minimum number of edges $X \subseteq E(G^c)$ such that $\tau(G + X) \geq k$.

We use decomposition and contraction methods to approach the problem. This decomposition is described in Section 2. The main result is proved in Section 3.

2. Some properties involving $\tau(G)$.

Let X be a nonempty set. A **partition** $(P_1, P_2, ..., P_m)$ of X satisfies:

(a) $P_i \neq \emptyset$, $1 \leq i \leq m$;
(b) $P_i \cap P_j = \emptyset$, $i \neq j$ and $1 \leq i, j \leq m$;
(c) $\bigcup_{i=1}^{m} P_i = X$.

For an integer $r \geq 1$, let $\mathcal{T}_r$ denote the family of all graphs G with $\tau(G) \geq r$. Lemma 2.1 below summarizes some observations.

Lemma 2.1 Let G be a connected graph, and let r, r' be integers with $r' \geq r > 0$.

(i) Let H be a subgraph of G and $H \in \mathcal{T}_{r'}$. Then $G/H \in \mathcal{T}_r$ if and only if $G \in \mathcal{T}_r$.

(ii) If $G \in \mathcal{T}_r$, and if $e \in E(G^c)$, then $G + e \in \mathcal{T}_r$.

(iii) If $G \in \mathcal{T}_r$ and if $e \in E(G)$, then $G/e \in \mathcal{T}_r$.

(iv) If H_1 and H_2 are two subgraphs of G such that $H_1, H_2 \in \mathcal{T}_r$ and $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2 \in \mathcal{T}_r$.

Proof: (i) Since $H \in \mathcal{T}_{r'}$ and since $r' \geq r$, H has r edge-disjoint spanning trees $T_1, \cdots, T_r$. Since $G/H \in \mathcal{T}_r$, G/H has disjoint spanning trees $T'_1, \cdots, T'_r$. Note that each $T''_i = G[E(T_i) \cup E(T'_i)]$ is a spanning tree of G, and so $G \in \mathcal{T}_r$.

Conversely, suppose that G has r edge-disjoint spanning trees, say T_1, T_2, $\cdots$, and T_r. Then $T_i/(E(T_i) \cap E(H))$ is a spanning connected subgraph of G/H $(1 \leq i \leq r)$, and so G/H has r edge-disjoint spanning trees. Thus, $G/H \in \mathcal{T}_r$.

(ii) Any spanning tree of G is also a spanning tree of $G + e$.

(iii) Let $T_1, \cdots, T_r$ be edge-disjoint spanning trees of G. Let $T'_i = T_i$ if $e \notin E(T_i)$ and $T'_i = T_i/e$ if $e \in E(T_i)$, for $1 \leq i \leq r$. Then $T'_1, \cdots, T'_r$ are edge-disjoint spanning subgraphs of G/e, and so $G/e \in \mathcal{T}_r$.

(iv) Let $G = H_1 \cup H_2$. Since $H_1 \in \mathcal{T}_r$, and by Lemma 2.1(iii), $G/H_2 \in \mathcal{T}_r$. Since $H_2 \in \mathcal{T}_r$, and by Lemma 2.1(i), $G = H_1 \cup H_2 \in \mathcal{T}_r$. $\square$

Let G be a nontrivial connected graph. For any $r \in \mathbf{N}$, a nontrivial subgraph H of G is called r**-maximal** if $H \in \mathcal{T}_r$ and if there is no subgraph K of G, such that K contains H properly and that $K \in \mathcal{T}_r$. An r-maximal subgraph H of G is called an r**-region** if $r = \tau(H)$ (See Example 3.5). Call a subgraph H of G a **region** if H is an r-region for some integer r. Define $\xi(G) = \max\{r \mid G \text{ has a subgraph as an } r\text{-region}\}$.

Lemma 2.2 Let H be a nontrivial subgraph of G. If $\tau(H) = r$, then there is always a region L of G with $E(H) \subseteq E(L)$ and with $\tau(L) \geq r$.

Proof: Let L be the union of all r-regions of G each of which contains H. Then by Lemma 2.1(iv) $L \in \mathcal{T}_r$, and so L is $\tau(L)$-maximal. □

Lemma 2.3 Let $r', r \in \mathbf{N}$, let H be an r'-region of G, and let K be an r-region of G. One of the following holds:

(i) $V(H) \cap V(K) = \emptyset$,

(ii) $r' = r$ and $H = K$,

(iii) $r' > r$ and H is a nonspanning subgraph of K,

(iv) $r' < r$ and H contains K as a nonspanning subgraph.

Proof: Suppose that Lemma 2.3(i) does not hold, and so $V(H) \cap V(K) \neq \emptyset$. Without loss of generality, we assume $r' \geq r$. By Lemma 2.1(i), $H \cup K \in \mathcal{T}_r$. Since K is an r-region, $H \cup K$ is a subgraph of K, and so H is a subgraph of K. This implies (ii)-(iv) of Lemma 2.3. □

Theorem 2.4 Let G be a nontrivial connected graph. Then

(a) there exist an integer $m \in \mathbf{N}$, and an m-tuple $(i_1, i_2, ..., i_m)$ of integers in $\mathbf{N}$ with

$$\tau(G) = i_1 < i_2 < ... < i_m = \xi(G), \tag{1}$$

and a sequence of edge subsets

$$E_m \subset ... \subset E_2 \subset E_1 = E(G); \tag{2}$$

such that each component of the induced subgraphs $G[E_j]$ is an r-region of G for some $r \in \mathbf{N}$ with $r \geq i_j$ $(1 \leq j \leq m)$, and such that at least one component H in $G[E_j]$ is an i_j-region of G;

(b) if H is a subgraph of G with $\tau(H) \geq i_j$, then $E(H) \subseteq E_j$;

(c) the integer m and the sequences (1) and (2) are uniquely determined by G.

Proof: Let $\mathcal{R}(G)$ denote the collection of all regions of G. By Lemma 2.2, $\mathcal{R}(G)$ is not empty. Since G is a finite graph,

$$|\mathcal{R}(G)| \text{ is finite.} \tag{3}$$

Define $sp(G)$ as

$$sp(G) = \{\tau(H) : \quad H \in \mathcal{R}(G) \text{ is nontrivial } \}.$$

By (3), $|sp(G)|$ is finite. Since $G \in \mathcal{R}(G)$, $|sp(G)| \geq 1$. Let $m = |sp(G)|$. We may assume that $sp(G) = \{i_1, i_2, ..., i_m\}$ with $i_1 < i_2 < ... < i_m$. By Lemma 2.1(i), we have

$$\tau(G) = i_1. \tag{4}$$

For each $j \in \{1, 2, ..., m\}$, define

$$E_j = \bigcup_{\tau(H) \geq i_j} E(H). \tag{5}$$

By the definition of $\mathcal{T}_r$,

$$\mathcal{T}_{i_1} \supset \mathcal{T}_{i_2} \supset ... \supset \mathcal{T}_{i_m}. \tag{6}$$

Hence by (5) and (6),

$$E_1 \supseteq E_2 \supseteq ... \supseteq E_m. \tag{7}$$

By (4),

$$E_1 = \bigcup_{\tau(H) \geq i_1} E(H) = \bigcup_{\tau(H) \geq \tau(G)} E(H) = E(G). \tag{8}$$

Fix $j \in \{1, 2, ..., m-1\}$. Since $i_j \in sp(G)$, there is an i_j-region K of G. Since $\tau(K) = i_j < i_{j+1}, E(K) - E_{j+1} \neq \emptyset$. Hence, $E_j \neq E_{j+1}$, and so (1) and (2) hold.

Fix $j \in \{1, 2, ..., m\}$. We prove the following claim first.

<u>**Claim A**</u> Every component of $G[E_j]$ is an r-region of G, for some $r \geq i_j$, where $1 \leq j \leq m$.

Let H be a nontrivial component of $G[E_j]$. By (5), we may assume that there are s regions H_t, $(1 \leq t \leq s)$ such that each H_t is an r_t-region, for some $r_t \geq i_j$, and such that

$$E(H) = \bigcup_{t=1}^{s} E(H_t).$$

Without loss of generality, we may assume that

$$r_1 \leq r_2 \leq \ldots \leq r_s.$$

Since H is connected, if $s \geq 2$, then H_1 must share a common vertex with some H_i for some $i \geq 2$, and so by Lemma 2.1(iv), $H_1 \cup H_i \in \mathcal{T}_{r_1}$, contrary to the fact that H_1 is r_1-maximal. Hence, we must have $s = 1$. Thus, Claim A is proved.

What is left is to show that $G[E_j]$ contains an i_j-region of G. Since $i_j \in sp(G)$, there is an i_j-region H of G. By (5), $E(H) \subseteq E_j$. We claim that H is a component of $G[E_j]$. Since H is connected, H is in a component K of $G[E_j]$. By Claim A, K is an r-region with $r \geq i_j$. It follows by Lemma 2.3 that $H = K$. Thus the claim follows and so (a) of Theorem 2.4 must hold. Theorem 2.4(b) follows from Lemma 2.2 and the proof above.

Since $\mathcal{R}(G)$ and $sp(G)$ are uniquely determined by G, the integer m, the m-tuple $(i_1, i_2, ..., i_m)$ and the sequence (2) are all uniquely determined by G. Therefore (c) of Theorem 2.4 follows. This proves Theorem 2.4. □

Corollary 2.5 If $(i_1, i_2, \cdots, i_m)$ is the tuple determined by G as defined in Theorem 2.4, then $(i_1, i_2, \cdots, i_{m-1})$ is the tuple determined by G/E_m. In particular, $i_{m-1} = \xi(G/E_m)$.

Proof: By Theorem 2.4, we know that each component of $G[E_m]$ is an i_m-region. The corollary follows from Lemma 2.1(i) and the definition of G/E_m. □

Proposition 2.6 Let $r', r \in \mathbf{N}$ and let H be an r'-region of G and K be an r-region of G.

(i) If $V(H) \cap V(K) = \emptyset$, then $(G/H)[E(K)]$ is also an r-region of G/H;

(ii) If K contains H as a nonspanning subgraph, then $r' > r$ and K/H is an r-region of G/H.

Proof: Let v_H denote the vertex of G/H onto which the subgraph H is contracted.

(i) Suppose that $V(H) \cap V(K) = \emptyset$. Then K is a subgraph of G/H. If K is not a region of G/H, then G/H has a region L' with $\tau(L') \geq r$ and $K \subset L'$. If $v_H \notin V(L')$, then L' is a subgraph of G, contrary to the fact that K is a region of G. Hence, $v_H \in V(L')$. Let $L = G[E(L') \cup E(H)]$. Then L is a subgraph of G containing both K and L. If $r' \geq r$, then by Lemma 2.1(i), $\tau(L) \geq r$ and so K is not a region, a contradiction. Similarly, if $r \geq r'$, then $\tau(L) \geq r'$ and so H is not a region, a contradiction. These contradictions establish Proposition 2.6(i).

(ii) Now suppose that K contains H as a nonspanning subgraph. By Lemma 2.3(iii), $r' > r$. By Lemma 2.1(i), $\tau(K/H) \geq r$. If G/H has a region L' containing K/H with $\tau(L') \geq r$, then by Lemma 2.1(i), $L = G[E(L')]$ is a subgraph of G containing K with $\tau(L) \geq r = \tau(K)$. Since K is a region, $K = L$, and so $K/H = L'$. This proves that K/H is an r-region of G/H. □

Corollary 2.7 Let $r', r \in \mathrm{N}$ and let H be an r'-region of G, and denote by v_H the vertex in G/H to which H is contracted.

(i) If K is an r-region of G/H not containing v_H, then $G[E(K)]$ is an r-region of G disjoint from H.

(ii) If K is an r-region of G/H containing v_H, and if $r' > r$, then $G[E(K) \cup E(H)]$ is an r-region of G.

Proof: (i) Suppose that $v_H \notin V(K)$. Then $G[E(K)] \cong K$, and so K can be regarded as a subgraph of G disjoint from H. Since $\tau(K) = r$, G has an s-region L containing K as a subgraph, where $s \geq t$. Then L (if $V(L) \cap V(H) = \emptyset$) or $L/(L \cap H)$ (if $V(L) \cap V(H) \neq \emptyset$) is a region of G/H containing K, by Proposition 2.6, and so we must have $L = K$.

(ii) Let $K'' = G[E(K) \cup E(H)]$ with $\tau(K'') = s$. By $r' > r$, both $K \in \mathcal{T}_r$ and $H \in \mathcal{T}_{r'} \subset \mathcal{T}_r$, and so by Lemma 2.1(i), $K'' \in \mathcal{T}_r$. This implies $s \geq r$. By Lemma 2.2, there is a region L of G containing K'' as a subgraph with $\tau(L) \geq s \geq r$. Note that H is a nonspanning subgraph of L. Apply Proposition 2.6(ii) to L and H to conclude that L/H is a $\tau(L)$-region of G/H containing K. Then apply Lemma

2.3 to L/H and K to conclude that $r \geq \tau(L)$, where equality holds if and only if $K = L/H$. It follows that $r = s = \tau(L)$ and $K = L/H$, and so $K'' = L$ is an r-region of G. □

3. The Main results.

Let G be a graph. The edge arboricity of G, $a(G)$, is the minimum number of edge-disjoint spanning forests whose union is G.

__Theorem 3.1__ (Nash-Williams [4], [5], Tutte [8]) Let G be a graph and let k be an integer. Then

(i) $a(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil$, where the maximum is taken over all induced subgraphs H of G with $|V(H)| \neq 2$.

(ii) If $|E(G)| \geq k(|V(G)|-1)$, then G has a subgraph H with $\tau(H) \geq k$.

(iii) $\tau(G) = \left\lfloor \min_{X \subseteq E(G)} \frac{|X|}{\omega(G-X)-\omega(G)} \right\rfloor$, where the minimum is over all subsets $X \subseteq E(G)$ such that $\omega(G-X) > \omega(G)$.

__Corollary 3.2__ $a(G) \geq i_m \geq a(G) - 1$.

__Proof__: Let L be a component of $G[E_m]$. By Theorem 2.4, every component of $G[E_m]$ has i_m edge-disjoint spanning trees, and no nontrivial subgraph of G with $i_m + 1$ edge-disjoint spanning trees. Thus by Theorem 3.1, $i_m = \tau(L) \leq |E(L)|/(|V(L)|-1) \leq a(G)$.

By Theorem 3.1(i), there is a subgraph H of G such that $|E(H)| \geq (a(G)-1)(|V(H)|-1)$. Therefore, by Theorem 3.1(ii), H (and so G) has a subgraph H' with $\tau(H') \geq (a(G)-1)$, and so by Lemma 2.2, G has a region K with $E(H') \subseteq E(K)$ and with $\tau(K) \geq (a(G)-1)$. By the definition of i_m in the proof of Theorem 2.4, we know that $i_m \geq a(G) - 1$. □

__Lemma 3.3__ Let G be a graph and let k be an integer with $k > \tau(G)$. If $k \geq a(G)$, then one can find $X \subseteq E(G^c)$ with $|X| = k(|V(G)|-1) - |E(G)|$ such that $G+X$ is the union of k edge-disjoint spanning trees.

Proof: By $a(G) \leq k$, there are edge-disjoint spanning forests $F_1, \cdots F_k$ such that $G = \cup_{i=1}^{k} F_i$. Set $X_0 = \emptyset$. For each i, $(1 \leq i \leq k)$, there is an edge set $X_i \subset E((G + (\cup_{j=0}^{i-1} X_j))^c)$ such that $F_i + X_i$ is a tree. Let $X = \cup_{j=1}^{k} X_j$. Then $G + X$ is the union of k edge-disjoint spanning trees, and so $|E(G)| + |X| = k(|V(G)| - 1)$. □

Let G be a graph and let $k \geq \tau(G)$ be an integer. Let $f(G,k)$ denote the minimum number of edges that must be added to G so that the resulting graph has k edge-disjoint spanning trees. By Theorem 2.4, G has a decomposition satisfying (1) and (2). If $k \leq i_m$, define $i(k) = \min\{i_j : i_j \geq k \text{ and } i_j \in sp(G)\}$; if $k > i_m$, define $i(k) = \infty$, and define $E_\infty = \emptyset$. Let $c_k(G)$ be the number of components of $G[E_{i(k)}]$, and let $w_k(G) = |V(G[E_{i(k)}])|$. Note that $c_k(G) = w_k(G) = 0$ if $i(k) = \infty$.

Theorem 3.4 Let G be a graph and let $k > \tau(G)$. Then

$$f(G,k) = k(|V(G)| - w_k(G) + c_k(G) - 1) - (|E(G)| - |E_{i(k)}|).$$

Proof: If $k > a(G)$, then by Corollary 3.2 and the definition of $i(k)$, we have $i(k) = \infty$, and so $c_k(G) = w_k(G) = 0$. Thus, by Lemma 3.3, $f(G,k) = k(|V(G)| - 1) - |E(G)|$. Theorem 3.3 holds in this case. In the following we assume that $k \leq a(G)$.

By Theorem 2.4, G has a decomposition satisfying (1) and (2). Let $G' = G/E_{i(k)}$. Then

$$|V(G')| = |V(G)| - (w_k(G) - c_k(G)) \text{ and } |E(G')| = |E(G)| - |E_{i(k)}|. \tag{9}$$

Claim $a(G') \leq k$.

Suppose that $a(G') > k$. By Corollary 3.2, we know that G' has an r-region L' with $r \geq k$. Let $H_1, \cdots, H_c$ be the components of $G[E_{i(k)}]$, and let v_i denote the vertex in G' to which H_i is contracted. By Theorem 2.4,

$$\tau(H_i) \geq k, \text{ for every } i = 1, 2, \cdots, c. \tag{10}$$

If L' does not contain any v_i, then by Corollary 2.7(i), $L' = G[E(L')]$ is an r-region of G. Since $r \geq k$, and by Theorem 2.4, $E(L') \subseteq E_{i(k)}$, then L' cannot be a subgraph of G', a contradiction.

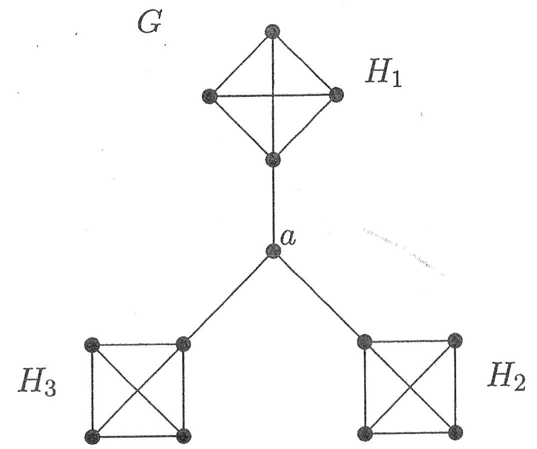
Hence, we may assume that $v_1, \cdots, v_t \in V(L')$ and $v_i \notin V(L')$ for $i \geq t+1$. Let $L = G[E(L') \cup (\cup_{i=1}^{t} E(H_i))]$, and let $k' = \min_{1 \leq i \leq t} \tau(H_i)$. Then by the definition of $E_{i(k)}$, $k' \geq k$. Thus, $\min\{r, k'\} \geq k$. By Lemma 2.1(i), L is a subgraph of G containing $H_1, \cdots, H_t$ as subgraphs, and $\tau(L) \geq \min\{r, k'\} \geq k$. Therefore, by the definition of $E_{i(k)}$, by $\tau(L) \geq k$, and by Theorem 2.4, $E(L) \subset E_{i(k)}$, contrary to the assumption that L' is a subgraph of G'. Thus, the claim follows.

By the Claim, G' satisfies the hypothesis of Lemma 3.3, and so by Lemma 3.3, there is an edge subset $X \subset E((G')^c)$ with

$$|X| = k(|V(G')| - 1) - |E(G')|, \tag{11}$$

such that $G'+X$ is the union of k edge-disjoint spanning trees. Thus, the number of additional edges represented in (11) is the minimum number of edges that must be added to G' to have k edge-disjoint spanning trees. Note that $G'+X = G/E_{i(k)}+X \cong (G+X)/E_{i(k)}$, and each component of $G[E_{i(k)}]$ is an r-region of G with $r \geq i(k) \geq k$. By Lemma 2.1(i), $\tau(G+X) \geq k$. By (9) and (11), Theorem 3.4 follows. □

Example 3.5 Let $V(K_{1,3}) = \{a, v_1, v_2, v_3\}$, where $d(a) = 3$, and $d(v_i) = 1$ $(1 \leq i \leq 3)$. Let G be a graph obtained from $K_{1,3}$ by replacing each v_i in $K_{1,3}$ by $H_i = K_4 (1 \leq i \leq 3)$ as shown below. Obviously, $\tau(G) = 1$, and G itself is a 1-region. Only H_1, H_2 and H_3 are 2-regions in G. If $r \geq 3$, G has no r-region, and so $\xi(G) = 2$. Therefore, $sp(G) = \{1, 2\}$. Thus, as stated in Theorem 2.4, $1 = i_1 < 2 = i_2$ are the integers uniquely determined by G. And $E_2 = \bigcup_{i=1}^{3} E(H_i) \subset E_1 = E(G)$ are the edge subsets uniquely determined by G.



Let $k = 2$. Then $i(k) = 2$, and so $|E_{i(k)}| = |E_2| = 18$, $c_k(G) = 3$, and $w_k(G) = |V(G[E_2])| = 12$. By Theorem 3.4, the minimum number of edges that must be added to G so that the resulting graph has 2 edge-disjoint spanning trees is

$$\begin{aligned} f(G,2) &= k(|V(G)| - w_k(G) + c_k(G) - 1) - (|E(G)| - |E_{i(k)}|) \\ &= 2(13 - 12 + 3 - 1) - (21 - 18) = 3. \end{aligned}$$

Note that there are more than one way to select three edges to add to G so that the resulting graph has 2 edge-disjoint spanning trees. In fact, we can choose an arbitrary vertex v_{H_i} from $V(H_i)$ $(1 \le i \le 3)$, and let $e_1 = v_{H_1}v_{H_2}$, $e_2 = v_{H_1}v_{H_3}$, and $e_3 = v_{H_2}v_{H_3}$ be the three new edges. Then the new graph obtained from G by adding e_1, e_2, and e_3 has 2 edge-disjoint spanning trees.

Remark. From Theorem 3.4 above, one can see that for a given graph G and a given integer k, the main task to find $f(G,k)$ is to find $E_{i(k)}$. Hobbs [2] developed a polynomial-time algorithm to compute the number i_m and to locate the subset E_m as defined in Theorem 2.4. As long as $E_m \ne E(G)$ and $i_m \ge k$, by Corollary 2.5, one can apply Hobbs' algorithm to the contraction G/E_m. There are at most m iterations before $E_{i(k)}$ is found. Once $E_{i(k)}$ is found, it is easy to compute $c_k(G)$ and $w_k(G)$, and so by Theorem 3.4 to compute

$f(G,k)$. Thus, this gives a polynomial-time algorithm to compute $f(G,k)$.

In the following, we shall derive a different expression, a min-max formula, for $f(G,k)$.

Define, for each subset $X \subseteq E(G)$,

$$f_k(G,X) = k[\omega(G-X)-1] - |X|,$$

and

$$F_k(G) = \max_{X\subseteq E(G)} \{f_k(G,X)\}. \tag{12}$$

Note that $F_k(G) \geq f_k(G,\emptyset) \geq 0$, and that $F_k(K_1) = 0$, for any $k \geq 1$. We shall show in Theorem 3.10 that $F_k(G) = f(G,k)$.

Lemma 3.6 Assume that $X \subseteq E(G)$ is an edge-subset with $f_k(G,X) = F_k(G)$, and that H is a component of $G - X$. If $X_H \subseteq E(H)$ is an edge-subset, then

$$f_k(G, X \cup H_X) = f_k(G,X) + f_k(H,X_H). \tag{13}$$

Proof: Let X, H and X_H be as assumed. Then

$$\begin{aligned} f_k(G, X \cup H_X) &= k[\omega(G - X \cup X_H) - 1] - |X| - |X_H| \\ &= k[\omega(G-X) - 1 + \omega(H - X_H) - 1] - |X| - |X_H| \\ &= f_k(G,X) + f_k(H,X_H). \end{aligned}$$

Corollary 3.7 If $X \subseteq E(G)$ satisfies $F_k(G) = f_k(G,X)$, then for every component H of $G-X$, $F_k(H) = 0$. In particular, $\tau(H) \geq k$.

Proof: By Lemma 3.6, for any $X_H \subseteq E(H)$, $f_k(H,X_H) = f_k(G, X\cup X_H) - F_k(G) \leq 0$, and so $F_k(H) = \max_{X_H\subseteq E(H)}\{f_k(H,X_H)\} = 0$.

To prove that $\tau(H) \geq k$, we may assume that $H \neq K_1$ since

$\tau(K_1) = \infty$. By the definition of $f_k(H, X_H)$, $F_k(H) = \max_{X_H \subseteq E(H)}\{f_k(H, X_H)\} = 0$ implies that

$$\max_{X_H \subseteq E(H)} \{k[\omega(H - X_H) - 1] - |X_H|\} = 0.$$

Therefore, for any $X_H \subseteq E(H)$ with $\omega(H - X_H) > 1$,

$$\frac{|X_H|}{\omega(H - X_H) - 1} \geq k.$$

By Theorem 3.1(iii), $\tau(H) \geq k$. □

Lemma 3.8 If G is connected, and if $F_k(G) = f_k(G, E(G))$, then $a(G) \leq k$.

Proof: Let H be an induced subgraph of G. Define $E_H = E(G) - E(H)$. Since the components of $G - E_H$ are H and $|V(G)| - |V(H)|$ isolated vertices,

$$\omega(G - E_H) = |V(G)| - |V(H)| + \omega(H). \tag{14}$$

By (12),

$$\begin{aligned}
F_k(G) &\geq f_k(G, E_H) \geq k(\omega(G - E_H) - 1) - |E_H| \\
&= k(|V(G)| - |V(H)| + \omega(H) - 1) - |E(G)| + |E(H)| \\
&\geq k(|V(G)| - |V(H)|) - |E(G)| + |E(H)| \\
&= k(|V(G)| - 1) - |E(G)| + k - (k|V(H)| - |E(H)|) \\
&= f_k(G, E(G)) - [k(|V(H)| - 1) - |E(H)|] \\
&= F_k(G) - [k(|V(H)| - 1) - |E(H)|].
\end{aligned}$$

It follows that

$$0 \geq -k(|V(H)| - 1) + |E(H)|,$$

and so

$$\frac{|E(H)|}{|V(H)| - 1} \leq k.$$

By Theorem 3.1(i), $a(G) \leq k$. □.

Lemma 3.9 Let G be a graph and let $E_0 \subset E(G)$ be such that $f_k(G, E_0) = F_k(G)$. Let $G_0 = G/(E(G) - E_0)$. Then

$$f_k(G_0, E_0) = F_k(G_0) = F_k(G).$$

Proof: Note that $\omega(G - E_0) = \omega(G_0 - E_0)$, and so by the assumption that $f_k(G, E_0) = F_k(G)$, we have

$$F_k(G_0) \geq f_k(G_0, E_0) = f_k(G, E_0) = F_k(G).$$

Choose $E_1 \subseteq E_0$, such that $F_k(G_0) = f_k(G_0, E_1)$. Then since $E_1 \subseteq E_0$, $\omega(G - E_1) = \omega(G_0 - E_1)$, and so

$$F_k(G) \geq f_k(G, E_1) = f_k(G_0, E_1) = F_k(G_0). \ \square$$

Next we prove a min-max theorem.

Theorem 3.10 $F_k(G) = f(G, k)$.

Proof: Let $E_0 \subseteq E(G)$ be an edge subset of $E(G)$ such that $f_k(G, E_0) = F_k(G)$, and let $G_0 = G/(E(G) - E_0)$ as defined in Lemma 3.9. By Lemma 3.9, $f_k(G_0, E(G_0)) = F_k(G_0) = F_k(G)$.

By Lemma 3.8, $a(G_0) \leq k$. Hence, G_0 is an edge-disjoint union of k spanning forests $F_1, F_2, \cdots, F_k$ of G_0. Let $|E(F_i)| = |V(G_0)| - 1 - s_i$, where $s_i \geq 0$ and $1 \leq i \leq k$. Then one can add s_i edges to F_i to form a spanning tree of G_0. Therefore, by adding an edge set X with $\sum_{i=1}^k s_i$ edges to G_0, the resulting graph $G_0 + X$ has k edge-disjoint spanning trees. Note that $|E(G_0)| = \sum_{i=1}^k |E(F_i)| = k(|V(G_0)| - 1) - \sum_{i=1}^k s_i$. Since $F_k(G) = F_k(G_0) = f_k(G_0, E(G_0)) = k(|V(G_0)| - 1) - |E(G_0)|$, $F_k(G) = F_k(G_0) = \sum_{i=1}^k s_i$. This shows that

$$F_k(G) = F_k(G_0) = f(G_0, k). \tag{15}$$

Let $H_1, H_2, \cdots, H_c$ be the components of $G - E_0$. By Corollary 3.7, $\tau(H_i) \geq k$, $1 \leq i \leq c$. Note that $G_0 + X = (G + X)/(E(G) - E_0) = (G + X)/\bigcup_{i=1}^c H_i = ((G + X)/\bigcup_{i=2}^c H_i)/H_1$. By repeatedly applying Lemma 2.1(i), we have $\tau(G+X) \geq k$, and so $F_k(G) \geq f(G,k)$.

Conversely, let X be a set of $f(G,k)$ edges that must be added to G such that $\tau(G+X) \geq k$. Let $W_i = X \cap E(H_i^c)$, and let $H_i' = H_i + W_i$. Since $\tau(H_i) \geq k$, by Lemma 2.1(ii), $\tau(H_i') \geq k$. Let $X_1 = \cup_{i=1}^c W_i$, and $X_0 = X - X_1$. Then $G_0 + X_0 = (G+X)/((E(G)+X)-(E_0+X_0)) = (G+X)/\bigcup_i^c H_i'$. By Lemma 2.1(i), $\tau(G_0+X_0) \geq k$. Therefore, by (15)

$$F_k(G) = f(G_0,k) \leq |X_0| \leq |X| = f(G,k). \ \square$$

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