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Hamiltonian connectedness in 3-connected line graphs

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Abstract

We investigate graphs G such that the line graph $L(G)$ is hamiltonian connected if and only if $L(G)$ is 3-connected, and prove that if each 3-edge-cut contains an edge lying in a short cycle of G, then $L(G)$ has the above mentioned property. Our result extends Kriesell's recent result in [M. Kriesell, All 4-connected line graphs of claw free graphs are hamiltonian-connected, J. Combin. Theory Ser. B 82 (2001) 306–315] that every 4-connected line graph of a claw free graph is hamiltonian connected. Another application of our main result shows that if $L(G)$ does not have an hourglass (a graph isomorphic to $K_5 - E(C_4)$, where C_4 is an cycle of length 4 in K_5) as an induced subgraph, and if every 3-cut of $L(G)$ is not independent, then $L(G)$ is hamiltonian connected if and only if $\kappa(L(G)) \geq 3$, which extends a recent result by Kriesell [M. Kriesell, All 4-connected line graphs of claw free graphs are hamiltonian-connected, J. Combin. Theory Ser. B 82 (2001) 306–315] that every 4-connected hourglass free line graph is hamiltonian connected.

Keywords: Hamiltonian connected; Collapsible graphs; Core graphs

1. Introduction

Graphs considered here are finite and loopless but may have multiple edges. Unless otherwise noted, we follow [1] for notations and terms. A graph G is **nontrivial** if $E(G) \neq \emptyset$. For a graph G and a vertex $v \in V(G)$, define $D_i(G) = \{v \in V(G) : d_G(v) = i\}$ and

$$E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}.$$

An edge cut X of G is **peripheral** if for some $v \in V(G)$, $X = E_G(v)$; and is **essential** if each side of $G - X$ has an edge. Let G be a graph and let $X \subseteq E(G)$ be an edge subset. The **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. For convenience, we use G/e for $G/\{e\}$ and $G/\emptyset = G$; and if H is a subgraph of G, we write G/H for $G/E(H)$.

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The **line graph** of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.

A graph G is **hamiltonian connected** if for every pair of vertices $u, v \in V(G)$, G has a spanning (u, v)-path (a path starting from u and ending at v). In [10], Thomassen conjectured that every 4-connected line graph is hamiltonian, and in 1986, Zhan proved:

Theorem 1.1 (*Zhan, [11]*). *If G is a 4-edge-connected graph, then the line graph $L(G)$ is hamiltonian connected.*

For a graph G, an induced subgraph H isomorphic to $K_{1,3}$ is called a **claw** of G, and the only vertex of degree 3 of H is the **center** of the claw. A graph G is **claw free** if it does not contain a claw. Let C_4 denote a 4-cycle in K_5. The graph $K_5 - E(C_4)$ is called an **hourglass**. A graph G is **hourglass free** if G does not have an induced subgraph isomorphic to $K_5 - E(C_4)$. Recently, Kriesell presented the following results.

Theorem 1.2. (i) *(Kriesell, [9]). Every 4-connected line graph of a claw free graph is hamiltonian connected.*
(ii) *(Kriesell, [9]). Every 4-connected hourglass free line graph is hamiltonian connected.*

It is well known that every hamiltonian connected graph with at least 4 vertices must be 3-connected. In this paper, we investigate such graphs G that $L(G)$ is hamiltonian connected if and only if $L(G)$ is 3-connected. To describe our finding, we need one more concept. Let G be a graph such that $\kappa(L(G)) \geq 3$ and $L(G)$ is not complete. The **core** of this graph G, denoted by G_0, is obtained by deleting all the vertices of degree 1 and contracting exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$. After deleting all the vertices of degree one, no new vertices of degree two arise, and hence the minimum degree of the core is at least three. The length of each path with internal vertices of degree 2 in G is at most two, and hence when we say contracting one edge xy or yz, no ambiguity arises.

Note that an essential edge cut in G corresponds to a vertex cut in $L(G)$; and vice versa when $L(G)$ is not complete. Our main result is the following

Theorem 1.3. *Let G be a connected graph with $|E(G)| \geq 4$. If every 3-edge-cut of the core G_0 has at least one edge lying in a cycle of length at most 3 in G_0, and if $\kappa(L(G)) \geq 3$, then $L(G)$ is hamiltonian connected.*

Theorem 1.3 clearly extends Theorem 1.1 and the following corollaries of Theorem 1.3 extend Theorem 1.2.

Corollary 1.4. *Let G be a graph with $|V(G)| \geq 4$. Suppose that $L(G)$ is hourglass free in which every 3-cut of $L(G)$ is not an independent set. If $\kappa(L(G)) \geq 3$, then $L(G)$ is hamiltonian-connected.*

A graph G is **almost claw free** if the vertices that are centers of claws in G are independent and if the neighborhoods of the center of each claw in G is 2-dominated (having 2 vertices in the neighborhoods of the center adjacent to other neighbors). Note that every claw free graph is an almost claw free graph and there exist almost claw free graphs that are not claw-free.

Corollary 1.5. *Every 4-connected line graph of an almost claw free graph is hamiltonian-connected.*

In Section 2, we introduce Catlin's reduction method and provide the mechanism needed in the proofs. Our main result is proved in Section 3 and Corollaries 1.4 and 1.5 are proved in Section 4.

2. Preliminaries

For a graph G, let $O(G) = \{v \in V(G) : d_G(v) \text{ is odd}\}$. A connected graph G is **eulerian** if $O(G) = \emptyset$. A spanning closed trail of G is also referred as a **spanning eulerian subgraph** of G. A subgraph H of G is **dominating** if $G - V(H)$ is edgeless. (Note the difference between a dominating vertex subset and a dominating subgraph.) If a closed trail C of G satisfies $E(G - V(C)) = \emptyset$, then C is a **dominating eulerian subgraph**. A well known relationship between dominating eulerian subgraphs in G and hamiltonian cycles in $L(G)$ is given by Harary and Nash-Williams.

Theorem 2.1 (*Harary and Nash-Williams, [8]*). *Let G be a connected graph with at least 3 edges. The line graph $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

We view a trail of G as a vertex-edge alternating sequence

$$v_0, e_1, v_1, e_2, \ldots, e_k, v_k \tag{1}$$

such that all the e_i's are distinct and such that for each $i = 1, 2, \ldots, k$, e_i is incident with both v_{i-1} and v_i. All the vertices in $\{v_1, v_2, \ldots, v_{k-1}\}$ are **internal vertices** of the trail in (1). For edges $e', e'' \in E(G)$, an (e', e'')-trail of G is a trail of G whose first edge is e' and whose last edge is e''. (Thus the trail in (1) is an (e_1, e_k)-trail). A **dominating** (e', e'')**-trail** of G is an (e', e'')-trail T of G such that every edge of G is incident with an internal vertex of T; and a **spanning** (e', e'')**-trail** of G is a dominating (e', e'')-trail T of G such that $V(T) = V(G)$. The following follows by a similar argument in the proof of Theorem 2.1.

Proposition 2.2. *Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian connected if and only if for any pair of edges $e', e'' \in E(G)$, G has a dominating (e', e'')-trail.*

A graph G is **collapsible** if for any even subset X of $V(G)$, G has a spanning connected subgraph R_X of G such that $O(R_X) = X$. Catlin [4] showed that every graph G has a unique subgraph H each of whose components is a maximal collapsible subgraph of G. The contraction G/H is the **reduction** of G. A graph G is **reduced** if G has no nontrivial collapsible subgraphs; or equivalently, if G equals the reduction of G. We summarize some results on Catlin's reduction method and other related facts below.

Theorem 2.3. *Let G be a graph and let H be a collapsible subgraph of G. Let v_H denote the vertex onto which H is contracted in G/H. Each of the following holds.*

(i) *(Catlin, Theorem 3 of [4]). G is collapsible if and only if G/H is collapsible. In particular, G is collapsible if and only if the reduction of G is K_1.*
(ii) *(Catlin, Theorem 8 of [4]). 2-cycles and 3-cycles are collapsible.*
(iii) *If G is collapsible, then for any pair of vertices $u, v \in V(G)$, G has a spanning (u, v)-trail.*
(iv) *For vertices $u, v \in V(G/H) - \{v_H\}$, if G/H has a spanning (u, v)-trail, then G has a spanning (u, v)-trail.*
(v) *(Catlin, Theorem 5 of [4]). Any subgraph of a reduced graph is reduced.*
(vi) *If G is collapsible, and if $e \in E(G)$, then G/e is also collapsible.*

Proof. (iii) Let $X = \{u, v\}$. Then $|X| \equiv 0 \pmod 2$, and a spanning connected subgraph R_X of G with $O(R_X) = \{u, v\}$ is a spanning (u, v)-trail.

(iv) Let Γ' be a spanning (u, v)-trail of G/H and let

$$X = \{w \in V(H) : w \text{ is incident with an odd number of edges in } \Gamma'\}.$$

Since v_H has even degree in Γ', $|X| \equiv 0 \pmod 2$. Let R'_X be a spanning connected subgraph of H with $O(R'_X) = X$. Then $\Gamma = G[E(\Gamma') \cup E(R'_X)]$ is a spanning (u, v)-trail in G.

(vi) follows by the definition of collapsible graphs. □

Let $\tau(G)$ denote the maximum number of edge-disjoint spanning trees of G. We assume that $\tau(K_1) = \infty$. Catlin showed the relationship between $\tau(G)$ and the edge-connectivity $\kappa'(G)$. Part (ii) of the next theorem is an observation made in [3,6].

Theorem 2.4. *Let G be a graph, H be a subgraph of G, and $k > 0$ be an integer.*

(i) *(Catlin, Theorem 5.1 of [2]). $\kappa'(G) \geq 2k$ if and only if for any edge subset $X \subseteq E(G)$ with $|X| \leq k$, $\tau(G - X) \geq k$.*
(ii) *If $\tau(H) \geq k$ and if $\tau(G/H) \geq k$, then $\tau(G) \geq k$.*

Theorem 2.5 (*Catlin and Lai, Theorem 4 of [7]*). *Let G be a graph with $\tau(G) \geq 2$ and let $e', e'' \in E(G)$. Then G has a spanning (e', e'')-trail if and only if $\{e', e''\}$ is not an essential edge cut of G.*

We define $F(G)$ be the minimum number of additional edges that must be added to G such that the resulting graph has two edge-disjoint spanning trees.

Theorem 2.6. *Let G be a graph.*

(i) *(Catlin, Han and Lai, Lemma 2.3 of [5]). If for any $H \subset G$ with $|V(H)| < |V(G)|$, H is reduced, and if $|V(G)| \geq 3$, then $F(G) = 2|V(G)| - |E(G)| - 2$.*
(ii) *(Catlin, Theorem 7 of [4]). If $F(G) \leq 1$, then G is collapsible if and only if $\kappa'(G) \geq 2$.*
(iii) *(Catlin, Han and Lai, Theorem 1.3 of [5]). Let G be a connected graph and t an integer. If $F(G) \leq 2$, then G is collapsible if and only if G cannot be contracted to a member in $\{K_2\} \cup \{K_{2,t} : t \geq 1\}$.*

We say that an edge $e \in E(G)$ is **subdivided** when it is replaced by a path of length 2 whose internal vertex, denoted by $v(e)$, has degree 2 in the resulting graph. The process of taking an edge e and replacing it by that length 2 path is called **subdividing** e. For a graph G and edges $e', e'' \in E(G)$, let $G(e')$ denote the graph obtained from G by subdividing e', and let $G(e', e'')$ denote the graph obtained from G by subdividing both e' and e''. Then,

$$V(G(e', e'')) - V(G) = \{v(e'), v(e'')\}.$$

The above definitions imply the following lemma.

Lemma 2.7. *For a graph G and edges $e', e'' \in E(G)$, each of the following holds.*

(i) *if $G(e', e'')$ has a spanning $(v(e'), v(e''))$-trail, then G has a spanning (e', e'')-trail.*
(ii) *if $G(e', e'')$ has a dominating $(v(e'), v(e''))$-trail, then G has a dominating (e', e'')-trail.*

Lemma 2.8. *Let G be a graph and $G' = G - D_1(G)$. If $\kappa(L(G)) \geq 3$ and $L(G)$ is not complete, then*

(i) *G' is nontrivial and $\delta(G') \geq \kappa'(G') \geq 2$.*
(ii) *G_0 is nontrivial and $\delta(G_0) \geq \kappa'(G_0) \geq 3$.*
(iii) *for $v \in V(G)$ with $d_G(v) = 1$ or $d_G(v) = 2$, $N_G(v) \subseteq V(G_0)$.*

Lemma 2.9. *Let G be a graph such that $\kappa(G) \geq 3$ and $L(G)$ is not complete and let G_0 be the core of G. If $G_0(e', e'')$ has a spanning $(v(e'), v(e''))$-trail for any $e', e'' \in E(G_0)$, then for any $e', e'' \in E(G)$, $G(e', e'')$ has a dominating $(v(e'), v(e''))$-trail.*

Proof. Let $e', e'' \in E(G)$. If $e' \in E(G_0)$, let $f' = e'$; if e' is incident with a vertex of degree 2, let f' be the corresponding new edge in G_0; if e' is incident to a vertex of degree 1, let f' be any edge in G_0 incident with the other vertex incident with e'. Similarly we define f''. Then a spanning $(v(f'), v(f''))$-trail in $G_0(f', f'')$ can be adjusted to a dominating $(v(e'), v(e''))$-trail in G. □

3. Proof of Theorem 1.3

We start with a few more lemmas.

Lemma 3.1. *Let G be a 3-edge-connected graph without loops, $v, v_1, u_1, u_2 \in V(G)$ be such that $d_G(v_1) = 3$ and $N_G(v_1) = \{v, u_1, u_2\}$, and for an integer $k \geq 1$ let $X' = \{u_1u_2, u_1v_i, u_2v_i : 1 \leq i \leq k\}$ be an edge subset of G and $W = G[X']$. Then each of the following holds.*

(i) *If $(G - vv_1)/W$ is nontrivial and $\tau((G - vv_1)/W) \geq 2$, then $\tau(G) \geq 2$.*
(ii) *If $G/W = K_1$, then $\tau(G) \geq 2$.*

Proof. (i) Let $H = (G - vv_1)/W$. As H is nontrivial, let T_1', T_2' be two edge-disjoint spanning trees of H. For $k = 1$ (see Fig. 1(a)), $T_1 = G[E(T_1') \cup \{vv_1, u_1u_2\}]$ and $T_2 = G[E(T_2') \cup \{v_1u_1, v_1u_2\}]$ are two edge-disjoint spanning trees of G. For $k \geq 2$ (see Fig. 1(b)), $T_1 = G[E(T_1') \cup \{vv_1, u_1u_2\} \cup \{u_2v_2, u_2v_3, \ldots, u_2v_k\}]$ and $T_2 = G[E(T_2') \cup \{v_1u_2\} \cup \{u_1v_1, u_1v_2, \ldots, u_1v_k\}]$ are two edge-disjoint spanning trees of G.

(ii) If $G/W = K_1$, then G is spanned by the vertex set $V(W) = \{v, v_1, \ldots, v_k, u_1, u_2\}$. Therefore, $v \in V(W)$. Since G has no loops, $v \neq v_1$ and so $v \in \{v_2, \ldots, v_k, u_1, u_2\}$ and the construction of T_1, T_2 in the proof of (i) still works with $E(T_1') = E(T_2') = \emptyset$. □

Lemma 3.2. *If G is a graph with $\tau(G) \geq 2$ and $\kappa'(G) \geq 3$, then $G(e', e'')$ is collapsible for any $e', e'' \in E(G)$.*

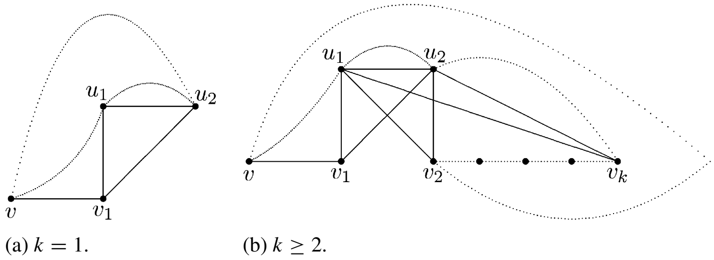


Fig. 1. G.

Proof. Since $\tau(G) \geq 2$, $F(G(e', e'')) \leq 2$. By Theorem 2.6(iii), $G(e', e'')$ is either collapsible, or the reduction of $G(e', e'')$ is a K_2 or a $K_{2,t}$ for some integer $t \geq 1$. Since $\kappa'(G) \geq 3$, $\kappa'(G(e', e'')) \geq 2$ and $G(e', e'')$ has at most two 2-edge-cuts. Thus $G(e', e'')$ can not be contracted to K_2 or $K_{2,t}$ for some integer $t \geq 1$, and so $G(e', e'')$ must be collapsible. □

Theorem 3.3. *Let G be a graph with $\kappa'(G) \geq 3$. If every 3-edge-cut of G has at least one edge in a 2-cycle or 3-cycle of G, then the graph $G(e', e'')$ is collapsible for any $e', e'' \in E(G)$.*

Proof. By contradiction, we assume that

$$G \text{ is a counterexample to Theorem 3.3 with } |V(G)| \text{ minimized.} \tag{2}$$

Thus G satisfies the hypotheses of Theorem 3.3 but for some $e', e'' \in E(G)$, $G(e', e'')$ is not collapsible.

Let G_1 be the reduction of $G(e', e'')$. The following observations (I), (II) and (III) follow from the assumption that $\kappa'(G) \geq 3$, from (2) and Theorem 2.3(i), and from the definition of $G(e', e'')$.

(I) The only edge cuts of size 2 in $G(e', e'')$ are $E_{G(e',e'')}(v(e'))$ and $E_{G(e',e'')}(v(e''))$.

(II) $G_1 \neq K_1$ and so G_1 is not collapsible.

(III) For every 3-edge-cut X_1 of G_1, there is a 3-edge-cut X of G such that

$$X = \begin{cases} (X_1 - f') \cup e' & \text{if } X_1 \text{ contains } f' \in E_{G_1}(v(e')) \text{ and } E_{G_1}(v(e'')) \cap X_1 = \emptyset \\ (X_1 - f'') \cup e'' & \text{if } X_1 \text{ contains } f'' \in E_{G_1}(v(e'')) \text{ and } E_{G_1}(v(e')) \cap X_1 = \emptyset \\ (X_1 - \{f', f''\}) \cup \{e', e''\} & \text{if } X_1 \text{ contains } f' \in E_{G_1}(v(e')) \text{ and } f'' \in E_{G_1}(v(e'')) \\ X_1 & \text{otherwise.} \end{cases}$$

In any case, we shall say that X is an edge-cut in G corresponding to the edge-cut X_1 in G_1, or vice versa. Let X be a 3-edge-cut of G such that at least one edge of X lies in a cycle C_X of G with $|E(C_X)| \leq 3$. This C_X is called a **short cycle related to the edge-cut** X. If $e' \in E(C_X)$, then call X an e'**-cut**. Similarly, we define an e''**-cut**.

Since G_1 is the reduction of $G(e', e'')$, we have either $G_1 = G(e', e'')$ or $G_1 \neq G(e', e'')$. Next we show that neither of these two cases is possible.

Case 1. $G_1 \neq G(e', e'')$.

Then there exists a nontrivial subgraph H of $G(e', e'')$, each of whose components is a maximal collapsible subgraph of $G(e', e'')$ such that $G_1 = G(e', e'')/H$. The definition of collapsible graphs implies that

$$\text{each component of } H \text{ is 2-edge-connected.} \tag{3}$$

If $v(e'), v(e'') \notin V(H)$, then $v(e'), v(e'') \in V(G_1)$ and by (3), $E_{G_1}(v(e')) \cup E_{G_1}(v(e'')) \subseteq E(G_1)$. Then $G/H = (G_1 - \{v(e'), v(e'')\}) \cup \{e', e''\}$ and G/H satisfies the conditions of Theorem 3.3 with $|V(G/H)| < |V(G)|$. By (2), $G_1 = (G/H)(e', e'')$ must be collapsible, contrary to (II).

If $v(e'), v(e'') \in V(H)$, then by (3), $E_{G_1}(v(e')) \cup E_{G_1}(v(e'')) \subseteq E(H)$. Thus $e', e'' \notin E(G_1) = E(G(e', e'')) - E(H)$ and so by (I), $\kappa'(G_1) \geq 3$. If G_1 has a 3-edge-cut X, then as $X \cap E(H) = \emptyset$ and by (III), X must be a 3-edge-cut of G. It follows by the assumption of Theorem 3.3 that X has a related short cycle C_X in G with $|E(C_X)| \leq 3$ and with $|E(C_X) \cap X| = 2$. Since C_X is a collapsible subgraph by Theorem 2.3(ii), $C_X \subseteq H$, and so $X \cap E(H) \neq \emptyset$, a contradiction. Thus $\kappa'(G_1) \geq 4$, and so by Theorem 2.4(i) and 2.6(ii), G_1 is collapsible, contrary to (II).

Therefore we assume without loss of generality that $v(e') \notin V(H)$ and $v(e'') \in V(H)$. Let $H_1 = (H - v(e'')) \cup e''$. Thus each component of H_1 is collapsible by the definition of collapsible graphs. Since e' is not in H_1,

$$G_1 = G(e', e'')/H = (G/H_1)(e') \quad \text{and} \quad \kappa'(G/H_1) \geq 3. \tag{4}$$

Claim 1. *Each of the following holds for the graph G/H_1.*

(i) *The graph G/H_1 must have 3-edge-cuts.*
(ii) *Every 3-edge-cut of G/H_1 is an e'-cut of G/H_1.*
(iii) *One of 3-edge-cuts of G/H_1 is peripheral.*

Proof of Claim 1. (i) If G/H_1 has no 3-edge-cuts, then by (4), $\kappa'(G/H_1) \geq 4$. By Theorem 2.4(i), $F((G/H_1)(e')) \leq 1$, and so by Theorem 2.6(ii), $G_1 = (G/H_1)(e')$ is collapsible, contrary to (II).

(ii) Let X be a 3-edge-cut of G/H_1. Since $G_1 = (G/H_1)(e')$, G_1 has a 3-edge-cut X_1 corresponding to X. If X is not an e'-cut, then $C_{X_1} = C_X$ is a collapsible subgraph of G_1 by Theorem 2.3(ii), contrary to the assumption that G_1 is reduced.

(iii) Suppose that all 3-edge-cuts are non-peripheral. As $\kappa'(G) \geq 3$, $(G/H_1)(e')$ has only one vertex of degree 2 and no vertex of degree 3. By Theorem 2.6(i), $F((G/H_1)(e')) = 2|V((G/H_1)(e'))| - |E((G/H_1)(e'))| - 2 \leq 2|V((G/H_1)(e'))| - (2|V((G/H_1)(e'))| - 2) - 2 = 0$. By Theorem 2.6(ii), $G(e', e'')$ is collapsible, contrary to the fact that $(G/H_1)(e')$ is reduced.

This completes the proof for Claim 1. □

By Claim 1, G/H_1 must have a peripheral 3-edge-cut which is also an e'-cut, *i.e.*, whose related short cycle contains e'. Then G/H_1 is isomorphic to the graph in Fig. 1(a) or (b), where $E_{G/H_1}(v_1)$ is a peripheral 3-edge-cut in G/H_1 and $e' \in \{u_1u_2, v_1u_1, v_1u_2\}$.

Let M be an edge subset of all triangles containing e' in G/H_1. By Claim 1(ii), each related short cycle of each 3-edge-cut contains e' and by the definition of M, it must be contained in the edge induced graph $(G/H_1)[M]$. If $(G/H_1)/M = K_1$, then by Lemma 3.1(ii), $\tau(G/H_1) \geq 2$ and so $G_1 = (G/H_1)(e')$ is collapsible by Lemma 3.2, contrary to (II).

Therefore we may assume that $(G/H_1)/M$ is a nontrivial 4-edge-connected graph. By Theorem 2.4(i), $\tau((G/H_1 - vv_1)/M) \geq 2$. By Lemma 3.1(i), $\tau(G/H_1) \geq 2$, and so by Theorem 2.6(i), $F[(G/H_1)(e')] \leq 1$. Thus by Theorem 2.6(ii), $G_1 = (G/H_1)(e')$ is collapsible, contrary to (II). This contradiction precludes Case 1.

Case 2. $G_1 = G(e', e'')$.

Claim 2. *Each of the following must hold.*

(i) *The graph G has at least three 3-edge-cuts.*
(ii) *Every 3-edge-cut of G is either an e'-cut or an e''-cut of G.*
(iii) *One of the 3-edge-cuts of G is peripheral.*

Proof of Claim 2. (i) As $\kappa'(G) \geq 3$, if G has at most two 3-edge-cuts, then we can add two new edges f_1, f_2 to G such that $\kappa'(G + \{f_1, f_2\}) \geq 4$. It follows by Theorem 2.4(i) that $\tau(G) \geq 2$. Thus by Lemma 3.2, $G(e', e'')$ is collapsible, contrary to (II).

(ii) Let X be a 3-edge-cut of G and suppose that the short cycle C_X related to X does not contain e' or e''. Since $G_1 = G(e', e'')$, G_1 has a 3-edge-cut X_1 corresponding to X. Then by Theorem 2.3(ii), C_X is a collapsible subgraph of G_1, contrary to the assumption that G_1 is reduced.

(iii) Assume that all 3-edge-cuts are non-peripheral. As $\kappa(G)' \geq 3$, $G(e', e'')$ has only two vertices of degree 2 and no vertex of degree 3. By Theorem 2.6(i), $F(G(e', e'')) = 2|V(G(e', e''))| - |E(G(e', e''))| - 2 \leq 2|V(G(e', e''))| - (2|V(G(e', e''))| - 2) - 2 = 0$. By Theorem 2.6(ii), $G(e', e'')$ is collapsible, contrary to the fact that $G(e', e'')$ is reduced.

This completes the proof for Claim 2. □

By Claim 2, we assume that G has a peripheral e'-cut. Then G is isomorphic to the graph in Fig. 1(a) or (b), where $E_G(v_1)$ is a peripheral 3-edge-cut in G.

Let M_1 be an edge subset of all triangles containing e' in G. With $z \mapsto z'$ being a graph isomorphism from W in Fig. 1(b) to W', we may assume that

$$E(W') = M_1 \cup \{v'v'_1\} = \{u'_1u'_2, u'_1v'_i, u'_2v'_i : 1 \le i \le k\} \cup \{v'v'_1\} \quad \text{and} \quad e' \in \{u'_1u'_2, v'_1u'_1, v'_1u'_2\}.$$

By Claim 2(ii), each related short cycle of any e'-cut of G must be contained in $G[M_1]$. Define $G_{11} = G/M_1$. If $G_{11} = K_1$, then by Lemma 3.1(ii), $\tau(G) \ge 2$ and so $G_1 = G(e', e'')$ is collapsible by Lemma 3.2, contrary to (II). Thus we may assume that G_{11} is nontrivial and $\kappa'(G_{11}) \ge 3$.

Claim 3. *Each of the following must hold.*

(i) *The graph G_{11} must have 3-edge-cuts.*
(ii) *Every 3-edge-cut of G_{11} must be an e''-cut of G.*
(iii) *G_{11} has a peripheral e''-cut.*

Proof of Claim 3. (i) If $\kappa'(G_{11}) \ge 4$, then by Theorem 2.4(i), $\tau(G_{11} - v'v'_1) = \tau(G/M_1 - v'v'_1) \ge 2$ and so by Lemma 3.1(i), $\tau(G) \ge 2$. Lemma 3.2 implies that $G(e', e'')$ is collapsible, contrary to (II).
(ii) As any edge-cut of G_{11} is also an edge-cut of G and $e' \notin E(G_{11})$, by Claim 2(ii), every 3-edge-cut of G_{11} must be an e''-cut of G.
(iii) By a similar argument as in the proof of Claim 1(iii), G_{11} has a peripheral e''-cut.

This completes the proof of Claim 3. □

By Claim 3, we assume that G_{11} has a peripheral e''-cut. Then G_{11} is isomorphic to the graph in Fig. 1(a) or (b), where $E_{G_{11}}(v_1)$ is a peripheral 3-edge-cut in G_{11} and $e'' \in \{u_1u_2, v_1u_1, v_1u_2\}$.

Let M_2 be an edge subset of all triangles containing e'' in G_{11}. By Claim 3(ii), each related short cycle of each 3-edge-cut must contain e''. And so the subgraph $W'' = G[M_2 \cup vv_1]$ is isomorphic to the graph in Fig. 1(b). With $z \mapsto z''$ being a graph isomorphism from W in Fig. 1(b) to W'', we may assume that

$$E(W'') = M_2 \cup \{v''v''_1\} = \{u''_1u''_2, u''_1v''_i, u''_2v''_i : 1 \le i \le k\} \cup \{v''v''_1\} \quad \text{and} \quad e'' \in \{u''_1u''_2, v''_1u''_1, v''_1u''_2\}.$$

Let $L = G_{11}/M_2 = G/(M_1 \cup M_2)$. Then by Claim 3(ii) and as $W'' = G_{11}[M_2]$ is maximal, we must have $\kappa'(L) \ge 4$ (similar argument as $\kappa'(G_{11})$). Since

$$L - \{v'v'_1, v''v''_1\} = G_{11}/M_2 - \{v'v'_1, v''v''_1\} = ((G_{11} - v'v'_1) - v''v''_1)/M_2,$$

it follows by Theorem 2.4(i) that $\tau(L - \{v'v'_1, v''v''_1\}) \ge 2$.

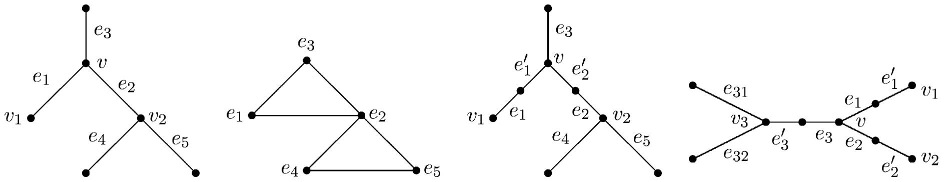
By applying Lemma 3.1(i) to $v''v''_1$ and M_2, $\tau(G_{11} - v'v'_1) \ge 2$. Since $G_{11} - v'v'_1 = (G - v'v'_1)/M_1$, by applying Lemma 3.1(i) again to $v'v'_1$ and M_1, $\tau(G) \ge 2$. Thus by Lemma 3.2, $G(e', e'')$ must be collapsible, contrary to (II). This contradiction precludes Case 2. □

Proof of Theorem 1.3. *Assume that $L(G)$ is not complete. By Lemma 2.8(ii), $\kappa'(G_0) \ge 3$. By Theorems 3.3 and 2.3(iii), $G_0(e', e'')$ has a spanning $(v(e'), v(e''))$-trail for any $e', e'' \in E(G_0)$. Then by Lemma 2.9, $G(e', e'')$ has a dominating $(v(e'), v(e''))$-trail for any $e', e'' \in E(G)$. By Lemma 2.7(ii) and Proposition 2.2, Theorem 1.3 is proved.* □

4. Applications

In this section we show that our main result, Theorem 1.3, implies Corollaries 1.4 and 1.5. For convenience, we restate them as Corollaries 4.1 and 4.2.

Corollary 4.1. *Let G be a graph with $|V(G)| \ge 4$. Suppose that $L(G)$ is hourglass free in which every 3-cut of $L(G)$ is not an independent set. If $\kappa(L(G)) \ge 3$, then $L(G)$ is hamiltonian-connected.*



(a) G_0 (or G). (b) $L(G)$. (c) G. (d) G.

Fig. 2.

Proof. We may assume that $L(G)$ is not a complete graph. Let G_0 denote the core of G. As $L(G)$ is not a complete graph and $\kappa(L(G)) \geq 3$, by Lemma 2.8(ii), G_0 is nontrivial and $\kappa'(G_0) \geq 3$. By Theorem 1.3, it suffices to show that every 3-edge-cut of G_0 has an edge lying in a cycle of length at most 3. Let $X = \{e_0, e_1, e_2\}$ be a 3-edge-cut of G_0. By the definition of G_0, we may assume that $X \subseteq E(G)$ and so X is an edge cut of G.

Case 1. Consider a non-peripheral 3-edge-cut X of G_0. Since every 3-cut of $L(G)$ is not an independent set, two of the corresponding vertices e_0, e_1, e_2 in $L(G)$ are adjacent. We may assume that e_1, e_2 are adjacent in $L(G)$ and so are in G. By the definition of G_0, e_1, e_2 are adjacent in G_0 (see Fig. 2). Since $\kappa'(G_0) \geq 3$, there is some edge e_3 incident with v and there are some edges e_4, e_5 incident with v_2 in G_0 (see Fig. 2(a)). By the definition of G_0, we may assume that $e_3, e_4, e_5 \in E(G)$ and e_3 is incident with v and e_4, e_5 are incident with v_2 in G.

Case 1.1. At least one of $\{e_1, e_2\}$ is not subdivided in G. Without loss of generality we assume that e_2 is not subdivided in G (see Fig. 2(a)). Since $L(G)$ is hourglass free and without loss of generality, we may assume that e_4 is adjacent to e_1 in $L(G)$. Thus e_4 is either incident with v or v_1 in G. In any case, e_2 is in a cycle of length at most 3 in G, so is in G_0.

Case 1.2. e_1, e_2 are subdivided to e_1, e'_1 and e_2, e'_2 respectively in G (see Fig. 2(c)), then $\{e_0, e_1, e_2\}$ is a 3-edge-cut of G and so the corresponding vertex set in $L(G)$ is a 3-cut of $L(G)$ which is not independent. We may assume without loss of generality that e_0 is incident with v_1 and $X'' = \{e_0, e'_1, e_2\}$ is a 3-edge-cut of G and so the corresponding vertex set in $L(G)$ is a 3-cut of $L(G)$ which is not independent. Since X'' is a 3-edge-cut of G, we must have that $v_2 = v_1$, or $e_0 = v_1v_2$ or $e_0 = v_1v$. If $v_1 = v_2$, then e_1 lies in a 2-cycle in G_0; if $e_0 = v_1v_2$ or $e_0 = v_1v$, then e_0 lies in a cycle of length at most 3 in G_0.

Case 2. Consider a peripheral 3-edge-cut X' of G_0. Let $X' = \{e_1, e_2, e_3\}$. Then there exists $v \in V(G_0)$ such that $E_{G_0}(v) = \{e_1, e_2, e_3\}$ and $e_i = vv_i$, $i = 1, 2, 3$. Since $\delta(G_0) \geq 3$ (Lemma 2.8(ii)), we may assume that v_3 is incident with e_{31} and e_{32} in $E(G_0) - \{e_1, e_2, e_3\}$. If at least one of $\{e_1, e_2, e_3\}$ is not subdivided in G, with the same argument as in Case 1.1, we can see that an edge in X' must be lying in a cycle of length at most 3 in G_0. If each of $\{e_1, e_2, e_3\}$ is subdivided in G (see Fig. 2(d)), then $e_{3i} = v_3v$ or v_3v_1, or v_3v_2 for $i = 1$ or $i = 2$. We can check that in each case X' has one edge lying in a cycle of length 2 in G_0. □

Corollary 4.2. *Every 4-connected line graph of an almost claw free graph is hamiltonian-connected.*

Proof. Let G be an almost claw free graph such that $L(G)$ is 4-connected. By Theorem 1.3, it suffices to show that every 3-edge-cut of G_0 must have an edge lying in a cycle of length at most 3. Since $L(G)$ is 4-connected, G has no essential 3-edge-cuts. By the definition of G_0, G_0 has no essential 3-edge-cuts either. Let X be a peripheral 3-edge-cut of G_0. If there are no edges of X in a 2-cycle or 3-cycle of G_0, then $G_0[X]$ must be a claw of G_0. Let $v \in V(G_0)$ be the center of the claw X. By the definition of G_0, $G_0[X]$ gives rise to a claw with center v in G. Since v is of degree 3 in G, the neighborhood of v in G can not be 2-dominated. So there must be at least one edge of X lying in a 2-cycle or a 3-cycle of G_0. By Theorem 1.3, $L(G)$ is hamiltonian connected. □

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