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Every line graph of a 4-edge-connected graph is $\mathbf{Z}_3$-connected

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ABSTRACT

We prove that every line graph of a 4-edge-connected graph is $\mathbf{Z}_3$-connected. In particular, every line graph of a 4-edge-connected graph has a nowhere zero 3-flow.

1. Introduction

Graphs considered in this paper are finite graphs with possible loops and multiple edges, and we follow Bondy and Murty [1] for undefined notations and terminology. We use $\mathbf{Z}$ to denote the group of all integers, and for an integer $n > 1$, $\mathbf{Z}_n$ to denote the cyclic group of order n. For a graph G and a vertex $v \in V(G)$, define

$$E_G(v) = \{e \in E(G) : \ e \text{ is incident with } v \text{ in } G\}.$$

Let G be a digraph, A be a nontrivial additive Abelian group with additive identity 0, and $A^* = A - \{0\}$. For an edge $e \in E(G)$ oriented from a vertex u to a vertex v, u is referred as the **tail** of e, while v the **head** of e. For a vertex $v \in V(G)$, the set of all edges incident with v being the tail (or the head, respectively) is denoted by $E^+(v)$ (or $E^-(v)$, respectively). We define

$$F(G, A) = \{f \mid f : E(G) \mapsto A\} \quad \text{and} \quad F^*(G, A) = \{f \mid f : E(G) \mapsto A^*\}.$$

For each $f \in F(G, A)$, the **boundary** of f is a function $\partial f : V(G) \mapsto A$ defined by $\partial f = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e)$, for each vertex $v \in V(G)$, where "$\sum$" refers to the addition in A. We define

$$Z(G, A) = \left\{ b \mid b : V(G) \mapsto A \quad \text{with} \sum_{v \in V(G)} b(v) = 0 \right\}.$$

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An undirected graph G is A-**connected**, if G has an orientation G' such that for every function $b \in Z(G, A)$, there is a function $f \in F^*(G', A)$ such that $\partial f = b$. It is well known (see [4]) that this property does not depend on the choice of the orientation of G. For an Abelian group A, let $\langle A \rangle$ denote the family of graphs that are A-connected.

An A-**nowhere-zero-flow** (abbreviated as an A-NZF) of G is a function $f \in F^*(G, A)$ such that $\partial f = 0$. For an integer $k \geq 2$, a k-**nowhere-zero-flow** (abbreviated as a k-NZF) of G is a function $f \in F^*(G, \mathbf{Z})$ such that $\partial f = 0$ and such that for every $e \in E(G)$, $0 < |f(e)| < k$. Tutte ([8], also see [3] and [9]) showed that a graph G has an A-NZF if and only if G has an $|A|$-NZF.

The concept of A-connectivity was introduced by Jaeger et al. in [4], where A-NZF is successfully generalized to A-connectivity. For a graph G, define

$$\Lambda_g(G) = \min\{k : \text{if } A \text{ is an abelian group of order at least } k, \text{ then } G \in \langle A \rangle\}.$$

From the definitions, if $\Lambda_g(G) \leq k$, then G has a k-NZF. The following conjectures have been proposed.

Conjecture 1.1 (*Tutte [8], and [3]*). *Every 4-edge-connected graph has a 3-NZF.*

Conjecture 1.2 (*Jaeger et al. [3]*). *If G is 5-edge-connected graph, then $\Lambda_g(G) \leq 3$.*

Both conjectures are still open. M. Kochol [5] showed that to prove Conjecture 1.1, it suffices to show that every 5-edge-connected graph has a 3-NZF. Thus Conjecture 1.2 implies Conjecture 1.1.

We shall follow [2] to define a line graph. The **line graph** of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where for an integer $k \in \{1, 2\}$, two vertices in $L(G)$ are joined by k edges in $L(G)$ if and only if the corresponding edges in G are sharing k common vertices in G. In other words, if e_1 and e_2 are adjacent but not parallel in G, then e_1 and e_2 are joined by one edge in $L(G)$; if e_1 and e_2 are parallel edges in G, then e_1 and e_2 are joined by two (parallel) edges in $L(G)$. Note that our definition for line is slightly different from the one defined in [1] (called an edge graph there), and when G is a simple graph, both definitions are consistent with each other. The main reason for us to adopt this definition in [2] instead of the traditional definition of a line graph is that when G is a multigraph, Corollary 1.5 will fail to hold if we use the traditional definition of line graphs. For example, let G denote the loopless connected graph with two vertices and with 4 edges. With the traditional definition of a line graph, the line graph of G will be a K_4, which is not $\mathbf{Z}_3$-connected. With the current definition, $L(G)$ is obtained from a K_4 by replacing each edge by a pair of parallel edges, which is clearly $\mathbf{Z}_3$-connected.

In [2], the following is proved.

Theorem 1.3 (*Chen et al., [2]*). *If every 4-edge-connected line graph has a 3-NZF, then every 4-edge-connected graph has a 3-NZF.*

The main purpose of this paper is to investigate when a line graph is $\mathbf{Z}_3$-connected or has a 3-NZF. By the definition of a line graph, for a vertex $v \in V(G)$, the edges incident with v in G induce a complete subgraph H_v in $L(G)$, and when $u, v \in V(G)$ with $u \neq v$, if G is simple, H_v and H_u are edge disjoint complete subgraphs of $L(G)$. Such an observation motivates the following definition.

For a connected graph G, a partition $(E_1, E_2, \ldots, E_k)$ of $E(G)$ is a **clique partition** of G if $G[E_i]$ is spanned by a complete graph for each $i \in \{1, 2, \ldots, k\}$. Furthermore, $(E_1, E_2, \ldots, E_k)$ is a (≥ 4)-**clique partition** of G, if for each $i \in \{1, 2, \ldots, k\}$, $G[E_i]$ is spanned by a K_{n_i} with $n_i \geq 4$; and a K_m-**partition** if for each $i \in \{1, 2, \ldots, k\}$, $G[E_i]$ is spanned by a K_m. Note that if G is simple, and if $(E_1, E_2, \ldots, E_k)$ of $E(G)$ is a clique partition of G, then $|V(G[E_i]) \cap V(G[E_j])| \leq 1$ where $i \neq j$ and $i, j \in \{1, 2, \ldots, k\}$.

Our main result is the following.

Theorem 1.4. *If G is 4-edge-connected and G has a (≥ 4)-clique partition, then $\Lambda_g(G) \leq 3$.*

The corollary below follows from Theorem 1.4 and from the observations made above.

Corollary 1.5. *Each of the following holds.*

(i) *If $\kappa'(G) \geq 4$, then $\Lambda_g(L(G)) \leq 3$.*

(ii) *Every line graph of a 4-edge-connected graph has a 3-NZF.*

We display the prerequisites in Section 2 and present the proof of the main result in Section 3.

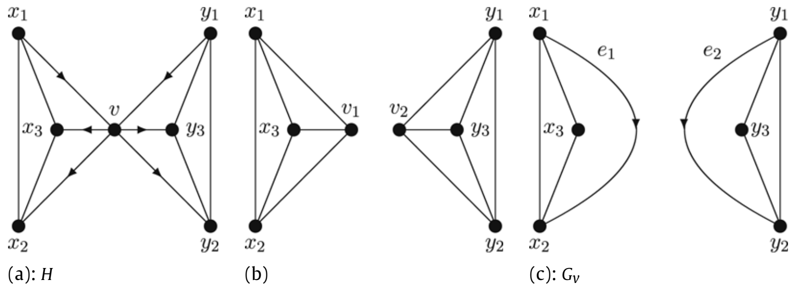

Fig. 1.

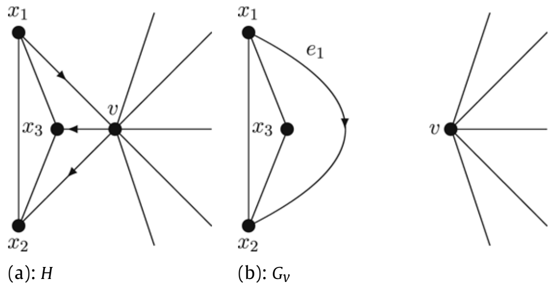

Fig. 2.

2. Prerequisites

Throughout this section, we use the notation that $\mathbf{Z}_3 = \{0, 1, 2\}$ with the mod 3 addition. Let G be a graph and let $X \subseteq E(G)$ be an edge subset. The **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. For convenience, we use G/e for $G/\{e\}$ and $G/\emptyset = G$; and if H is a subgraph of G, we write G/H for $G/E(H)$.

Theorem 2.1. *Let G be a graph and A be an Abelian group. Each of the following holds.*

(i) *(Proposition 3.2 of [6]) Let H be a subgraph of G and $H \in \langle \mathbf{Z}_3 \rangle$, then $G/H \in \langle \mathbf{Z}_3 \rangle$ if and only if $G \in \langle \mathbf{Z}_3 \rangle$.*

(ii) *([4] and Lemma 3.3 of [6] For an integer $n \geq 1$ and an Abelian group A, the n-cycle $C_n \in \langle A \rangle$ if and only if $|A| \geq n + 1$. (Thus $\Lambda_g(C_n) = n + 1$.)*

(iii) *(Corollary 3.5 of [6]) For $n \geq 5$, $\Lambda_g(K_n) = 3$.*

(iv) *(Lemma 2.1 of [7]) If for every edge e in a spanning tree of G, G has a subgraph $H_e \in \langle A \rangle$ with $e \in E(H_e)$, then $G \in \langle A \rangle$.*

Lemma 2.2. *Let G be a graph, and let G' denote the graph obtained from G by contracting the 2-cycles of G (if there are any) and then contracting all loops of the resulting graph (if there are any). If $G' \in \langle \mathbf{Z}_3 \rangle$, then $G \in \langle \mathbf{Z}_3 \rangle$.*

Proof. This follows from Theorem 2.1(ii) and (i). □

Lemma 2.3. *Let G be a graph and $H \cong K_4$ a subgraph of G and $v \in V(H)$ (see Figs. 1(a) and 2(a)). If $d_G(v) = 6$ and G has another subgraph $H' \cong K_4$ with $V(H) \cap V(H') = \{v\}$, then let G_v be the graph obtained from G by splitting the vertex $v \in V(G)$ into v_1, v_2 (see Fig. 1 (b)), and by first deleting x_3v_1, y_3v_2 and then contracting v_1x_1, v_2y_1 (see Fig. 1 (c)); if $d_G(v) > 6$, let G_v be the graph obtained from G by splitting the vertex $v \in V(G)$ into v_1, v_2 (still using v to denote it in Fig. 2), deleting the edge v_1x_3, and then contracting v_1x_1 (see Fig. 2(c)). If $G_v \in \langle \mathbf{Z}_3 \rangle$, then $G \in \langle \mathbf{Z}_3 \rangle$.*

Proof. If $d_G(v) = 6$, using the notation in Fig. 1, we may assume that G_v is oriented and the edge e_1 is oriented from x_1 to x_2, e_2 is from y_1 to y_2 in G_v. Then restore G from G_v by preserving the orientation of G_v and by orienting the edges incident with v as follows: from v to x_2 and x_3, from v to y_2 and y_3, and from x_1, y_1 to v.

Let $b \in Z(G, \mathbf{Z}_3)$. We consider three cases below.

Case 1. $b(v) = 0$.

Let

$$b'(z) = \begin{cases} b(z) & \text{if } z \in V(G_v) - \{x_3, y_3\} \\ b(z) + 1 & \text{if } z = x_3 \\ b(z) + 2 & \text{if } z = y_3. \end{cases}$$

Then $b' \in Z(G_v, \mathbf{Z}_3)$. Since $G_v \in \langle \mathbf{Z}_3 \rangle$, there exists $f_1 \in F^*(G_v, \mathbf{Z}_3)$ such that $\partial f_1 = b'$ under the given orientation of G_v. Let $f \in F^*(G, \mathbf{Z}_3)$ be given by

$$f(e) = \begin{cases} f_1(e) & \text{if } e \in E(G) - \{x_1v, vx_3, vx_2, y_1v, vy_3, vy_2\} \\ f_1(e_1) & \text{if } e \in \{x_1v, vx_2\} \\ f_1(e_2) & \text{if } e \in \{y_1v, vy_2\} \\ 1 & \text{if } e = vx_3 \\ 2 & \text{if } e = vy_3. \end{cases}$$

Then, for each $z \in V(G)$,

$$\partial f(z) = \begin{cases} \partial f_1(z) = b'(z) = b(z) & \text{if } z \in V(G) - \{x_3, v, y_3\} \\ \partial f_1(x_3) - f(vx_3) = b(x_3) + 1 - 1 = b(x_3) & \text{if } z = x_3 \\ \partial f_1(y_3) - f(vy_3) = b(y_3) + 2 - 2 = b(y_3) & \text{if } z = y_3 \\ 1 + 2 = 0 = b(v) & \text{if } z = v. \end{cases}$$

It follows that $\partial f = b$.

Case 2. $b(v) = \alpha \in \{1, 2\} \subset Z_3$.

Let

$$b'(z) = \begin{cases} b(z) & \text{if } z \in V(G_v) - \{x_3, y_3\} \\ b(z) + 2\alpha & \text{if } z = x_3 \\ b(z) + 2\alpha & \text{if } z = y_3. \end{cases}$$

Then $b' \in Z(G_v, \mathbf{Z}_3)$. Since $G_v \in \langle \mathbf{Z}_3 \rangle$, there exists $f_1 \in F^*(G_v, \mathbf{Z}_3)$ such that $\partial f_1 = b'$ under the given orientation of G_v. Let $f \in F^*(G, \mathbf{Z}_3)$ be given by

$$f(e) = \begin{cases} f_1(e) & \text{if } e \in E(G) - \{x_1v, vx_3, vx_2, y_1v, vy_3, vy_2\} \\ f_1(e_1) & \text{if } e \in \{x_1v, vx_2\} \\ f_1(e_2) & \text{if } e \in \{y_1v, vy_2\} \\ 2\alpha & \text{if } e = vx_3 \\ 2\alpha & \text{if } e = vy_3. \end{cases}$$

Then, for each $z \in V(G)$,

$$\partial f(z) = \begin{cases} \partial f_1(z) = b'(z) = b(z) & \text{if } z \in V(G) - \{x_3, v, y_3\} \\ \partial f_1(x_3) - f(vx_3) = b(x_3) + 2\alpha - 2\alpha = b(x_3) & \text{if } z = x_3 \\ \partial f_1(y_3) - f(vy_3) = b(y_3) + 2\alpha - 2\alpha = b(y_3) & \text{if } z = y_3 \\ 2\alpha + 2\alpha = \alpha = b(v) & \text{if } z = v. \end{cases}$$

It follows that $\partial f = b$.

If $d_G(v) > 6$, using the notation in Fig. 2, we may assume that G_v is oriented and the edge e_1 is oriented from x_1 to x_2. Then restore G from G_v by preserving the orientation of G_v and by orienting the edges incident with v as follows: from v to x_2 and x_3, and from x_1 to v.

Let $b \in Z(G, \mathbf{Z}_3)$ and

$$b'(z) = \begin{cases} b(z) & \text{if } z \in V(G_v) - \{x_3, v\} \\ b(z) + 1 & \text{if } z = x_3 \\ b(z) - 1 & \text{if } z = v. \end{cases}$$

Then $b' \in Z(G_v, \mathbf{Z}_3)$. Since $G_v \in \langle \mathbf{Z}_3 \rangle$, there exists $f_1 \in F^*(G_v, \mathbf{Z}_3)$ such that $\partial f_1 = b'$ under the given orientation of G_v. Let $f \in F^*(G, \mathbf{Z}_3)$ be given by

$$f(e) = \begin{cases} f_1(e) & \text{if } e \in E(G) - \{x_1v, vx_3, vx_2\} \\ f_1(e_1) & \text{if } e \in \{x_1v, vx_2\} \\ 1 & \text{if } e = vx_3. \end{cases}$$

Then, for each $z \in V(G)$,

$$\partial f(z) = \begin{cases} \partial f_1(z) = b'(z) = b(z) & \text{if } z \in V(G) - \{x_3, v\} \\ \partial f_1(x_3) - f(vx_3) = b(x_3) + 1 - 1 = b(x_3) & \text{if } z = x_3 \\ \partial f_1(v) + f(vx_3) = b(v) - 1 + 1 = b(v) & \text{if } z = v. \end{cases}$$

It follows that $\partial f = b$. □

Lemma 2.4. *Let G be a simple graph and $H \cong K_4$ a subgraph of G and $v \in V(H)$ (see Figs. 1(a)and 2(a)). Suppose that $(E_1, E_2, \ldots, E_k)$ $(k \geq 2)$ is a K_4-partition of G. Define G_v as in Lemma 2.3, and obtain a graph G' by contracting repeatedly cycles of length ≤ 2 in G_v until no such cycles exist. Then G' has a K_4-clique partition. Moreover, if $\kappa'(G') \leq 3$, then we must have $\kappa'(G') = 3$ and for any 3-edge-cut X of G', there exists $u \in V(G')$ such that $X \subseteq E_{G'}(u)$.*

Proof. Since G is simple, when $i \neq j$,

$$|V(G[E_i]) \cap V(G[E_j])| \leq 1.$$

By the definition of G_v and G', if $d_G(v) = 6$, G' can be obtained by first splitting v into v_1 and v_2 and then contracting both K_4 cliques of the resulting graph containing v_1 or v_2; if $d_G(v) > 6$, G' can be obtained by first splitting v into v_1 and v_2 and then contracting the K_4 clique of the resulting graph containing v_1. Therefore, in either case, G' has a K_4-clique partition.

Suppose that $\kappa'(G') \leq 3$. Let X be an edge cut of G' with $|X| \leq 3$. Since every edge of G' must be in one of the K_4 cliques, X must contain an edge cut of a K_4, and so $|X| = 3$, and there exists $u \in V(G')$ such that $X \subseteq E_{G'}(u)$. □

Lemma 2.5. *Let G be a loopless graph spanned by a complete graph $K_n (n \geq 4)$ and R a nonempty subset of $E(G)$. Then $G/R \in \langle \mathbf{Z}_3 \rangle$.*

Proof. Since G is loopless and R is not empty, G/R must have a 2-cycle or is a trivial graph. If $n = 4$, we contract this 2-cycle in G/R. Then the resulting graph has at most 2 vertices and so is $\mathbf{Z}_3$-connected. If $n > 4$, we can argue by Theorem 2.1(i) and by induction on n and contract the 2-cycle in G/R to reduce the order of G so that induction hypothesis can be applied. □

3. Proof of Theorem 1.4

Proof of Theorem 1.4. Note that by Theorem 2.1(ii) and (iii), Theorem 1.4 holds if $|V(G)| \leq 5$, and so we assume that $|V(G)| \geq 6$. By Theorem 2.1(ii) and (iv), for each i with $1 \leq i \leq k$, $\Lambda_g(G[E_i]) \leq 4$. Again by Theorem 2.1(iv), $\Lambda_g(G) \leq 4$. It suffices to show that $\Lambda_g(G) \neq 4$.

We argue by contradiction. Suppose that there exists a graph G with $\kappa'(G) \geq 4$ and with a (≥ 4)-clique partition $(E_1, E_2, \ldots, E_k)$, such that $\Lambda_g(G) = 4$. Therefore we may choose such a graph that

$$G \text{ is not } \mathbf{Z}_3\text{-connected.} \tag{1}$$

and that

$$|V(G)| + |E(G)| \text{ is minimized.} \tag{2}$$

Claim 1. *G does not have a nontrivial subgraph H such that $H \in \langle \mathbf{Z}_3 \rangle$.*

Proof of Claim 1. Suppose that G has a nontrivial maximal subgraph $H \in \langle \mathbf{Z}_3 \rangle$. Then there must exist some E_i such that $E_i \cap E(H) \neq \emptyset$. Let $D = G[E(H) \cup E_i]$. Then $D/H \cong G[E_i]/(E_i \cap E(H))$. Since $E_i \cap E(H) \neq \emptyset$ and since $G[E_i]$ is spanned by a complete graph, by Lemma 2.5, $D/H \in \langle \mathbf{Z}_3 \rangle$. Since $H \in \langle \mathbf{Z}_3 \rangle$, by Theorem 2.1(i), $D \in \langle \mathbf{Z}_3 \rangle$. But since H is a subgraph of D and since H is maximal, we must have $H = D$, and so $E_i \subseteq E(D) = E(H)$. Hence we may assume that there exists a smallest integer m with $0 \leq m < k$, such that $E_i \subseteq E(H)$ for each $i \geq m+1$ and $E_i \cap E(H) = \emptyset$ for each $i < m+1$. Therefore, $(E_1, E_2, \cdots, E_m)$ is a ($\geq$ 4)-clique partition of G/H, and $\kappa'(G/H) \geq 4$. By (2) and since H is nontrivial, $G/H \in \langle \mathbf{Z}_3 \rangle$. By Theorem 2.1(i) and since $H \in \langle \mathbf{Z}_3 \rangle$, we conclude that $G \in \langle \mathbf{Z}_3 \rangle$, contrary to (1). □

Claim 2. *G is simple, and for each $i \in \{1, 2, \ldots, k\}$, $G[E_i] \cong K_4$.*

Proof of Claim 2. By Theorem 2.1(ii) and (iii), loops, 2-cycles and K_m with $m \geq 5$ are in $\langle \mathbf{Z}_3 \rangle$. Therefore Claim 2 below follows immediately from Claim 1. □

Claim 3. *$\delta(G) \geq 4$ and so $k \geq 4$ where k is the number of cliques to which $E(G)$ is decomposed.*

Proof of Claim 3. By Claim 2, G is simple and so any two distinct K_4 cliques of G can have at most one vertex in common. By the assumption that $\kappa'(G) \geq 4$, we establish Claim 3. □

Claim 4. *$\kappa(G) \geq 2$.*

Proof of Claim 4. If G has a cut vertex, then by (2), each block of G is in $\langle \mathbf{Z}_3 \rangle$ and so by Theorem 2.1(i), $G \in \langle \mathbf{Z}_3 \rangle$, contrary to (1). □

Claim 5. *For any $v \in V(G)$, G has a vertex 2-cut (a vertex cut with 2 vertices) containing v.*

Proof of Claim 5. By Claims 2 and 3, $(E_1, E_2, \ldots, E_k)$, $(k \geq 4)$ is a K_4-partition of G. Pick $v \in V(G)$ such that

$$v \in V(G[E_{l_1}]) \cap V(G[E_{l_2}]) \cap \cdots \cap V(G[E_{l_m}]), \ (m \geq 2).$$

Split v and perform the operation as in Lemma 2.3 to get graph G_v, and contract 2-cycles and loops in G_v. Denote the resulting graph by G'. Then G' also has a K_4-partition by Lemma 2.4.

By Claim 4, G' is connected. If $\kappa'(G') \geq 4$, then by (2), $G' \in \langle \mathbf{Z}_3 \rangle$. By Lemma 2.1(i), and by Lemma 2.3, $G \in \langle \mathbf{Z}_3 \rangle$, contrary to (1).

Thus $\kappa'(G') \leq 3$, and so $\kappa'(G') = 3$. By Lemma 2.4, if X is a 3-edge-cut of G', then there exists $u \in V(G')$ such that $X \subseteq E_{G'}(u)$. Since X is a 3-edge-cut of G', it follows that u is a cut vertex of G' and $u \neq v$, and so $\{u, v\}$ is a vertex 2-cut of G. □

Let $W = \{w_1, w_2\}$ be a vertex cut of G and $W'_1, W'_2, \ldots,$ are components of $G - W$. Define $G_i = G[V(W'_i) \cup W]$ to be the subgraph induced by $V(W'_i) \cup W$ and we call each G_i a W-**component** of G. For each vertex 2-cut W of G, let $S(W)$ denote a specified W-component such that $|V(S(W))|$ is minimized, among all W-components of G.

Choose a subgraph $H \in \{S(W) : W \text{ is a 2-cut of } G\}$ such that $|V(H)|$ is the smallest among them. Then for some vertex 2-cut $W = \{w, w'\}$ of G, $H = S(W)$.

Since H is a W-component, we have $V(H) - W \neq \emptyset$ and so we can pick a vertex $v \in V(H) - W$. By Claim 5, G has a vertex 2-cut $W' = \{v, v'\}$ where $v' \in V(G')$.

Case 1. $v' \in V(H)$.

If $v' = w$ (or $v' = w'$, respectively), then $W'' = \{v, w\}$ (or $\{v, w'\}$, respectively) is a vertex 2-cut of G and $|S(W'')| < |V(H)|$, contrary to the choice of H. If $v' \in V(H) - \{w, w'\}$ and $\{v, v'\}$ separates w and w' in H, then $\{v, v'\}$ is not a 2-cut of G. Therefore, W' does not separate w, w' in H, and so a W'-component of G which does not contain w and w' would be a proper subgraph of H, contrary to the choice of H, (see Fig. 3).

Case 2. $v' \notin V(H)$.

By Claim 4, v must be a cut vertex of H separating w and w' in H, and so $W'' = \{v, w\}$ is also a vertex 2-cut of G, and a W''-component that does not contain w' is a violation to the choice of H, (see Fig. 4).

Thus neither of the cases is possible. The contradictions establish Theorem 1.4. □

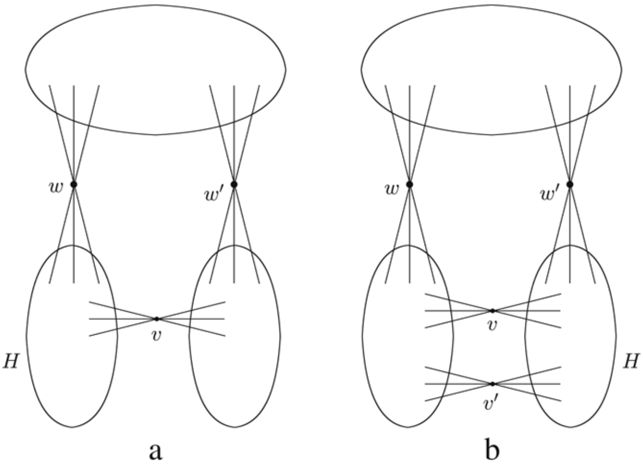

Fig. 3.

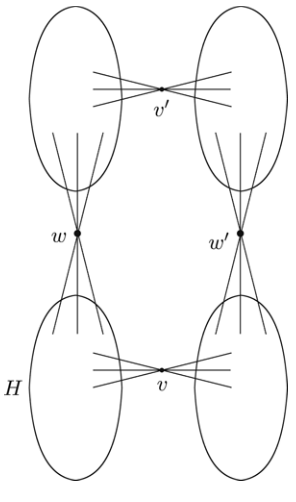

Fig. 4.

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