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On mod $(2p+1)$-orientations of graphs

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ABSTRACT

It is shown that every $(2p+1)\log_2(|V(G)|)$-edge-connected graph G has a mod $(2p+1)$-orientation, and that a $(4p+1)$-regular graph G has a mod $(2p+1)$-orientation if and only if $V(G)$ has a partition (V^+, V^-) such that $\forall U \subseteq V(G)$,

$$|\partial_G(U)| \geqslant (2p+1)\big||U \cap V^+| - |U \cap V^-|\big|.$$

These extend former results by Da Silva and Dahab on nowhere zero 3-flows of 5-regular graphs, and by Lai and Zhang on highly connected graphs with nowhere zero 3-flows.

1. Introduction

We consider finite loopless graphs. Undefined notations and terminology will follow [1]. Throughout this paper, $\mathbf{Z}$ denotes the set of all integers, and $\mathbf{Z}^+$ the set of all nonnegative integers. For a positive integer m, $\mathbf{Z}_m$ denotes the set of integers modulo m, as well as the additive cyclic group on m elements.

If D is an orientation of a graph G, and if $S \subseteq V(G)$, then $D^+(S)$ denotes the set of edges with tails in S and $\delta^+(S) = |D^+(S)|$, $D^-(S)$ denotes the set of edges with heads in S and $\delta^-(S) = |D^-(S)|$. When $S = \{v\}$, then $d_D^+(v) = \delta^+(\{v\})$ is the out-degree of v and $d_D^-(v) = \delta^-(\{v\})$ is the in-degree of v. For a function $f: E(G) \mapsto \{1, -1\}$, define $\partial f(v) = \sum_{w \in D^+(\{v\})} f(v,w) - \sum_{w \in D^-(\{v\})} f(w,v)$. Note that when $f \equiv 1$, $\partial f(v) = d_D^+(v) - d_D^-(v)$.

Let $k > 0$ be an integer, and assume that G has a fixed orientation D. A *mod k-orientation* of G is a function $f: E(G) \mapsto \{1, -1\}$ such that $\forall v \in V(G)$, $\partial f(v) \equiv 0 \pmod{k}$. The collection of all graphs admitting a mod k-orientation is denoted by M_k. Note that by definition, $K_1 \in M_k$.

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For a graph G, a function $b : V(G) \mapsto \mathbf{Z}_m$ is a *zero sum* function in $\mathbf{Z}_m$ if $\sum_{v \in V(G)} b(v) \equiv 0 \pmod m$. The set of all zero sum functions in $\mathbf{Z}_m$ of G is denoted by $Z(G, \mathbf{Z}_m)$. When $k = 2p + 1 > 0$ is an odd number, we define M^o_{2p+1} to be the collection of graphs such that $G \in M^o_{2p+1}$ if and only if $\forall b \in Z(G, \mathbf{Z}_{2p+1})$, $\exists f : E(G) \mapsto \{1, -1\}$ such that $\forall v \in V(G)$, $\partial f(v) \equiv b(v) \pmod{2p+1}$. The following proposition can be easily verified.

Proposition 1.1. *$G \in M^o_{2p+1}$ if and only if $\forall b \in Z(G, \mathbf{Z}_{2p+1})$, G has an orientation D with the property that $\forall v \in V(G)$, $d^+_D(v) - d^-_D(v) \equiv b(v) \pmod{2p+1}$.*

Conjecture 1.2. *Let G be a graph, and $p > 0$ be an integer.*

(i) *(Tutte [14].) If G is 4-edge-connected, then $G \in M_3$.*
(ii) *(Jaeger [7].) If G is $(4p)$-edge-connected, then $G \in M_{2p+1}$.*
(iii) *(Jaeger et al. [8].) If G is 5-edge-connected, then $G \in M^o_3$.*

All these conjectures are still open. It is well known that it suffices to prove Conjecture 1.2(ii) for $(4p + 1)$-regular graphs. The following related conjectures have also been proposed.

Conjecture 1.3. *Let G be a graph, and $p > 0$ be an integer.*

(i) *(Jaeger [7].) There exists a smallest integer $k \geqslant 4$ such that if G is k-edge-connected, then $G \in M_3$.*
(ii) *There exists a smallest integer $f(p)$ such that every $f(p)$-edge-connected graph G is in M^o_{2p+1}. (In* [9], *we also conjectured that $f(p) = 4p + 1$.)*

There have been some results done on attacking these conjectures. The following theorem briefly summarizes these progresses.

Theorem 1.4. *Let G be a graph.*

(i) *(Grötzsch [4], Steinberg and Younger [13].) Every 4-edge-connected projective planar graph is in M_3.*
(ii) *(Lai and Li [10].) Every 5-edge-connected planar graph is in M^o_3.*
(iii) *(Lai and Zhang [11].) Every $4\lceil \log_2(|V(G)|) \rceil$-edge-connected graph is in M_3.*

We shall derive a necessary and sufficient condition for a $(4p + 1)$-regular graph to have a mod $(2p + 1)$-orientation, and show that for any integer $p > 0$, every $(2p + 1)\lceil \log_2(|V(G)|) \rceil$-edge-connected graph is in M^o_{2p+1}.

2. Orientation of graphs

Let G be a graph. For a subset $S \subseteq V(G)$, $E(S)$ denotes the set of edges in G with both ends in S and $\partial_G(S)$ denotes the set of edges with just one end in S. When $S = \{v\}$, we denote $E_G(v) = \partial_G(\{v\})$. If $S_1, S_2 \subseteq V(G)$ and $S_1 \cap S_2 = \emptyset$, then $E(S_1, S_2)$ denotes the set of edges in $E(G)$ with one end in S_1 and the other end in S_2.

Let $c : V(G) \mapsto \mathbf{Z}^+$ be a function. For each $v \in V(G)$, define $X_v = \{v^1, v^2, \ldots, v^{c(v)}\}$. We assume that these X_v's are disjoint and so if $v \neq v'$ and $v, v' \in V(G)$, then $X_v \cap X_{v'} = \emptyset$. Construct a new bipartite graph G_c whose vertex bipartition is $(E(G), \bigcup_{v \in V(G)} X_v)$. An edge in G_c joins a vertex $e \in E(G)$ and a vertex $v^i \in X_v$ if and only if v is incident with e in G. Note that any perfect matching M of G_c corresponds to an orientation D of G in such a way that an edge in M joins e and v^i in G_c if and only if e is oriented away from v in D. Thus G_c has a perfect matching if and only if G has an orientation D such that $d^+_D(v) = c(v), \forall v \in V(G)$. By the Marriage Theorem of Frobenius ([3], or Corollary 2.5 on p. 185 in [12]), or by Hall's Theorem [6] on system of distinct representatives, we obtained the following theorem of Hakimi.

Theorem 2.1. *(See Hakimi, [5].) Let G be a graph, and let* $c: V(G) \mapsto \mathbf{Z}$ *be a function. Then G has an orientation D such that* $d_D^+(v) = c(v), \forall v \in V(G)$ *if and only if*

$$\forall S \subseteq V(G), \quad |E(S)| \leqslant \sum_{v \in S} c(v) \leqslant |E(S)| + |\partial_G(S)|. \tag{1}$$

Theorem 2.1 can also be stated in terms of net out-degree at each vertex in an orientation.

Corollary 2.2. *Let G be a graph and* $b: V(G) \mapsto \mathbf{Z}$ *be a function such that* $\sum_{v \in V(G)} b(v) = 0$ *and* $b(v) \equiv d_G(v) \pmod 2$, $\forall v \in V(G)$. *Then G has an orientation D such that* $d_D^+(v) - d_D^-(v) = b(v)$, $\forall v \in V(G)$ *if and only if*

$$\forall S \subseteq V(G), \quad \left|\sum_{v \in S} b(v)\right| \leqslant |\partial_G(S)|. \tag{2}$$

Proof. Let G be a graph with an orientation D. For a subset $S \subseteq V(G)$, let $\partial_D^+(S)$ denote the set of edges oriented from a vertex in S to a vertex not in S, or a boundary edge oriented from a vertex in S; and let $\partial_D^-(S) = \partial_G(S) - \partial_D^+(S)$.

Suppose that G has an orientation D such that $d_D^+(v) - d_D^-(v) = b(v)$, $\forall v \in V(G)$. Let $c(v) = d_D^-(v) + b(v)$, $\forall v \in V(G)$. By Theorem 2.1, G has an orientation D such that $d_D^+(v) = c(v)$ (which is equivalent to $d_D^+(v) - d_D^-(v) = b(v)$) if and only if (1) holds. Furthermore, (1) holds if and only if the following holds:

$$\forall S \subseteq V(G), \quad |E(S)| \leqslant \sum_{v \in S} d_D^-(v) + \sum_{v \in S} b(v) \leqslant |E(S)| + |\partial_G(S)|.$$

Substituting $\sum_{v \in S} d_D^-(v)$ by $|E(S)| + |\partial_D^-(S)|$ in the inequalities above, we conclude that (1) holds if and only if

$$\forall S \subseteq V(G), \quad -|\partial_G(S)| \leqslant -|\partial_D^-(S)| \leqslant \sum_{v \in S} b(v) \leqslant |\partial_G(S)| - |\partial_D^-(S)| \leqslant |\partial_G(S)|,$$

and so (2) must follow.

Conversely, suppose that (2) holds and that there is a function $b: V(G) \mapsto \mathbf{Z}$ such that $\sum_{v \in V(G)} b(v) = 0$ and $b(v) \equiv d_G(v) \pmod 2$, $\forall v \in V(G)$. Define a function $c: V(G) \mapsto \mathbf{Z}$ as follows

$$c(v) = \frac{b(v) + d_G(v)}{2}, \quad \forall v \in V(G).$$

Since $\sum_{v \in V(G)} b(v) = 0$, $\sum_{v \in V(G)} c(v) = |E(G)|$. Then $\forall S \subseteq V(G)$,

$$2\sum_{v \in S} c(v) = \sum_{v \in S} b(v) + \sum_{v \in S} d_G(v) = \sum_{v \in S} b(v) + 2|E(S)| + |\partial_G(S)|.$$

Hence by (2), $|E(S)| \leqslant \sum_{v \in S} c(v) \leqslant |\partial_G(S)| + |E(S)|$. It follows by Theorem 2.1 that G has an orientation D with $d_D^+(v) = c(v) = \frac{b(v)+d_G(v)}{2}$, which is equivalent to $d_D^+(v) - d_D^-(v) = b(v)$, as $d_G(v) = d_D^+(v) + d_D^-(v)$. $\square$

Theorem 2.3. *Let* $p > 0$ *be an integer, and let G be a* $(4p+1)$*-regular graph. The following are equivalent.*

(i) $G \in M_{2p+1}$.
(ii) $V(G)$ *has a partition* (V^+, V^-) *such that* $\forall U \subseteq V(G)$,

$$|\partial_G(U)| \geqslant (2p+1)\big||U \cap V^+| - |U \cap V^-|\big|.$$

Proof. In the proofs of both sufficiency and necessity, we let $b : V(G) \mapsto \mathbf{Z}$ be a map satisfying both $b^{-1}(\{2p+1\}) = V^+$ and $b^{-1}(\{-2p-1\}) = V^-$, when V^+ and V^- are defined in G. Note that as G is a $(4p+1)$-regular graph, $\forall v \in V(G)$, $b(v) \equiv d_G(v) \pmod 2$.

Now we show that (ii) implies (i). Take $U = V(G)$. Then $\partial(U) = \emptyset$, and so we must have $|V^+| = |V^-|$. It follows that $\sum_{v\in V(G)} b(v) = 0$. For any $S \subseteq V(G)$, by (ii) with $U = S$,

$$\left|\sum_{v\in S} b(v)\right| = (2p+1)\big||S\cap V^+| - |S\cap V^-|\big| \leqslant |\partial_G(S)|,$$

and so by Corollary 2.2, $G \in M_{2p+1}$. Hence (i) must hold.

Conversely, suppose (i) holds. Then G has a mod $(2p+1)$-orientation D. Since G is $(4p+1)$-regular, it follows that $\forall v \in V(G)$, either $d^+_D(v) = 3p+1$ or $d^+_D(v) = p$. Define $V^+ = \{v \in V(D){:}\ d^+_D(v) = 3p+1\}$ and $V^- = V(D) - V^+$. By Corollary 2.2, for any $U \subseteq V(G)$,

$$(2p+1)\big||U\cap V^+| - |U\cap V^-|\big| = \left|\sum_{v\in U} b(v)\right| \leqslant |\partial_G(U)|,$$

and so (ii) must hold. □

Theorem 2.4. *(See Da Silva and Dahab, [2].) Let G be a 5-regular graph. Then $G \in M_3$ if and only if $V(G)$ has a partition (V^+, V^-) with $|V^+| = |V^-|$ such that $\forall U \subseteq V(G)$,*

$$|\partial_G(U)| \geqslant 3\big||U\cap V^+| - |U\cap V^-|\big|.$$

Proof. Take $p = 1$ in the theorem above. □

3. mod $(2p+1)$-orientation of graphs

Let $T \subseteq V(G)$ be a vertex subset. The *contraction* G/T is obtained from G by identifying all vertices in T into a single vertex v_T, and then removing all edges in $E(T)$. Note that if G is loopless, then any contraction G/T is also loopless.

Proposition 3.1. *Let G be a graph. The following are equivalent.*

(i) $G \in M^o_{2p+1}$.

(ii) *For any function $b : V(G) \mapsto \mathbf{Z}$ satisfying both*

$$\sum_{v\in V(G)} b(v) \equiv 0 \pmod{2p+1} \tag{3}$$

and

$$b(v) \equiv d_G(v) \pmod 2, \quad \forall v \in V(G), \tag{4}$$

G has an orientation D such that $d^+_D(v) - d^-_D(v) \equiv b(v) \pmod{2p+1}$, $\forall v \in V(G)$.

Proof. Suppose that $G \in M^o_{2p+1}$. Let $b : V(G) \mapsto \mathbf{Z}$ be a function satisfying (3) and (4). For each $v \in V(G)$, choose $b'(v) \in \mathbf{Z}$ with $0 \leqslant b'(v) \leqslant 2p$ and with $b'(v) \equiv b(v) \pmod{2p+1}$. By (3), $b' \in Z(G, \mathbf{Z}_{2p+1})$. Since $G \in M^o_{2p+1}$, by Proposition 1.1, G has an orientation D such that $d^+_D(v) - d^-_D(v) \equiv b(v) \pmod{2p+1}$, $\forall v \in V(G)$.

Suppose (ii) holds. Let $b' \in Z(G, \mathbf{Z}_{2p+1})$. We may assume that $\forall v \in V(G)$, $b'(v) \equiv i_v \pmod{2p+1}$ for some $i_v \in \mathbf{Z}$ with $0 \leqslant i_v \leqslant 2p$. Define $b : V(G) \mapsto \mathbf{Z}$ as follows

$$b(v) = \begin{cases} i_v & \text{if } i_v \equiv d_G(v) \pmod 2, \\ i_v + 2p + 1 & \text{if } i_v + 1 \equiv d_G(v) \pmod 2. \end{cases}$$

Then both (3) and (4) hold. By (ii), G has an orientation D such that such that $d^+_D(v) - d^-_D(v) \equiv b(v) \equiv b'(v) \pmod{2p+1}$, $\forall v \in V(G)$. It follows by Proposition 1.1 that $G \in M^o_{2p+1}$. □

Lemma 3.2. *Let $b : V(G) \mapsto \mathbf{Z}$ be a function satisfying* (3) *and* (4). *Then $\exists b' : V(G) \mapsto \mathbf{Z}$ satisfies* (3), (4),

$$b'(v) \equiv b(v) \pmod{2p+1}, \quad \forall v \in V(G), \tag{5}$$

$$\sum_{v \in V(G)} b'(v) = 0, \tag{6}$$

and

$$\max\{b'(v):\ v \in V(G)\} - \min\{b'(v):\ v \in V(G)\} \leqslant 4p+2. \tag{7}$$

Proof. Since b satisfies (4), $\sum_{v \in V(G)} b(v) \equiv \sum_{v \in V(G)} d_G(v) \equiv 0 \pmod 2$. This, together with (3), implies

$$\sum_{v \in V(G)} b(v) \equiv 0 \pmod{4p+2}. \tag{8}$$

Among all functions $b' : V(G) \mapsto \mathbf{Z}$ satisfying (3)–(5), choose one such that $|\sum_{v \in V(G)} b'(v)|$ is the smallest. We claim that $\sum_{v \in V(G)} b'(v) = 0$. If not, then by (8), $\sum_{v \in V(G)} b'(v)$ must be a multiple of $4p+2$. Without loss of generality, we may assume that $\sum_{v \in V(G)} b'(v) > 0$ and that $b'(v_1) = \max\{b'(v):\ v \in V(G)\}$. Define $b'' : V(G) \mapsto \mathbf{Z}$ as follows

$$b''(v) = \begin{cases} b'(v) & \text{if } v \neq v_1, \\ b'(v_1) - (4p+2) & \text{if } v = v_1. \end{cases}$$

Then b'' satisfies (3)–(5), but $|\sum_{v \in V(G)} b'(v)| = |\sum_{v \in V(G)} b''(v)| + 4p + 2$, contrary to the choice of b'. Therefore, b' must satisfy (6) as well.

Among all functions $b' : V(G) \mapsto \mathbf{Z}$ satisfying (3)–(6), choose one such that

$$\max\{b'(v):\ v \in V(G)\} - \min\{b'(v):\ v \in V(G)\} \text{ is minimized,} \tag{9}$$

and subject to (9)

$$\left|\{z:\ b'(z) = \max\{b'(v):\ v \in V(G)\}\}\right| \text{ is minimized.} \tag{10}$$

Relabelling the vertices if needed, we assume that

$$b'(v_1) \geqslant b'(v_2) \geqslant \cdots \geqslant b'(v_n), \tag{11}$$

where $n = |V(G)|$. If $b'(v_1) - b'(v_n) \leqslant 4p+2$, then we are done. Suppose that $b'(v_1) - b'(v_n) > 4p+2$. Define $b''' : V(G) \mapsto \mathbf{Z}$ as follows

$$b'''(v) = \begin{cases} b'(v) & \text{if } v \notin \{v_1, v_n\}, \\ b'(v_1) - (4p+2) & \text{if } v = v_1, \\ b'(v_n) + (4p+2) & \text{if } v = v_n. \end{cases}$$

Then b''' also satisfies (3)–(6). If $b'''(v_2) > b'''(v_1)$, then $\max\{b'''(v):\ v \in V(G)\} = \max\{b'(v):\ v \in V(G)\}$, and so as $b'''(v_1) < \max\{b'''(v):\ v \in V(G)\}$, this is contrary to the choice of (10). Therefore, we assume that $b'''(v_1) > b'''(v_2)$. Note now that for any i with $2 \leqslant i \leqslant n-1$, $b'''(v_i) - b'''(v_1) > 4p+2$ if and only if $b'(v_i) > b'(v_1)$, and so the occurrence of $b'''(v_i) - b'''(v_1) > 4p+2$ would be contrary to (11). Hence $b'''(v_1) \geqslant b'''(v_i)$. Similarly, $b'''(v_i) \geqslant b'''(v_n)$. Thus $\max\{b'(v):\ v \in V(G)\} - \min\{b'(v):\ v \in V(G)\} = \max\{b'''(v):\ v \in V(G)\} - \min\{b'(v):\ v \in V(G)\} + 8p + 4$, contrary to (9). This proves the lemma. □

Theorem 3.3. *Let G be a graph with $n = |V(G)|$. If $G \notin M^o_{2p+1}$, then each of the following holds.*

(i) *$V(G)$ can be expressed as a disjoint union $V(G) = V_1 \cup V_2$ with $|V_1| = k$, $|V_2| = n-k$, and*

$$\left\lceil \frac{|E(V_1, V_2)| + 1}{k} \right\rceil + \left\lceil \frac{|E(V_1, V_2)| + 1}{n-k} \right\rceil \leqslant 4p+2. \tag{12}$$

(ii) *$V(G)$ can be expressed as a disjoint union $V(G) = V_1 \cup V_2$ with $|V_1| = k$, $|V_2| = n - k$, and*

$$\left|E(V_1, V_2)\right| \leqslant \frac{(4p+2)k(n-k)}{n} - 1. \tag{13}$$

Proof. (i) If $G \notin M^o_{2p+1}$, then by Proposition 3.1, there exist a function $b: V(G) \mapsto \mathbf{Z}$ satisfying (3) and (4) but G does not have an orientation D such that $d^+_D(v) - d^-_D(v) \equiv b(v) \pmod{2p+1}$, $\forall v \in V(G)$. By Lemma 3.2, we may assume that b also satisfies (5)–(7). By Corollary 2.2, there must be a subset $V_1 \subseteq V(G)$ such that

$$\left|\sum_{v \in V_1} b(v)\right| > \left|\partial_G(V_1)\right|. \tag{14}$$

Let $V_2 = V(G) - V_1$. Then $\partial_G(V_1) = \partial_G(V_2) = E(V_1, V_2)$. By (6), $|\sum_{v \in V_1} b(v)| = |\sum_{v \in V_2} b(v)|$. Thus by (14) and (6), we may assume, without loss of generality, that

$$\sum_{v \in V_1} b(v) \geqslant \left|\partial_G(V_1)\right| + 1 \quad \text{and} \quad \sum_{v \in V_2} b(v) \leqslant -\left|\partial_G(V_2)\right| - 1. \tag{15}$$

Let $k = |V_1|$. Then $|V_2| = n - k$. By (15),

$$\left|\partial_G(V_1)\right| + 1 \leqslant \sum_{v \in V_1} b(v) \leqslant k \max\left\{b(v):\ v \in V(G)\right\},$$

and so

$$\max b(v) \geqslant \left\lceil \frac{|\partial_G(V_1)| + 1}{k} \right\rceil. \tag{16}$$

Similarly, we have

$$\min b(v) \leqslant -\left\lceil \frac{|\partial_G(V_2)| + 1}{n-k} \right\rceil. \tag{17}$$

Since b also satisfies (7), combining (16) and (17), we obtain (12). This proves (i).

(ii) Suppose that $G \notin M^o_{2p+1}$. By (i), $V(G)$ has a partition (V_1, V_2) such that (12) holds. By (12),

$$\frac{|E(V_1, V_2)| + 1}{k} + \frac{|E(V_1, V_2)| + 1}{n-k} \leqslant 4p + 2,$$

and so

$$(n-k)\left(\left|E(V_1, V_2)\right| + 1\right) + k\left(\left|E(V_1, V_2)\right| + 1\right) \leqslant (4p+2)k(n-k).$$

Thus (13) follows also. □

Corollary 3.4. *For any positive $p \in \mathbf{Z}$, $K_{4p+1} \in M^o_{2p+1}$.*

Proof. Let $n = 4p + 1$. If $K_n \notin M^o_{2p+1}$, then by Theorem 3.3, $V(K_n)$ can be partitioned into two subsets V_1 and V_2 with $|V_1| = k$ and $|V_2| = n - k$ such that (12) holds. Since $|E(V_1, V_2)| = k(n-k)$, we have

$$\left\lceil \frac{|E(V_1, V_2)| + 1}{k} \right\rceil + \left\lceil \frac{|E(V_1, V_2)| + 1}{n-k} \right\rceil = \left\lceil \frac{k(n-k) + 1}{k} \right\rceil + \left\lceil \frac{k(n-k) + 1}{n-k} \right\rceil$$
$$= (n - k + 1) + (k + 1) = n + 2 > 4p + 2,$$

contrary to (12). Thus we must have $K_{4p+1} \in M^o_{2p+1}$. □

Lemma 3.5. *Let $h(\lambda) = \frac{\log_2(\lambda)}{1-\lambda} + \frac{\log_2(1-\lambda)}{\lambda}$ be a function defined on the interval $(0, 1)$. Then $h(\lambda) \leqslant h(1/2) = -4$.*

Proof. Note that the derivative

$$h'(\lambda) = \frac{1}{\ln(2)\lambda(1-\lambda)} + \frac{\log_2\lambda}{(1-\lambda)^2} - \frac{1}{\ln(2)\lambda(1-\lambda)} - \frac{\log_2(1-\lambda)}{\lambda^2} = \frac{\log_2\lambda}{(1-\lambda)^2} - \frac{\log_2(1-\lambda)}{\lambda^2}.$$

Consider $g(\lambda) = \lambda^2\log_2(\lambda) - (1-\lambda)^2\log_2(1-\lambda)$. Suppose that $\lambda \leqslant \frac{1}{2}$. Set $t = \frac{1}{\lambda}$. Then $t \geqslant 2$, and so $(t-1) - \log_2(t) \geqslant 0$. It follows that

$$\begin{aligned} g(\lambda) &= \frac{1}{t^2}\left(\log_2\left(\frac{1}{t}\right) - (t-1)^2\log_2\left(\frac{t-1}{t}\right)\right) = \frac{1}{t^2}\left((t-1)^2\log_2\left(\frac{t}{t-1}\right) - \log_2(t)\right) \\ &= \frac{1}{t^2}\left((t-1)\log_2\left(\left(1+\frac{1}{t-1}\right)^{t-1}\right) - \log_2(t)\right) \geqslant \frac{1}{t^2}\big((t-1) - \log_2(t)\big) \geqslant 0. \end{aligned}$$

It follows that $h'(\lambda) \geqslant 0$ when $\lambda \leqslant \frac{1}{2}$. Similarly, $h'(\lambda) \leqslant 0$ when $\lambda \geqslant \frac{1}{2}$. This proves that $h(\lambda) \leqslant h(1/2) = -4$. □

Theorem 3.6. *Let n, p be positive integers, and let $f(n) = \frac{(2p+1)n\lceil\log_2(n)\rceil}{2}$ be a function. If G is a graph with n vertices and if $|E(G)| \geqslant f(n)$, then G has a nontrivial subgraph H with $H \in M^o_{2p+1}$.*

Proof. Since $f(1) = 0$ and $f(2) = 2p+1$, the theorem holds when $n \in \{1, 2\}$. We argue by induction to prove the theorem and assume that the theorem holds for all smaller values of n, and that $n \geqslant 3$.

Suppose now $|E(G)| \geqslant f(n)$ but $G \notin M^o_{2p+1}$. By Theorem 3.3, $V(G)$ can be partitioned into a disjoint subsets V_1 and V_2 with $|V_1| = k$ and $|V_2| = n-k$ such that (13) holds.

If one of the induced subgraphs $G[V_1]$ and $G[V_2]$ contains a desirable H, then theorem holds. Therefore we assume that neither $G[V_1]$ nor $G[V_2]$ contains a nontrivial subgraph H in M^o_{2p+1}. By induction hypothesis, we must have

$$\big|E\big(G[V_1]\big)\big| < f(k) \quad \text{and} \quad \big|E\big(G[V_2]\big)\big| < f(n-k). \tag{18}$$

By (13), $|E(V_1, V_2)| < \frac{(4p+2)k(n-k)}{n}$, and so by (18),

$$\begin{aligned} \big|E(G)\big| &= \big|E(V_1)\big| + \big|E(V_2)\big| + \big|E(V_1, V_2)\big| < f(k) + f(n-k) + \frac{(4p+2)k(n-k)}{n} \\ &= \frac{(2p+1)k}{2}\log_2(k) + \frac{(2p+1)(n-k)}{2}\log_2(n-k) + \frac{(4p+2)k(n-k)}{n}. \end{aligned}$$

Let $k = \lambda n$. Then we have, by Lemma 3.5,

$$\begin{aligned} \big|E(G)\big| &< \frac{(2p+1)n\lambda}{2}\log_2(n\lambda) + \frac{(2p+1)n(1-\lambda)}{2}\log_2\big(n(1-\lambda)\big) + (4p+2)n\lambda(1-\lambda) \\ &= (2p+1)n\left(\frac{\lambda\log_2(n)}{2} + \frac{\lambda\log_2(\lambda)}{2} + \frac{(1-\lambda)\log_2(n)}{2} + \frac{(1-\lambda)\log_2(1-\lambda)}{2}\right) \\ &\quad + (4p+2)n\lambda(1-\lambda) \\ &= \frac{(2p+1)n\log_2(n)}{2} + (2p+1)n\frac{\lambda(1-\lambda)}{2}\left(\frac{\log_2(\lambda)}{1-\lambda} + \frac{\log_2(1-\lambda)}{\lambda} + 4\right) \\ &\leqslant \frac{(2p+1)n\log_2(n)}{2} + (2p+1)n\frac{\lambda(1-\lambda)}{2}(-4+4) \\ &= \frac{(2p+1)n\log_2(n)}{2} = f(n), \end{aligned}$$

contrary to the assumption that $|E(G)| \geqslant f(n)$. □

Proposition 3.7. *(See [9].) For any integer $p \geqslant 1$, if H is a subgraph of G, and if $H, G/H \in M^o_{2p+1}$, then $G \in M^o_{2p+1}$.*

The following corollary sharpens Theorem 1.4(iii) when $p = 1$.

Corollary 3.8. *Let G be a graph with n vertices. If* $\kappa'(G) \geqslant (2p+1)\log_2(n)$*, then* $G \in M^o_{2p+1}$.

Proof. By contradiction, we assume that G is a counterexample with $|V(G)|$ minimized. By Theorem 3.6, G has a nontrivial subgraph $H \in M^o_{2p+1}$. Since $\kappa'(G/H) \geqslant \kappa'(G)$, by the minimality of G, $G/H \in M^o_{2p+1}$. It follows by the facts that $H \in M^o_{2p+1}$ and $G/H \in M^o_{2p+1}$, and by Proposition 3.7 that $G \in M^o_{2p+1}$. □

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