



## Transforming a graph into a 1-balanced graph

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### ABSTRACT

Let  $G$  be a non-trivial, loopless graph and for each non-trivial subgraph  $H$  of  $G$ , let  $g(H) = \frac{|E(H)|}{|V(H)| - \omega(H)}$ . The graph  $G$  is 1-balanced if  $\gamma(G)$ , the maximum among  $g(H)$ , taken over all non-trivial subgraphs  $H$  of  $G$ , is attained when  $H = G$ . This quantity  $\gamma(G)$  is called the *fractional arboricity* of the graph  $G$ . The value  $\gamma(G)$  appears in a paper by Picard and Queyranne and has been studied extensively by Catlin, Grossman, Hobbs and Lai. The quantity  $\gamma(G) - g(G)$  measures how much a given graph  $G$  differs from being 1-balanced. In this paper, we describe a systematic method of modifying a given graph to obtain a 1-balanced graph on the same number of vertices and edges. We obtain this by a sequence of iterations; each iteration re-defining one end-vertex of an edge in the given graph. After each iteration, either the value  $\gamma$  of the new graph formed is less than that of the graph from the previous iteration or the size of the maximal  $\gamma$ -achieving subgraph of the new graph is smaller than that of the graph in the previous iteration. We show that our algorithm is polynomial in time complexity. Further ways to decrease the number of iterations are also discussed.

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### 1. Background

In this paper, graphs are allowed to have multiple edges but not loops. We use the terminology of [8] for graphs. Throughout the paper, we assume that the graph  $G$  is a non-trivial graph.

The *average degree* of a graph  $G$  is defined as the sum of the degrees of the vertices divided by the total number of vertices in  $G$ . Since the sum of the degrees of the vertices in  $G$  is equal to  $2|E(G)|$ , the average degree of  $G$  is  $\frac{2|E(G)|}{|V(G)|}$ .

Variations on the average degree of a graph have produced a surprising variety of useful concepts. For example, the ratio  $\frac{|E(G)|}{|V(G)|}$  is widely used in the study of random graphs [9,19,28,2,30,23,24,1,13], while the ratio  $\frac{|E(G)|}{|V(G)|-1}$  has important uses in graph survivability [11,7,16]. Moreover,  $\frac{|E(G)|}{3|V(G)|-6}$  is important in studies of rigid frameworks [22,10]. If we let  $f(G)$  stand for any of these ratios, those graphs  $G$  for which  $f(H) \leq f(G)$  for every subgraph  $H$  of  $G$  play a major role. But the terminology used for graphs satisfying this subgraph property for these ratios and others like them is not at all uniform. For example, those satisfying the condition on subgraphs with the ratio  $\frac{|E(G)|}{|V(G)|}$  are called “balanced,” while those satisfying the condition on subgraphs with the second ratio are variously called “uniformly dense” in [5], “strongly balanced” in [29], “molecular” in [26], etc. In a recent paper, Kannan [18] proposed a uniform terminology based on the term “balanced.” Thus, for example, she describes most balanced graphs as 0-balanced, and uniformly dense graphs are called “1-balanced.” We adopt Kannan’s terminology here.

The concept of 1-balanced graphs has been extended to matroids in a natural way in [26,20,5] among others, and a deeper analysis of the concept is provided by the study of “principal partitions in graphs” that was originated by Kishi and

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Kajitani [21]. The cycle matroid of a graph is 1-balanced if and only if the graph is 1-balanced. In the following paragraphs, we recall the definition of 1-balanced graphs that can be derived from the matroidal definition.

The number of components of  $G$  is denoted by  $\omega(G)$ . Let

$$g(H) = \frac{|E(H)|}{|V(H)| - \omega(H)}$$

for all non-trivial subgraphs  $H$  of  $G$ . Let

$$\gamma(G) = \max_{H \subseteq G} g(H),$$

where the maximum is taken over all non-trivial subgraphs  $H$  of  $G$ . Also, let

$$\eta(G) = \min_{X \subseteq E} \frac{|X|}{\omega(G - X) - \omega(G)},$$

where the minimum is taken over all edge sets  $X$  such that  $\omega(G - X) > \omega(G)$ . If  $X = E(G)$ , then  $\frac{|X|}{\omega(G - X) - \omega(G)} = g(G)$ . Thus,

$$\eta(G) \leq g(G) \leq \gamma(G).$$

A graph  $G$  is 1-balanced [5] if  $\gamma(G) = g(G)$ . A subgraph  $H$  of graph  $G$  is  $\gamma$ -achieving if  $g(H) = \gamma(G)$ . If  $G$  is connected, then  $g(G) = \frac{|E(G)|}{|V(G)| - 1}$ . It can be noticed that any graph contains a connected  $\gamma$ -achieving subgraph (Refer [16] for a proof for this fact). Thus, it can be seen that a connected graph  $G$  is 1-balanced if and only if  $\frac{|E(H)|}{|V(H)| - 1} \leq \frac{|E(G)|}{|V(G)| - 1}$  for all non-trivial subgraphs  $H$  of  $G$ . The terminology “1-balanced” is due to the presence of “1” in the above fact.

The quantity  $\gamma(G)$  appeared in [27,5], while  $\eta(G)$  was introduced by Gusfield [11] in the reciprocal form. Later,  $\eta(G)$  was extended to matroids by Cunningham [7]. Both  $\gamma(G)$  and  $\eta(G)$  were studied extensively in [5].

From now on, we assume that the graph  $G$  is connected. The quantity  $\gamma(G)$  is called the *fractional arboricity* of  $G$  due to the covering result derived in [5]:

**Theorem 1** ([5], Theorem 4). For any graph  $G$  and any natural numbers  $s$  and  $t$ ,  $\gamma(G) \leq \frac{s}{t}$  if and only if there is a family  $\mathcal{T}$  of  $s$  spanning trees in  $G$  such that each edge of  $G$  lies in at least  $t$  trees of  $\mathcal{T}$ .

Thus if  $\gamma(G) = \frac{s}{t}$ , by deleting some edges from the spanning trees in  $\mathcal{T}$ , we have a collection of  $s$  forests in  $G$  such that each edge of  $G$  is in exactly  $t$  of the forests. This fact is used later in the paper. A subgraph  $H$  of  $G$  is called a  $\gamma$ -achieving subgraph if  $g(H) = \gamma(G)$ . For any graph  $G$ , it is shown in [5] that there is only one maximal  $\gamma$ -achieving subgraph.

The quantity  $\eta(G)$  is called the *strength* or *edge-toughness* of the graph  $G$  since it measures the vulnerability of the graph  $G$  in terms of edge-deletions. A packing result involving  $\eta(G)$  is due to Catlin et al. [5].

**Theorem 2** ([5], Theorem 4). For any graph  $G$  and any natural numbers  $s$  and  $t$ ,  $\eta(G) \geq \frac{s}{t}$  if and only if there is a family  $\mathcal{T}$  of  $s$  spanning trees in  $G$  such that each edge of  $G$  lies in at most  $t$  trees of  $\mathcal{T}$ .

The following theorem appeared in [5] generalized to matroids.

**Theorem 3** ([5], Theorem 6). For any connected graph  $G$  of order  $n > 1$ , the following are equivalent:

- (1)  $|E(G)| = \gamma(G)(n - 1)$  i. e.,  $G$  is 1-balanced;
- (2)  $|E(G)| = \eta(G)(n - 1)$ ;
- (3)  $\eta(G) = \gamma(G)$ .

Ruciński and Vince [29] (and later Catlin et al. [4] independently) proved that for any given positive integers  $m, n$  with  $n - 1 \leq m \leq n(n - 1)/2$ , there is a simple, connected, 1-balanced graph on  $n$  vertices and  $m$  edges. By gluing together two graphs of suitable uniform densities at a vertex, Catlin et al. [3] observed that for any rational numbers  $x$  and  $y$  with  $1 \leq x \leq y$ , there is a graph  $G$  with  $\eta(G) = x$  and  $\gamma(G) = y$ . Hence there is a large collection of graphs which are not 1-balanced, and the quantity  $\gamma(G) - \eta(G)$  can be arbitrarily large. In view of Theorem 3,  $\gamma(G) - \eta(G) > 0$  if and only if  $\gamma(G) - g(G) > 0$ .

In the literature, there are algorithms to check if a given graph  $G$  is 1-balanced or not; see for example [27,7,15,12,6]. Let  $|V(G)| = n$  and  $|E(G)| = m$ . The algorithm in [15] finds both  $\gamma(G)$  and  $\eta(G)$  in  $O(m^3 n^4)$  computations. In the third section, we use the algorithm in [15] to find the maximal  $\gamma$ -achieving subgraph of  $G$ .

A 1-balanced graph is regarded as a minimally vulnerable network since a knowledgeable enemy (ignoring edge-connectivity) would find no edge set attractive to attack; see [7,16]. In fact, they are addressed as *bland graphs* in [16]. Hence constructing 1-balanced graphs would prove to be useful in many real-world situations.

Typically in real-world situations, the network owners do not want to dismantle the existing network completely and construct a new network that is 1-balanced. Rather, they are willing to budget modest amounts each year to gradually transform the network into one that is closer in some sense to being 1-balanced. In this paper, we find a first solution for this problem. We provide an algorithm for transforming any given connected graph into a 1-balanced graph. Specifically, we

measure closeness of a graph  $G$  to a 1-balanced graph by the difference  $\gamma(G) - g(G)$  and monotonically reduce this difference to 0. The main part of the algorithm is in the fourth section, where we show that if a graph  $G$  is not 1-balanced then we can construct a new graph  $G'$  by re-defining the adjacency of an edge from  $G$  such that either  $\gamma(G') < \gamma(G)$  or  $\gamma(G') = \gamma(G)$  and  $G'$  has fewer  $\gamma$ -achieving subgraphs than  $G$  has. Replacing  $G$  by  $G'$ , we repeat the process until a 1-balanced graph is obtained.

In the last section, we provide a conjecture whose truth would decrease the number of steps required to transform a graph into a 1-balanced graph. We also present a theorem in support of our conjecture.

A loopless matroid  $M$  with rank  $r$  is *cyclically orderable* if there is an ordering  $(e_1, e_2, \dots, e_m)$  of all the elements in  $E(M)$  such that every  $r$  cyclically consecutive elements in the ordering form a base of  $M$ . Kajitani et al. conjectured [17, page 190] that  $M$  is cyclically orderable if and only if  $M$  is 1-balanced. This conjecture is recently reported to be proved by Jan van den Henvel and Stéphane Thomassé [31]. Thus our main result indicates that for any connected loopless graph  $G$ , there is a sequence of edge-switchings that will give rise to a graph whose cycle matroid is cyclical orderable.

## 2. Preliminaries

**Lemma 4** ([16], Theorems 2 and 3). *A graph  $G$  is 1-balanced if and only if for all non-trivial, induced, connected subgraphs  $H$  of  $G$ ,  $g(H) \leq g(G)$ .*

**Lemma 5** ([5], Theorem 16). *Let  $G$  be a graph. Suppose  $g(H_1) = g(H_2) = \gamma(G)$  for subgraphs  $H_1, H_2$  of  $G$ . Then  $g(H_1 \cup H_2) = \gamma(G)$ . Furthermore, if  $H_1 \cap H_2$  has an edge, then  $g(H_1 \cap H_2) = \gamma(G)$ .*

As an important consequence of Lemma 5, we note that  $G$  has a unique maximal  $\gamma$ -achieving subgraph without isolated vertices and that each component of this subgraph is a maximal, connected  $\gamma$ -achieving subgraph and is vertex induced. This fact is used frequently in the paper. In the next section, we show how the algorithm in [15] can be used to find the maximal  $\gamma$ -achieving subgraph of a graph.

**Lemma 6** ([14]). *Let  $p_1/q_1, p_2/q_2, \dots, p_k/q_k$  be fractions in which  $p_i$  is a real number and  $q_i$  is a positive real number for each  $i \in \{1, 2, \dots, k\}$ . Then*

$$\min_{1 \leq i \leq k} \frac{p_i}{q_i} \leq \frac{p_1 + p_2 + \dots + p_k}{q_1 + q_2 + \dots + q_k} \leq \max_{1 \leq i \leq k} \frac{p_i}{q_i},$$

with equality on both sides if and only if the fractions  $p_i/q_i, i \in \{1, 2, \dots, k\}$ , are all the same.

For any subgraph  $H$  of  $G$ , we define the *rank* of  $H$  as the size of the maximal forest contained in  $H$ . We denote the rank of  $H$  as  $\rho(H)$ . Thus  $\rho(H) = |V(H)| - \omega(H)$ . The following lemma follows easily.

**Lemma 7.** *If the graph  $G$  is not 1-balanced, then the rank of the maximal  $\gamma$ -achieving subgraph cannot be more than  $|V(G)| - 2$ .*

**Proof.** If the rank of the maximal  $\gamma$ -achieving subgraph  $H$  is  $|V(G)| - 1$ , then  $H$  is spanning and induced. This implies  $H = G$ , or in other words,  $G$  is 1-balanced, which is a contradiction.  $\square$

If  $G$  is a graph and  $H$  is a non-trivial subgraph of  $G$ , let  $G/H$  be the graph obtained by contracting the edges of  $H$ . If the graph  $H$  is induced,  $G/H$  does not have loops. Moreover,  $F \subset E(G)$  such that  $F \cap E(H) = \emptyset$  corresponds in a natural way to an edge set in  $G/H$ . For convenience, we denote the corresponding edge sets of  $G - E(H)$  and  $G/H$  as the same, although, if needed in the context, we specify the graph in which the edge set belongs.

The following lemma is proved in [5] in the case that  $H$  is a maximal  $\gamma$ -achieving subgraph of  $G$ . Here, we add the condition "connected" to  $H$ .

**Lemma 8.** *Let  $G$  be a graph that is not 1-balanced and let  $H$  be a maximal connected  $\gamma$ -achieving subgraph of  $G$ . Let  $v$  be the vertex in  $G/H$  obtained by contracting the edges of  $H$ . If  $\hat{H}$  is a connected subgraph of  $G/H$  containing the vertex  $v$ , then  $g(\hat{H}) < \gamma(G)$ .*

**Proof.** Since  $H$  is a maximal connected  $\gamma$ -achieving subgraph of  $G$  and the graph  $G[E(H) \cup E(\hat{H})]$  is a connected subgraph of  $G$  strictly containing  $H$ , we have

$$g(G[E(H) \cup E(\hat{H})]) < \gamma(G). \tag{1}$$

On the other hand,

$$|E(H) \cup E(\hat{H})| = |E(H)| + |E(\hat{H})|$$

and

$$|V(G[E(H) \cup E(\hat{H})])| = |V(H)| + |V(\hat{H})| - 1.$$

Thus

$$g(G[E(H) \cup E(\widehat{H})]) = \frac{|E(H)| + |E(\widehat{H})|}{|V(H)| + |V(\widehat{H})| - 2} \geq \min \left\{ \frac{|E(H)|}{|V(H)| - 1}, \frac{|E(\widehat{H})|}{|V(\widehat{H})| - 1} \right\}$$

by Lemma 6.

But

$$\frac{|E(H)|}{|V(H)| - 1} = g(H) = \gamma(G)$$

and

$$\frac{|E(\widehat{H})|}{|V(\widehat{H})| - 1} = g(\widehat{H}).$$

Hence

$$g(G[E(H) \cup E(\widehat{H})]) \geq \min\{\gamma(G), g(\widehat{H})\}. \tag{2}$$

If  $g(\widehat{H}) \geq \gamma(G)$ , then by (2) we have  $g(G[E(H) \cup E(\widehat{H})]) \geq \gamma(G)$ , a contradiction to (1). Hence  $g(\widehat{H}) < \gamma(G)$ .  $\square$

### 3. Finding the maximal $\gamma$ -achieving subgraph of a graph

In this section, we give a method to find the maximal  $\gamma$ -achieving subgraph of a connected graph  $G$ . We first present a theorem which is used later in the section to find the maximal  $\gamma$ -achieving subgraph. Suppose  $s, t$  are integers such that  $\gamma(G) = \frac{s}{t}$ . In view of the paragraph following Theorem 1, there is a family  $\mathfrak{F}$  of  $s$  forests in  $G$  such that each edge of  $G$  lies in exactly  $t$  forests of  $\mathfrak{F}$ . If  $H$  is a subgraph of  $G$ , let  $\mathfrak{F}_H := \{F \cap H : F \in \mathfrak{F}\}$ . A forest  $F$  in a graph  $G$  is maximal if  $|V(F)| - \omega(F) = |V(G)| - \omega(G)$ .

**Theorem 9.** *Let  $G$  be a graph with  $\gamma(G) = \frac{s}{t}$ , where  $s$  and  $t$  are positive integers. Let  $\mathfrak{F}$  be a family of  $s$  forests such that each edge of  $G$  appears in exactly  $t$  forests of  $\mathfrak{F}$ . Let  $H$  be the maximal  $\gamma$ -achieving subgraph of  $G$ . Then  $\mathfrak{F}_H$  is a collection of  $s$  maximal forests in  $H$ . Moreover,  $H$  is the maximal subgraph without isolated vertices satisfying this property.*

**Proof.** Let  $\mathfrak{F} = \{F_1, \dots, F_s\}$  and let  $F'_i := F_i \cap H$  for  $i = 1, \dots, s$ . Since  $g(H) = \gamma(G)$ , we have

$$g(H) = \frac{s}{t}. \tag{3}$$

For  $i = 1, \dots, s$ , we have  $|V(F'_i)| - \omega(F'_i) \leq |V(H)| - \omega(H)$  since  $F'_i$  is a forest in  $H$ . Suppose  $|V(F'_j)| - \omega(F'_j) < |V(H)| - \omega(H)$  for some  $j \in \{1, \dots, s\}$ , then we have

$$t|E(H)| \leq \sum_{i=1}^s (|V(H)| - \omega(F'_i)) < s(|V(H)| - \omega(H)).$$

Therefore,

$$g(H) = \frac{|E(H)|}{|V(H)| - \omega(H)} < \frac{s}{t},$$

a contradiction to (3). Thus,  $|V(F'_i)| - \omega(F'_i) = |V(H)| - \omega(H)$  for  $i = 1, \dots, s$  and so,  $F'_1, \dots, F'_s$  are maximal forests in  $H$ .

Let  $H'$  be a maximal subgraph of  $G$  without isolated vertices such that  $\mathfrak{F}_{H'}$  is a collection of  $s$  maximal forests in  $H'$ . Then  $H \subseteq H'$  since  $\mathfrak{F}_H$  is a collection of maximal forests in  $H$ . So,

$$t|E(H')| = s(|V(H')| - \omega(H')),$$

implying  $g(H') = \frac{|E(H')|}{|V(H')| - \omega(H')} = \frac{s}{t} = \gamma(G)$ . Thus  $H' \subseteq H$  since  $H$  is the maximal  $\gamma$ -achieving subgraph. Therefore,  $H = H'$ .  $\square$

The algorithm in [15] finds a family  $\mathfrak{F} = \{F_1, \dots, F_s\}$  as specified in Theorem 9 with a time complexity of  $O(|E(G)|^3|V(G)|^4)$ . Using this family, the maximal  $\gamma$ -achieving subgraph can be found as follows: By Theorem 9, the maximal  $\gamma$ -achieving subgraph of  $G$  is the union of all the non-trivial subgraphs of  $G$  that are induced by the vertex sets of the form  $\cap_{i=1}^s U_i$ , where  $U_i$  is the vertex set of a component of  $F_i$ , for  $i = 1, \dots, s$ .

#### 4. Transforming a graph into a 1-balanced graph

**Theorem 10.** *If  $G$  is a connected graph that is not 1-balanced, then there exists a connected graph  $G'$  with the vertex set  $V(G)$  such that*

- (a)  $G - e = G' - e'$  for some  $e \in E(G)$ ,  $e' \in E(G')$  such that  $e$  and  $e'$  have a common end-vertex; and
- (b)  $\gamma(G') \leq \gamma(G)$ , and if  $\gamma(G') = \gamma(G)$ , then all the  $\gamma$ -achieving subgraphs of  $G'$  are  $\gamma$ -achieving subgraphs of  $G$ . Furthermore, the size of the maximal  $\gamma$ -achieving subgraph of  $G'$  is smaller than that of  $G$ .

**Proof.** Let  $H$  be a maximal connected  $\gamma$ -achieving subgraph of  $G$ . Then,  $H \neq G$  since  $G$  is not 1-balanced. Let  $f = uv$  be an edge in  $G$  with  $u \in V(H)$  and  $v \notin V(H)$ . There is such an edge since  $H \neq G$  and  $G$  is connected. Let  $e = uw$  be an edge in  $H$  incident to  $u$ . Form a new graph  $G'$  from  $G$  by removing the edge  $e$  and adding a new edge  $e' = vw$ . Clearly,  $V(G) = V(G')$  and (a) is satisfied.

To check (b), in view of Lemma 4, we show that if  $H'$  is a non-trivial, induced, connected subgraph of  $G'$ , then

$$g(H') \leq \gamma(G) \quad \text{if } e' \notin E(H') \quad \text{and} \tag{4}$$

$$g(H') < \gamma(G) \quad \text{if } e' \in E(H'). \tag{5}$$

Before proving (4) and (5), we show that if (4) and (5) are true, then (2) holds. By (4) and (5) and the definition of  $\gamma$ , we conclude that  $\gamma(G') \leq \gamma(G)$ . Further, if  $\gamma(G') = \gamma(G)$ , by (5), any connected subgraph of  $G$  containing  $e'$  cannot be a  $\gamma$ -achieving subgraph of  $G'$ . Hence, any  $\gamma$ -achieving subgraph  $H'$  of  $G'$  does not contain  $e'$  and, being a subgraph of  $G$ ,  $H'$  is a  $\gamma$ -achieving subgraph of  $G$ . On the other hand,  $H$  is a  $\gamma$ -achieving subgraph of  $G$ ; hence the maximal  $\gamma$ -achieving subgraph of  $G$  contains  $e$ . Therefore all  $\gamma$ -achieving subgraphs of  $G'$  are  $\gamma$ -achieving subgraphs in  $G$  and thus they do not contain  $e$  and  $e'$ . We conclude that the maximal  $\gamma$ -achieving subgraph of  $G'$  is properly contained in the maximal  $\gamma$ -achieving subgraph of  $G$ .

*Proof of (4) and (5):* Let  $H'$  be an induced, connected subgraph of  $G'$ . If  $H'$  does not contain  $e'$ , then  $H'$  is a subgraph of  $G$ . Thus  $g(H') \leq \gamma(G)$  and (4) is verified.

Let us now suppose that  $H'$  contains the edge  $e'$ . Then  $v, w \in V(H')$ .

*Case (i):*  $u \in V(H')$ . In this case,  $H'' := G[V(H')]$  is connected since any path  $P'$  in  $H'$  containing the edge  $e' = vw$  can be modified to obtain a walk  $P''$  in  $H''$  by replacing the edge  $vw$  by the edges  $vu, uv$  in that order. Also,  $H''$  contains the edge  $e$  but not  $e'$ . Hence,  $H'$  and  $H''$  are both connected and have the same number of edges on the same number of vertices, so

$$g(H') = g(H''). \tag{6}$$

But  $H'' \neq H$  since  $H''$  contains the edge  $f = uv$ . Note that  $H$  is a maximal connected  $\gamma$ -achieving subgraph and  $H''$  is a connected subgraph of  $G$  whose vertex set intersects with  $V(H)$  and with  $V(G - H)$ . By Lemma 5, we have

$$g(H'') < \gamma(G).$$

Therefore by (6),

$$g(H') < \gamma(G).$$

*Case (ii):*  $u \notin V(H')$ . Then  $f = uv \notin E(H')$ . Let  $E_1 = E(H') \cap E(H)$  and  $E_2 = E(H') - E_1$ . Thus  $e' \in E_2$ . Note that  $f \notin E_2$ . Let  $H_1 = G[E_1]$ .

Let  $\widehat{G} = G/H$  and  $\widehat{G}' = G'/(H - e)$ . Let  $H_2 = \widehat{G}[E_2]$ . Then  $H_2$  is connected and isomorphic to  $\widehat{H} := \widehat{G}[E_2 - e' + f]$ . Thus

$$g(H_2) = g(\widehat{H}) = \frac{|E_2|}{\rho_{\widehat{G}}(E_2)}. \tag{7}$$

By Lemma 8,  $g(\widehat{H}) < \gamma(G)$ . Thus

$$g(H_2) < \gamma(G). \tag{8}$$

If  $H_1$  is a graph with no edges, then

$$g(H') = \frac{|E_2|}{\rho_{G'}(E_2)}. \tag{9}$$

But  $\rho_{G'}(E_2) \geq \rho_{\widehat{G}}(E_2)$ . Hence we have

$$g(H') \leq \frac{|E_2|}{\rho_{\widehat{G}}(E_2)} = g(H_2) < \gamma(G) \tag{10}$$

by (7) and (8). Thus (5) holds. Therefore, we assume that  $E_1 \neq \emptyset$ . We have

$$g(H_1) \leq \gamma(G) \tag{11}$$

since  $H_1$  is a subgraph of  $G$ . Note that  $|V(H')| = |V(H_1)| + |V(H_2)| - 1$ . By Lemma 6 we have

$$g(H') = \frac{|E_1| + |E_2|}{|V(H_1)| + |V(H_2)| - 2} \leq \max_{i=1,2} \frac{|E_i|}{|V(H_i)| - 1}, \tag{12}$$

with equality if and only if  $\frac{|E_1|}{|V(H_1)|-1} = \frac{|E_2|}{|V(H_2)|-1}$ .

But,

$$\frac{|E_1|}{|V(H_1)| - 1} \leq \frac{|E_1|}{|V(H_1)| - \omega(H_1)} = g(H_1) \tag{13}$$

and

$$\frac{|E_2|}{|V(H_2)| - 1} = g(H_2) \tag{14}$$

since  $H_2$  is connected. Thus by (12)–(14),

$$g(H') \leq \max_{i=1,2} \{g(H_1), g(H_2)\}, \tag{15}$$

with equality if and only if  $g(H_1) = g(H_2)$  and  $g(H_1) = \frac{|E_1|}{|V(H_1)|-1}$ . But  $g(H_2) < \gamma(G)$  by (8) and  $g(H_1) \leq \gamma(G)$  by (11). Thus  $g(H') < \gamma(G)$  by (5) and (15) holds.  $\square$

Now, we describe an algorithm to modify a given graph  $G$  that is not 1-balanced into a graph that is 1-balanced. Since  $G$  is not 1-balanced, we have  $|V(G)| > 2$ , because all graphs on 2 vertices are 1-balanced. Let  $i = 1$  initially and  $G_1 = G$ . The algorithm proceeds as follows: Pick a maximal connected  $\gamma$ -achieving subgraph  $H_i$  of  $G_i$ . Let  $e_i = u_i w_i \in E(H_i)$  such that  $u_i$  is adjacent to a vertex  $v_i \in V(G_i) - V(H_i)$ . Let  $G_{i+1} = G_i - e_i + v_i w_i$ . If  $G_{i+1}$  is 1-balanced, we are done. Otherwise, replace  $i$  with  $i + 1$  and repeat the procedure.

The algorithm terminates when a 1-balanced graph is obtained. By Theorem 10, for  $i \geq 1$ , we have

- (i)  $\gamma(G_{i+1}) \leq \gamma(G_i)$  and
- (ii) if  $\gamma(G_{i+1}) = \gamma(G_i)$  then the size of the maximal  $\gamma$ -achieving subgraph of  $G_{i+1}$  is less than that of  $G_i$ .

Thus, we have an integer  $k$  and integers  $0 = i_0 < i_1 < \dots < i_k$  such that

- $\gamma(G_{i_j+1}) = \dots = \gamma(G_{i_{j+1}})$  for  $j = 0, \dots, k - 1$ ,
- $\gamma(G_{i_j}) > \gamma(G_{i_{j+1}})$  for  $j = 1, \dots, k$  and
- $\gamma(G_{i_k+1}) = g(G)$ , and the algorithm terminates.

Note that there are  $i_k$  iterations in the algorithms. We now calculate the following in order to obtain an upper bound for  $i_k$ .

- (1)  $l := \max\{i_{(j+1)} - i_j : j = 0, \dots, k - 1\}$  = maximum possible number of consecutive steps  $i$  with the same value of  $\gamma(G_{i+1})$ .
- (2)  $k$ .

The total number of iterations  $i_k$  is bounded by  $(l + 1)k$ . There is a factor of  $l + 1$  in this bound in order to count the last step. (Note that after the last iteration, the value of  $\gamma$  decreases to  $g(G)$ ).

(1) For each  $j \in \{0, \dots, k - 1\}$ , the maximal  $\gamma$ -achieving subgraph of  $G_{i_{(j+1)}}$  is contained in the maximal  $\gamma$ -achieving subgraph of  $G_{i_{j+1}}$ . Thus  $i_{(j+1)} - i_j$  is less than the rank of  $G_{i_{j+1}}$ . Since  $G_{i_{j+1}}$  is not 1-balanced, by Lemma 7, the rank of the maximal  $\gamma$ -achieving subgraph of  $G_{i_{j+1}}$  is at most  $|V(G)| - 2$ . Therefore,  $i_{(j+1)} - i_j \leq |V(G)| - 2$  and by the definition of  $l$ , we have  $l \leq |V(G)| - 2$ .

(2) Suppose  $\gamma(G_{i+1}) < \gamma(G_i)$  for some  $i \geq 1$ . Then

$$\begin{aligned} \gamma(G_i) - \gamma(G_{i+1}) &= \frac{|E(H_i)|}{|V(H_i)| - 1} - \frac{|E(H_{i+1})|}{|V(H_{i+1})| - 1} \\ &= \frac{|E(H_i)|(|V(H_{i+1})| - 1) - |E(H_{i+1})|(|V(H_i)| - 1)}{(|V(H_i)| - 1)(|V(H_{i+1})| - 1)} \\ &\geq \frac{1}{(|V(G)| - 1)(|V(G)| - 2)} \end{aligned} \tag{16}$$

since the numerator is greater than 1 and in the denominator,  $|V(H_i)| < |V(G)|$  and  $|V(H_{i+1})| \leq |V(G)|$ . By (16) we need at most  $(\gamma(G) - g(G))(|V(G)| - 1)(|V(G)| - 2)$  such iterations.

If  $H$  is a  $\gamma$ -achieving subgraph of  $G$ , then

$$\gamma(G) - g(G) = \frac{|E(H)|}{|V(H)| - 1} - g(G) < |E(G)|. \tag{17}$$

Thus

$$k \leq |E(G)|(|V(G)| - 1)(|V(G)| - 2).$$

Therefore,

$$(l + 1)k \leq |E(G)|(|V(G)| - 1)^2(|V(G)| - 2) = O(|E(G)||V(G)|^3).$$

Since at each iteration the maximal  $\gamma$ -achieving subgraph is calculated with  $O(|E(G)|^3|V(G)|^4)$  time complexity using the algorithm in [15], the overall time complexity of our algorithm is  $O(|E(G)|^4|V(G)|^7)$ .

The following corollary proves the existence of 1-balanced graphs on  $n$  vertices and  $m$  edges for all  $m \geq n - 1$ . This result was obtained by Ruciński and Vince [29] for  $n - 1 \leq m \leq \frac{n(n-1)}{2}$ .

**Corollary 11.** For integers  $m, n$  such that  $m \geq n - 1$ , there is a connected 1-balanced graph on  $n$  vertices and  $m$  edges.

**Proof.** Taking any connected graph on  $n$  vertices and  $m$  edges and applying Theorem 10 repetitively, we obtain a 1-balanced graph on  $n$  vertices and  $m$  edges.  $\square$

## 5. Minimizing the number of iterations

In Section 3, we saw how one can find the maximal  $\gamma$ -achieving subgraph of a graph. Also 1-balanced graphs are dealt with in a more generalized setting by Narayanan [25] and a polynomial time algorithm to find all the  $\gamma$ -achieving subgraphs of a graph in this generalized setting can be found in [25, pages 412–421].

In this section, we point out how knowing all the  $\gamma$ -achieving subgraphs of a graph helps in reducing the number of iterations in achieving a 1-balanced graph.

In the previous section, we showed that at most  $|E(G)||V(G)|^3$  iterations are needed to achieve a 1-balanced graph; however, the number of iterations could be very much less. The estimate is the best we could obtain that could be expressed in terms of known parameters of the graph. The reason for this is that there is no good estimate for  $k$  in the discussion following Theorem 10. However, in the next paragraph, we discuss how to minimize the number of consecutive iterations with the same  $\gamma$  value.

Suppose a graph  $G$  is not 1-balanced and there are two  $\gamma$ -achieving subgraphs  $H_1, H_2$  having at least one common edge. Changing one end vertex of an edge from  $H_1 \cap H_2$  to a vertex outside  $H_1 \cup H_2$  not only decreases the densities of both  $H_1$  and  $H_2$  at the same time, but also decreases the densities of  $H_1 \cap H_2$  and  $H_1 \cup H_2$ , which are also  $\gamma$ -achieving. We take care of this idea by addressing the case in which there is a nested sequence of  $\gamma$ -achieving subgraphs. At any point of the algorithm, the collection of all minimal  $\gamma$ -achieving subgraphs is a collection  $C$  of pairwise edge-disjoint  $\gamma$ -achieving subgraphs. Since each of these subgraphs has to be reduced by an edge move, there have to be at least  $|C|$  iterations before the  $\gamma$  value decreases. We conjecture that exactly  $|C|$  iterations are enough to decrease the value of  $\gamma$ . As a consequence of the following theorem, we show that at most  $2|C|$  iterations are enough.

**Theorem 12.** Let  $G$  be a connected non-1-balanced graph and let  $H_1 \subseteq H_2 \subseteq \dots \subseteq H_k$  be a sequence of connected  $\gamma$ -achieving subgraphs of  $G$  such that  $H_k$  is a maximal connected  $\gamma$ -achieving subgraph of  $G$ . Then after two iterations, each consisting of changing one end vertex of one edge, a new graph  $G'$  can be obtained with  $\gamma(G') \leq \gamma(G)$ , and if  $\gamma(G') = \gamma(G)$ , then all  $\gamma$ -achieving subgraphs of  $G'$  are  $\gamma$ -achieving subgraphs in  $G$ , and for  $i = 1, \dots, k$ , the subgraph  $G'[V(H_i)]$  is not  $\gamma$ -achieving in  $G'$ .

**Proof.** Since  $G$  is connected, there exists an edge  $e$  with end-points  $u \in H_k$  and  $v \notin H_k$ . If  $u \in H_1$ , the theorem holds by Theorem 10. The number of iterations is only one. If  $u \notin H_1$ , let  $\bar{u}$  be a vertex in  $H_1$ . Let  $\bar{G} := G - e + \bar{u}v$ . Let  $\bar{e}$  be the new edge  $\bar{u}v$ .

**Claim.**  $\gamma(G) = \gamma(\bar{G})$  and both  $G$  and  $\bar{G}$  have the same  $\gamma$ -achieving subgraphs.

**Proof of Claim.** Note that  $H_i$  with  $1 \leq i \leq k$  are subgraphs of  $\bar{G}$ . Since  $g(H_i) = \gamma(G)$  for  $1 \leq i \leq k$ , we have

$$\gamma(\bar{G}) \geq \gamma(G). \tag{18}$$

Let  $\bar{H}$  be a connected  $\gamma$ -achieving subgraph of  $\bar{G}$ . We show that  $\bar{H}$  is a subgraph of  $G$ . This proves the claim.

For a contradiction, suppose  $\bar{H} \not\subseteq G$ . Then  $\bar{e} \in E(\bar{H})$ . Let  $E_1 = E(H_k) \cap E(\bar{H})$  and  $E_2 = E(\bar{H}) - E_1$ . Then

$$|E(\bar{H})| = |E_1| + |E_2|$$

and

$$|V(\bar{H})| = |V(E_1)| + |V(\bar{H}/E_1)| - \omega(\bar{G}[E_1]).$$



Therefore we have

$$g(\bar{H}) = \frac{|E(\bar{H})|}{|V(\bar{H})| - 1} = \frac{|E_1| + |E_2|}{|V(E_1)| + |V(\bar{H}/E_1)| - \omega(\bar{G}[E_1]) - 1} \leq \max \left\{ \frac{|E_1|}{|V(E_1)| - \omega(\bar{G}[E_1])}, \frac{|E_2|}{|V(\bar{H}/E_1)| - 1} \right\}, \quad (19)$$

with equality if and only if  $\frac{|E_1|}{|V(E_1)| - \omega(\bar{G}[E_1])} = \frac{|E_2|}{|V(\bar{H}/E_1)| - 1}$  by Lemma 6. But  $\omega(\bar{G}[E_1]) = \omega(G[E_1])$ , so

$$\frac{|E_1|}{|V(E_1)| - \omega(\bar{G}[E_1])} = \frac{|E_1|}{|V(E_1)| - \omega(G[E_1])} = g(G[E_1]). \quad (20)$$

But  $G[E_1]$  is a subgraph of  $G$ , so

$$g(G[E_1]) \leq \gamma(G). \quad (21)$$

By (20) and (21), we have

$$\frac{|E_1|}{|V(E_1)| - \omega(\bar{G}[E_1])} \leq \gamma(G). \quad (22)$$

Since  $V((\bar{H} \cup H_k)/H_k) \subseteq V(\bar{H}/E_1)$  and  $E((\bar{H} \cup H_k)/H_k) = E_2$ , we have

$$\frac{|E_2|}{|V(\bar{H}/E_1)| - 1} \leq \frac{|E_2|}{|V((\bar{H} \cup H_k)/H_k)| - 1} = g((\bar{H} \cup H_k)/H_k). \quad (23)$$

The graph  $(\bar{H} \cup H_k)/H_k$  is isomorphic to  $G[(V(\bar{H} \cup H_k))/H_k]$  which is a connected subgraph of  $G/H_k$  containing the vertex obtained from the contracted edges. Thus,

$$g((\bar{H} \cup H_k)/H_k) < \gamma(G) \quad (24)$$

by Lemma 8. By (23) and (24), we have

$$\frac{|E_2|}{|V(\bar{H}/E_1)| - 1} < \gamma(G). \quad (25)$$

By (19), (22) and (25),  $\gamma(\bar{G}) = g(\bar{H}) < \gamma(G)$ , a contradiction to (18). Thus,  $H'$  is a subgraph of  $G$ . Thus the claim.

In  $\bar{G}$ , let  $\bar{w}$  be a vertex adjacent to  $\bar{u}$ . By Theorem 10,  $G' = \bar{G} - \bar{u}\bar{w} + \bar{w}\bar{v}$  is a graph with the vertex set  $V(\bar{G})$  such that  $\gamma(G') \leq \gamma(G)$ , and if  $\gamma(G') = \gamma(G)$ , then all  $\gamma$ -achieving subgraphs of  $G'$  are  $\gamma$ -achieving subgraphs in  $G$ , and for  $i = 1, \dots, k$ , the subgraph  $G'[V(H_i)]$  is not  $\gamma$ -achieving in  $G'$ .  $\square$

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